TURKISH JOURNAL OF SCIENCE
VOLUME 5, ISSUE 3, 233–241
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Weak Structures on Pythagorean Fuzzy Soft Topological Spaces

Taha Yasin ÖZTÜRK[a]

[a] *Department of Mathematic, Kafkas University, 36100, Kars, Turkey*

Abstract. In this paper, we initiate the topological structures of pythagorean fuzzy soft semi-open sets and pythagorean fuzzy soft semi-closed sets. Furthermore, the concept of pythagorean fuzzy soft semi-interior, pythagorean fuzzy soft semi-closure are presented. Also some related properties are investigated.

1. Introduction

Molodtsov [20] has presented soft-set theory as a new mathematical method for working with complexity, imprecise and uncertainly defined objects, and overcoming incompatibility with parameterization methods, where theories such as fuzzy sets, intuitionistic fuzzy sets theory, rough set theory fall short. The soft set theory proved useful in a number of areas, not restricted to decision-making [8, 26], data analysis [6, 34], forecasting [29] and so on. Topological structures for soft sets are studied and explored in [1, 2, 10, 11]. In [21] Molodtsov et al. listed a variety of directions for the implementation of soft sets, such as smoothness of functions, game theory, operational analysis, Riemann integration, Perron integration, probability and calculation theory for modeling problems in architecture, computer science, economics, social sciences.

The concept of fuzzy sets was initiated by Zadeh [33]. Intuitionistic fuzzy set (IFS) and intuitionistic L-fuzzy sets (ILFS) were initiated and discussed by Atanassov [3] to generalize the idea of fuzzy set. Maji et al. developed the idea of intuitionistic fuzzy soft sets [18], a generalization of fuzzy soft sets [17] and standard soft sets[19]. Coker [7] has presented and researched the concept of intuitionistic fuzzy topological spaces. Li et al. [16] initiated intuitionistic fuzzy topological constructs of intuitionistic fuzzy soft sets. They discussed the notions of intuitionistic fuzzy soft open (closed) sets, intuitionistic fuzzy soft interior (closure) and intuitionistic fuzzy soft base in intuitionistic fuzzy soft topological spaces. Recently, Hussain [12] initiated the idea of an intuitionistic fuzzy soft boundary and discussed the features and properties of the intuitionistic fuzzy soft boundary in general as well as the intuitionistic fuzzy soft interior and intuitionistic fuzzy soft closure. He also studied some weak structures on intuitionistic fuzzy soft topological spaces [13].

Yager [30, 31] introduced Pythagorean fuzzy set(PFS) as an expansion of Atanassov's intuitionistic fuzzy set and provided Pythagorean membership ratings for multi-criteria decision-making (MCDM) implementations. The main features of the PFS are that the sum of the membership degree and non-membership squares for each alternative is less than or equal to 1. Obviously, PFSs have more power than IFSs to model

Corresponding author: TYÖ mail address: taha36100@hotmail.com ORCID:0000-0003-2402-6507
Received: 2 December 2020; Accepted: 18 December 2020; Published: 30 December 2020
Keywords. (Pythagorean fuzzy soft set, pythagorean fuzzy soft topological spaces, pythagorean fuzzy soft semi-open(closed) sets, pythagorean fuzzy soft semi-interior, pythagorean fuzzy soft semi-closure,)
2010 *Mathematics Subject Classification*. 03E72,3E75, 94D05, 54C05
Cited this article as: Öztürk TY. Weak Structures on Pythagorean Fuzzy Soft Topological Spaces. Turkish Journal of Science. 2020, 5(3), 233-241.

the vagueness of realistic MCDM issues. The Pythagorean fuzzy soft set theory was defined by Peng et al.[24], and its significant properties were studied. Pythagorean fuzzy topology introduced by Olgun et al [22]. Also Riaz et al. [27] and Yolcu and Ozturk [32] studied on Pythagorean fuzzy soft topological spaces. Pythagorean fuzzy set theory is one of the most studied topics of recent times [4, 5, 9, 14, 23, 25, 28].

In this paper, we initiate and define the topological structures of pythagorean fuzzy soft semi-open sets and pythagorean fuzzy, soft semi-closed sets. We also investigate the properties of pythagorean fuzzy soft semi-interior, pythagorean fuzzy soft semi-closure, and discuss the relationship between them.

2. Preliminaries

Definition 2.1. *[33] Let X be an universe. A fuzzy set F in X,* $F = \{(x, \mu_F(x)) : x \in X\}$, *where* $\mu_F : X \to [0, 1]$ *is the membership function of the fuzzy set F;* $\mu_F(x) \in [0, 1]$ *is the membership of* $x \in X$ *in f. The set of all fuzzy sets over X will be denoted by* $FS(X)$.

Definition 2.2. *[3] An intuitionistic fuzzy set F in X is* $F = \{(x, \mu_F(x), v_F(x)) : x \in X\}$, *where* $\mu_F : X \to [0, 1]$, $v_F : X \to [0, 1]$ *with the condition* $0 \le \mu_F(x) + v_F(x) \le 1$, $\forall x \in X$. *The numbers* $\mu_F, v_F \in [0, 1]$ *denote the degree of membership and non-membership of x to F, respectively. The set of all intuitionistic fuzzy sets over X will be denoted by* $IFS(X)$.

Definition 2.3. *[20] Let E be a set of parameters and X be the universal set. A pair* (F, E) *is called a soft set over X, where F is a mapping* $F : E \to \mathcal{P}(X)$. *In other words, the soft set is a parameterized family of subsets of the set X.*

Definition 2.4. *[17] Let E be a set of parameters and X be the universal set. A pair* (F, E) *is called a fuzzy soft set over X, If* $F : E \to FS(X)$ *is a mapping from E into set of all fuzzy sets in X, where* $FS(X)$ *is set of all fuzzy subset of X.*

Definition 2.5. *[18] Let X be an initial universe E be a set of parameters. A pair* (F, E) *is called an intuitionistic fuzzy soft set over X, where F is a mapping given by,* $F : E \to IFS(X)$.

In general, for every $e \in E$, $F(e)$ *is an intuitionistic fuzzy set of X and it is called intuitionistic fuzzy value set of parameter e. Clearly,* $F(e)$ *can be written as a intuitionistic fuzzy set such that* $F(e) = \{(x, \mu_F(x), v_F(x)) : x \in X\}$

Definition 2.6. *[30] Let X be a universe of discourse. A pythagorean fuzzy set (PFS) in X is given by,* $P = \{(x, \mu_P(x), v_P(x)) : x \in X\}$ *where,* $\mu_P : X \to [0, 1]$ *denotes the degree of membership and* $v_p : X \to [0, 1]$ *denotes the degree of nonmembership of the element* $x \in X$ *to the set P with the condition that* $0 \le (\mu_P(x))^2 + (v_P(x))^2 \le 1$.

Definition 2.7. *[24] Let X be the universal set and E be a set of parameters. The pythagorean fuzzy soft set is defined as the pair* (F, E) *where,* $F : E \to PFS(X)$ *and* $PFS(X)$ *is the set of all Pythagorean fuzzy subsets of X. If* $\mu_F^2(x) + v_F^2(x) \le 1$ *and* $\mu_F(x) + v_F(x) \le 1$, *then pythagorean fuzzy soft sets degenerate into intuitionistic fuzzy soft sets.*

Definition 2.8. *[24] Let* $A, B \subseteq E$ *and* (F, A), (G, B) *be two pythagorean fuzzy soft sets over X.* (F, A) *is said to be pythagorean fuzzy soft subset of* (G, B) *denoted by* $(F, A)\tilde{\subseteq}(G, B)$ *if,*

1. $A \subseteq B$
2. $\forall e \in A$, *F(e) is a pythagorean fuzzy subset of G(e) that is,* $\forall x \in U$ *and* $\forall e \in A$, $\mu_{F(e)}(x) \le \mu_{G(e)}(x)$ *and* $v_{F(e)}(x) \ge v_{G(e)}(x)$. *If* $(F, A)\tilde{\subseteq}(G, B)$ *and* $(G, B)\tilde{\subseteq}(F, A)$ *then* (F, A), (G, B) *are said to be equal.*

Definition 2.9. *[24] Let* (F, E) *two pythagorean fuzzy soft sets over X. The complement of* (F, E) *is denoted by* $(F, E)^c$ *and is defined by*

$$(F, E)^c = \{(e, (x, v_{F(e)}(x), \mu_{F(e)}(x)) : x \in X) : e \in E\}$$

Definition 2.10. *[15] a) A pythagorean fuzzy soft set (F,E) over the universe X is said to be null pythagorean fuzzy soft set if $\mu_{F(e)}(x)=0$ and $v_{F(e)}(x)=1$; $\forall e\in E$, $\forall x\in X$. It is denoted by $\widetilde{0}_{(X,E)}$.*

b) A pythagorean fuzzy soft set (F,E) over the universe X is said to be absolute pythagorean fuzzy soft set if $\mu_{F(e)}(x)=1$ and $v_{F(e)}(x)=0$; $\forall e\in E$, $\forall x\in X$. It is denoted by $\widetilde{1}_{(X,E)}$.

Definition 2.11. *[15] Let (F,A) and (G,B) be two pythagorean fuzzy soft sets over the universe set X, E be a parameter set and $A,B\subseteq E$.Then,*

a) Extended union of (F,A) and (G,B) is denoted by $(F,E)\widetilde{\cup}_E(G,B)=(H,C)$ where $C=A\cup B$ and (H,C) defined by;

$$(H,C)=\{(e,(x,\mu_{H(e)}(x),v_{H(e)}(x)):x\in X):e\in E\}$$

where

$$\mu_{H(e)}(x)=\begin{cases}\mu_{F(e)}(x), & \text{if } e\in A-B\\ \mu_{G(e)}(x), & \text{if } e\in B-A\\ \max\{\mu_{F(e)}(x),\mu_{G(e)}(x)\}, & \text{if } e\in A\cap B\end{cases}$$

$$v_{H(e)}(x)=\begin{cases}v_{F(e)}(x), & \text{if } e\in A-B\\ v_{G(e)}(x), & \text{if } e\in B-A\\ \min\{\mu_{F(e)}(x),\mu_{G(e)}(x)\}, & \text{if } e\in A\cap B\end{cases}$$

b) Extended intersection of (F,A) and (G,B) is denoted by $(F,E)\widetilde{\cap}_E(G,B)=(H,C)$ where $C=A\cup B$ and (H,C) defined by;

$$(H,C)=\{(e,(x,\mu_{H(e)}(x),v_{H(e)}(x)):x\in X):e\in E\}$$

where

$$\mu_{H(e)}(x)=\begin{cases}\mu_{F(e)}(x), & \text{if } e\in A-B\\ \mu_{G(e)}(x), & \text{if } e\in B-A\\ \min\{\mu_{F(e)}(x),\mu_{G(e)}(x)\}, & \text{if } e\in A\cap B\end{cases}$$

$$v_{H(e)}(x)=\begin{cases}v_{F(e)}(x), & \text{if } e\in A-B\\ v_{G(e)}(x), & \text{if } e\in B-A\\ \max\{\mu_{F(e)}(x),\mu_{G(e)}(x)\}, & \text{if } e\in A\cap B\end{cases}$$

Let X be an initial universe and $PFS(X)$ denote the family of pythagorean fuzzy sets over X and $PFSS(X,E)$ be the family of all pythagorean fuzzy soft sets over X with parameters in E.

Definition 2.12. *[32]Let $X\neq\emptyset$ be a universe set and $\widetilde{\tau}\subset PFSS(X,E)$ be a collection of pythagorean fuzzy soft sets over X, then τ is said to be on pythagorean fuzzy soft topology on X if*

(i) $\widetilde{0}_{(X,E)}$, $\widetilde{1}_{(X,E)}$ belong to $\widetilde{\tau}$,

(ii) The union of any number of pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,

(iii) The intersection of any two pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triple $(X,\widetilde{\tau},E)_p$ is called an pythagorean fuzzy soft tpological space over X. Every member of τ is called a pythagorean fuzzy soft open set in X.

Definition 2.13. *[32] a) Let X be an initial universe set, E be the set of parameters and $\widetilde{\tau}=\left\{\widetilde{0}_{(X,E)},\widetilde{1}_{(X,E)}\right\}$. Then $\widetilde{\tau}$ is called a pythagorean fuzzy soft indiscrete topology on X and $(X,\widetilde{\tau},E)_p$ is said to be a pythagorean fuzzy soft indiscrete space over X.*

b) Let X be an initial universe set, E be the set of parameters and $\widetilde{\tau}$ be the collection of all pythagorean fuzzy soft sets which can be defined over X. Then $\widetilde{\tau}$ is called a pythagorean fuzzy soft discrete topology on X and $(X,\widetilde{\tau},E)_p$ is said to be a pythagorean fuzzy soft discrete space over X.

Definition 2.14. *[32] Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X. A pythagorean fuzzy soft set (F,E) over X is said to be a pythagorean fuzzy soft closed set in X, if its complement $(F,E)^c$ belongs to $\widetilde{\tau}$.*

Proposition 2.15. *[32]Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X. Then, the following properties hold.*

(i) $\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}$ are pythagorean fuzzy soft closed set over X.

(ii) The intersection of any number of pythagorean fuzzy soft closed set is a pythagorean fuzzy soft closed set over X.

(iii) The union of any two pythagorean fuzzy soft closed set is a pythagorean fuzzy soft closed set over X.

Definition 2.16. *[32]Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and (F, E) be a pythagorean fuzzy soft sets over X. The pythagorean fuzzy soft closure of (F, E) denoted by $pcl(F, E)$ is the intersection of all pythagorean fuzzy soft closed super sets of (F, E).*

Clearly $pcl(F, E)$ is the smallest pythagorean fuzzy soft closed set over X which contain (F, E).

Definition 2.17. *[32]Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(H, E) \in PFSS(X, E)$. The pythagorean fuzzy soft interior of (H, E), denoted by $pint(H, E)$, is the union of all the pythagorean fuzzy soft open sets contained in (H, E).*

3. Main Results

Definition 3.1. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset}PFSS(X, E)$. If there exists a pythagorean fuzzy soft open set (G, E) such that $(G, E)\widetilde{\subset}(F, E)\widetilde{\subset}pcl(G, E)$, then (F, E) is called pythagorean fuzzy soft semi-open set over X.*

Definition 3.2. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset}PFSS(X, E)$. (F, E) is pythagorean fuzzy soft semi-closed set if and only if its complement $(F, E)^c$ is pythagorean fuzzy soft semi-open set.*

Remark 3.3. *It is obvious that a pythagorean fuzzy soft open set is pythagorean fuzzy soft semi-open set. But the converse is not true in general. This is shown in following example.*

Example 3.4. *Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and*

$$\widetilde{\tau} = \left\{\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}, (F_1, E), (F_2, E), (F_3, E)\right\}$$

where $(F_1, E), (F_2, E), (F_3, E)$ pythagorean fuzzy soft sets over X, defined as;

$$\begin{aligned}
(F_1, E) &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.4, 0.6), (x_2, 0.3, 0.7)\}) \\ (e_2, \{(x_1, 0.5, 0.4), (x_2, 0.7, 0.6)\}) \end{array} \right\} \\
(F_2, E) &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.5, 0.5), (x_2, 0.4, 0.5)\}) \\ (e_2, \{(x_1, 0.6, 0.4), (x_2, 0.8, 0.3)\}) \end{array} \right\} \\
(F_3, E) &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.7, 0.3), (x_2, 0.7, 0.2)\}) \\ (e_2, \{(x_1, 0.8, 0.2), (x_2, 0.9, 0.2)\}) \end{array} \right\}
\end{aligned}$$

It is clear that $\widetilde{\tau}$ is a pythagorean fuzzy soft topological spaces and $(X, \widetilde{\tau}, E)_p$ is pythagorean fuzzy soft topological spaces. The pythagorean fuzzy soft closed sets as follow;

$$\begin{aligned}
\left(\widetilde{0}_{(X,E)}\right)^c &= \widetilde{1}_{(X,E)} \\
\left(\widetilde{1}_{(X,E)}\right)^c &= \widetilde{0}_{(X,E)} \\
(F_1,E)^c &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.6, 0.4), (x_2, 0.7, 0.3)\}) \\ (e_2, \{(x_1, 0.4, 0.5), (x_2, 0.6, 0.7)\}) \end{array} \right\} \\
(F_2,E)^c &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.5, 0.5), (x_2, 0.5, 0.4)\}) \\ (e_2, \{(x_1, 0.4, 0.6), (x_2, 0.3, 0.8)\}) \end{array} \right\} \\
(F_3,E)^c &= \left\{ \begin{array}{l} (e_1, \{(x_1, 0.3, 0.7), (x_2, 0.3, 0.7)\}) \\ (e_2, \{(x_1, 0.3, 0.8), (x_2, 0.2, 0.9)\}) \end{array} \right\}
\end{aligned}$$

Now we consider a pythagorean fuzzy soft set (G,E) over X defined by,

$$(G,E) = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.6, 0.4), (x_2, 0.7, 0.3)\}) \\ (e_2, \{(x_1, 0.7, 0.3), (x_2, 0.8, 0.2)\}) \end{array} \right\}$$

Then there exist a pythagorean fuzzy open set (F_2,E) such that $(F_2,E)\widetilde{\subset}(G,E)\widetilde{\subset}pcl(F_2,E) = \widetilde{1}_{(X,E)}$.

Hence (G,E) is a pythagorean fuzzy soft semi-open set, but (G,E) is not pythagorean fuzzy soft open set.

Proposition 3.5. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. Then (F,E) is pythagorean fuzzy soft semi-open set if and only if $(F,E)\widetilde{\subset}pcl\,(pint(F,E))$.*

Proof. ($\Rightarrow$)Suppose that (F,E) is pythagorean fuzzy soft semi-open set, then there exists a pythagorean fuzzy soft open set (G,E) such that $(G,E)\widetilde{\subset}(F,E)\widetilde{\subset}pcl(G,E)$. Now $(G,E)\widetilde{\subset}pint(F,E)$ implies that $pcl(G,E)\widetilde{\subset}pcl\,(pint(F,E))$. Therefore (

$$F,E)\widetilde{\subset}pcl(G,E)\widetilde{\subset}pcl\,(pint(F,E))$$

.

($\Leftarrow$) Suppose that $(F,E)\widetilde{\subset}pcl\,(pint(F,E))$. Take $(G,E) = pint(F,E)$, we have

$$(G,E)\widetilde{\subset}(F,E)\widetilde{\subset}pcl(G,E)$$

This complete this proof. □

Theorem 3.6. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X . Then an arbitrary union of pythagorean fuzzy soft semi-open sets is pythagorean fuzzy soft semi-open set.*

Proof. Let $\{(F_i,E) : i \in I\}$ be a collection of pythagorean fuzzy soft semi-open sets and $(G,E) = \bigcup_{i\in I}(F_i,E)$. Since each (F_i,E) is PFS semi-open, then there exist a pythagorean fuzzy soft open set (H_i,E) such that $(H_i,E)\widetilde{\subset}(F_i,E)\widetilde{\subset}pcl(H_i,E)$ and so $\bigcup_{i\in I}(H_i,E)\widetilde{\subset}\bigcup_{i\in I}(F_i,E)\widetilde{\subset}\bigcup_{i\in I}pcl(H_i,E)\widetilde{\subset}pcl\left(\bigcup_{i\in I}(H_i,E)\right)$. Let $(H,E) = \bigcup_{i\in I}(H_i,E)$. Then (H,E) is pythagorean fuzzy soft open and $(H,E)\widetilde{\subset}\bigcup_{i\in I}(F_i,E)\widetilde{\subset}pcl(H,E)$. Therefore, $\bigcup_{i\in I}(F_i,E)$ is a pythagorean fuzzy soft semi-open set. □

Corollary 3.7. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X . Then the family of all pythagorean fuzzy soft semi-open sets are a pythagorean fuzzy soft supra topology on X.*

Proposition 3.8. *Let (F,E) be a pythagorean fuzzy soft semi-open set and (G,E) be a pythagorean fuzzy soft set in $(X,\widetilde{\tau},E)_p$. Suppose $(F,E)\widetilde{\subset}(G,E)\widetilde{\subset}pcl(F,E)$. Then (G,E) is a pythagorean fuzzy soft semi-open set over X.*

Proof. (F,E) be a pythagorean fuzzy soft semi-open set implies that there exist a pythagorean fuzzy soft open set (H,E) such that $(H,E)\widetilde{\subset}(F,E)\widetilde{\subset}pcl(H,E)$. Now $(H,E)\widetilde{\subset}(G,E)$ and $pcl(F,E)\widetilde{\subset}pcl(H,E)$ implies that $(G,E)\widetilde{\subset}pcl(H,E)$. Therefore $(H,E)\widetilde{\subset}(G,E)\widetilde{\subset}pcl(H,E)$. Hence (G,E) is a pythagorean fuzzy soft semi-open set in X. □

Proposition 3.9. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset}PFSS(X, E)$. Then (F, E) is pythagorean fuzzy soft semi-closed if and only if there exist a pythagorean fuzzy soft closed set (G, E) such that $pint(G, E)\widetilde{\subset}(F, E)\widetilde{\subset}(G, E)$.*

Proof. This proof is clear that from the definition of pythagorean fuzzy soft semi-closed set. □

Proposition 3.10. *Every pythagorean fuzzy soft closed set is pythagorean fuzzy soft semi-closed set in a pythagorean fuzzy soft topological spaces $(X, \widetilde{\tau}, E)_p$.*

Proof. Straightforward. □

Remark 3.11. *The converse of Proposition 3.10 may not be provide in general. It is shown in following example.*

Example 3.12. *Consider the Example 3.4.*

$$(G, E)^c = \left\{ \begin{array}{l} (e_1, \{(x_1, 0.4, 0.6), (x_2, 0.3, 0.7)\}) \\ (e_2, \{(x_1, 0.3, 0.7), (x_2, 0.2, 0.8)\}) \end{array} \right\}$$

is pythagorean fuzzy soft semi-closed set. But $(G, E)^c$ is not pythagorean fuzzy soft closed set.

Theorem 3.13. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E) \in PFSS(X, E)$. Then (F, E) is pythagorean fuzzy soft semi-closed set if and only if $pint\,(pcl(F, E))\,\widetilde{\subset}(F, E)$.*

Proof. (⇒) Suppose that (F, E) is a pythagorean fuzzy soft closed set, then by Proposition 3.9, there exists a pythagorean fuzzy soft closed set (G, E) such that $pint(G, E)\widetilde{\subset}(F, E)\widetilde{\subset}(G, E)$. This follows that $pcl(F, E)\widetilde{\subset}pcl(G, E) = (G, E)$. Thus $pint\,(pcl(F, E))\,\widetilde{\subset}pint(G, E)$. Therefore, $pint\,(pcl(F, E))\,\widetilde{\subset}pint(G, E)\widetilde{\subset}(F, E)$.

(⇐) Suppose that (F, E) be a pythagorean fuzzy soft set in $(X, \widetilde{\tau}, E)_p$ such that $pint\,(pcl(F, E))\,\widetilde{\subset}(F, E)$. We take $pcl(F, E) = (G, E)$. Then $pint(G, E)\widetilde{\subset}(F, E)\widetilde{\subset}(G, E)$. This implies that (F, E) is a pythagorean fuzzy soft semi-closed set. □

Theorem 3.14. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X . Then an arbitrary intersection of pythagorean fuzzy soft semi-closed sets is pythagorean fuzzy soft semi-closed set.*

Proof. Suppose that $\{(F_i, E) : i \in I\}$ be a collection of pythagorean fuzzy soft semi-closed sets. Since each $i \in I$, (F_i, E) is a pythagorean fuzzy soft semi-closed set, then by Proposition 3.9, there exist a pythagorean fuzzy soft closed set (G_i, E) such that $pint(G_i, E)\widetilde{\subset}(F_i, E)\widetilde{\subset}(G_i, E)$. This follows that $\bigcap_{i\in I} (pint(G_i, E))\,\widetilde{\subset} \bigcap_{i\in I} (F_i, E)\widetilde{\subset} \bigcap_{i\in I} (G_i, E)$. We take $\bigcap_{i\in I}(G_i, E) = (G, E)$. Then by Theorem 2.15, (G, E) is a pythagorean fuzzy soft closed set and hence $\bigcap_{i\in I}(F_i, E)$ is a pytahgorean fuzzy soft semi-closed set. □

Theorem 3.15. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X, (F, E) be a pythagoren fuzzy soft semi-closed set and (G, E) be a pythagorean fuzzy soft set over X. If $pint(F, E)\widetilde{\subset}(G, E)\widetilde{\subset}(F, E)$, then (G, E) is a pythagorean fuzzy soft semi-closed set.*

Proof. Since (F, E) is a pythagorean fuzzy soft semi-closed set, then by Prosoposition 3.9, tehere exists an pythagorean fuzzy soft closed set (H, E) such that $pint(H, E)\widetilde{\subset}(F, E)\widetilde{\subset}(H, E)$. Then $(G, E)\widetilde{\subset}(H, E)$. Also $pint\,(pint(H, E)) = pint(H, E)\widetilde{\subset}pint(F, E)$. This implies that $pint(H, E)\widetilde{\subset}(G, E)$.

Therefore, $pint(H, E)\widetilde{\subset}(G, E)\widetilde{\subset}(H, E)$. Hence (G, E) is a pythagorean fuzzy soft semi-closed set. □

Definition 3.16. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E) \in PFSS(X, E)$.*

1. *The pythagorean fuzzy soft semi-interior of (F, E), denoted by $spint(F, E)$, is the union of all the pythagorean fuzzy soft semi-open sets contained in (F, E).*
   Clearly, $spint(F, E)$ is the largest pythagorean fuzzy soft semi-open set over X contained in (F, E).

2. *The pythagorean fuzzy soft semi-closure of (F,E), denoted by $spcl(F,E)$, is the intersection of all the pythagorean fuzzy soft semi-closed sets contains (F,E).*
   Clearly, $spcl(F,E)$ is the smallest pythagorean fuzzy soft semi-closed set over X which contains (F,E).

Remark 3.17. *It is clear that, If (F,E) be a pythagorean fuzzy soft set, then*

$$pint(F,E)\widetilde{\subset}spint(F,E)\widetilde{\subset}(F,E)\widetilde{\subset}spcl(F,E)\widetilde{\subset}pcl(F,E)$$

.

Theorem 3.18. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E),(G,E)\in PFSS(X,E)$.Then,*

1. $spint(\widetilde{0}_{(X,E)}) = spcl(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ *and* $spint(\widetilde{1}_{(X,E)}) = spcl(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
2. (F,E) *is a pythagorean fuzzy soft semi-open(semi-closed) set if and only if* $spint(F,E) = (F,E)$ ($spcl(F,E) = (F,E)$).
3. $spint\left(spint(F,E)\right) = (F,E)$.
4. $(F,E)\widetilde{\subset}(G,E)$ *implies* $spint(F,E)\widetilde{\subset}spint(G,E)$ *and* $spcl(F,E)\widetilde{\subset}spcl(G,E)$,
5. *(i)* $spint(F,E)\widetilde{\cap}_E spint(G,E) = spint\left((F,E)\widetilde{\cap}_E(G,E)\right)$
   (ii) $spcl(F,E)\widetilde{\cap}_E spcl(G,E)\widetilde{\supseteq}spcl\left((F,E)\widetilde{\cap}_E(G,E)\right)$
6. $spint(F,E)\widetilde{\cup}_E spint(G,E)\widetilde{\subseteq}spint\left((F,E)\widetilde{\cup}_E(G,E)\right)$
   $spcl(F,E)\widetilde{\cup}_E spcl(G,E) = spcl\left((F,E)\widetilde{\cup}_E(G,E)\right)$

Proof. *(1)-(4) follow directly from the definition of pythagorean fuzzy soft semi-interior and pythagorean fuzzy soft semi-closure .*

(5) (i) By (4), we have $\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}(F,E)$, $\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}(G,E)$ *implies*

$$spint\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}spint(F,E),\ spint\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}spint(G,E),$$

so that $spint\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}spint(F,E)\widetilde{\cap}_E spint(G,E)$. *Also, since* $spint(F,E)\widetilde{\subseteq}(F,E)$ *and* $spint(G,E)\widetilde{\subseteq}(G,E)$ *implies* $spint(F,E)\widetilde{\cap}_E spint(G,E)\widetilde{\subseteq}\left((F,E)\widetilde{\cap}_E(G,E)\right)$.

Thus $spint(F,E)\widetilde{\cap}_E spint(G,E)$ *is a pythagorean fuzzy soft semi-open subsets of* $\left((F,E)\widetilde{\cap}_E(G,E)\right)$.
Hence $spint(F,E)\widetilde{\cap}_E spint(G,E)\widetilde{\subseteq}spint\left((F,E)\widetilde{\cap}_E(G,E)\right)$. *Thus* $spint(F,E)\widetilde{\cap}_E spint(G,E) = \left((F,E)\widetilde{\cap}_E(G,E)\right)$.

(ii) By (4), we have $\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}(F,E)$, $\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}(G,E)$ *implies*

$$spcl\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}spcl(F,E),\ spcl\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}spcl(G,E),$$

so that $spcl\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subseteq}spcl(F,E)\widetilde{\cap}_E spcl(G,E)$.

(6) The proof is similar to (5) by using property that $(F,E)\widetilde{\subseteq}\left((F,E)\widetilde{\cup}_E(G,E)\right)$, $(G,E)\widetilde{\subseteq}\left((F,E)\widetilde{\cup}_E(G,E)\right)$. □

Theorem 3.19. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\in PFSS(X,E)$.Then,*

1. $(spint(F,E))^c = spcl\left((F,E)^c\right)$.
2. $(pcl(F,E))^c = spint\left((F,E)^c\right)$.
3. $spint\left(pint(F,E)\right) = pint\left(spint(F,E)\right) = pint(F,E)$.
4. $spcl\left(pcl(F,E)\right) = pcl\left(spcl(F,E)\right) = pcl(F,E)$.

Proof. (1) $spint(F,E)\widetilde{\subseteq}(F,E)$ implies that $(F,E)^c\widetilde{\subseteq}\left(spint(F,E)\right)^c$. Now by Theorem 3.18 (2), and since $(spint(F,E))^c$ is a pythagorean fuzzy soft semi-closed set, we have $spcl\left((F,E)^c\right)\widetilde{\subseteq}spcl\left((spint(F,E))^c\right) = (spint(F,E))^c$. For the reverse inclusion, $(F,E)^c\widetilde{\subseteq}spcl\left((F,E)^c\right)$ implies that $\left(spcl\left((F,E)^c\right)\right)^c\widetilde{\subseteq}\left((F,E)^c\right)^c = (F,E)$. $spcl\left((F,E)^c\right)$ being pythagorean fuzzy soft semi-closed implies that $(spcl\left((F,E)^c\right))^c$ is pythagorean fuzzy soft semi-open. Thus $(spcl\left((F,E)^c\right))^c \widetilde{\subseteq} spint(F,E)$ and hence $(spint(F,E))^c \widetilde{\subseteq} \left((spcl\left((F,E)^c\right))^c\right)^c = spcl\left((F,E)^c\right)$.

(2) It is similar to (1).

(3) By Remark 3.3, $pint(F,E)$ is a pythagorean fuzzy soft open set implies that it is pythagorean fuzzy soft semi-open set. Therefore, by Theorem 3.18(2), $spint\left(pint(F,E)\right) = pint(F,E)$. $pint(F,E) \widetilde{\subseteq} spint(F,E) = (F,E)$. This implies that $spint(pint(F,E)) = pint(F,E)$.

(4) $pcl(F,E)$ is pythagorean fuzzy soft closed set implies that it is pythagorean fuzzy soft semi-closed. Therefore $spcl\left(pcl(F,E)\right) = pcl(F,E)$. Then $(F,E) \widetilde{\subseteq} spcl(F,E) \widetilde{\subseteq} pcl(F,E)$. Hence $spcl(F,E) \widetilde{\subseteq} pcl\left(spcl(F,E)\right) \widetilde{\subseteq} spcl(F,E)$. This implies that $pcl\left(spcl(F,E)\right) = pcl(F,E)$. □

4. Conclusion

In this study,we presented topological structures of pythagorean fuzzy soft semi-open and pythagorean fuzzy soft semi-closed sets. We also investigated and explored some properties of pythagorean fuzzy soft semi-interior and pythagorean fuzzy soft semi-closure and discussed relationship between them. We hope that the findings in this paper will enhance and promote the further study in the pythagorean fuzzy soft set theory.

References

[1] Akdag, M. and Ozkan, A. Soft b-open sets and soft b-continuous functions. Mathematical Sciences 8(2), 2014, 124.
[2] Akdag, M. and Ozkan, A. On Soft Preopen Sets and Soft Pre Separation Axioms. Gazi University Journal of Science 27(4) 2014.
[3] Atanassov K. Intuitionistic fuzzy sets. Fuzzy Sets and Systems. 20, 1986, 87–96.
[4] Athira T.M., John S.J., & Garg H., Entropy and distance measures of Pythagorean fuzzy soft sets and their applications, Journal of Intelligent & Fuzzy Systems 37, 2019, 4071–4084, Doi:10.3233/JIFS-190217
[5] Athira T.M., John S.J., & Garg H. A novel entropy measure of Pythagorean fuzzy soft sets. AIMS Mathematics. 5(2), 2020 1050–1061.
[6] Chen D., Tsang E.C.C, Yeung D.S., Wang. X., The parameterization reduction of soft sets and its applications. Comput. Math. Appl. 49, 2005, 757-763.
[7] Coker D., An introduction of intuitionistic fuzzy topological spaces. Fuzzy Sets and Systems. 88, 1997, 81–89.
[8] Feng F., Jun Y.B., Liu X., Li L., An adjustable approach to fuzzy soft set based decision making. J. Comput. Appl. Math. 234, 2010, 10–20.
[9] Guleria A., Bajaj R.K. On pythagorean fuzzy soft matrices operations and their applications in decision making and medical diagnosis. Soft Computing. 23, 2019, 1889–7900, https://doi.org/10.1007/s00500-018-3419-z.
[10] Hussain S., A note on soft connectedness. Journal of Egyptian Mathematical Society. 23, 2015, 6–11.
[11] Hussain S., Ahmad B., Some properties of soft topological spaces. Computers and Mathematics with Applications. 62, 2011, 4058–4067.
[12] Hussain S. On some properties of Intuitionistic fuzzy soft boundary. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69(2), 2020, 39–50.
[13] Hussain S., On some weak structures in intuitionistic fuzzy soft topological spaces. Italian journal of pure and applied mathematics. 42, 2019, 512–525.
[14] Hussain A., Ali M.I. and Mahmood T. Pythagorean fuzzy soft rough sets and their applications in decision-making. Journal of Taibah University for Science. 14(1), 2020, 101–113, doi: 10.1080/16583655.2019.1708541
[15] Kirisci M. New type pythagorean fuzzy soft set and decision-making application, arXiv preprint arXiv:1904.04064, 2019.
[16] Li Z., Cui R., On the topological structurs of intuitionistic fuzzy soft sets. Annals of fuzzy Mathematics and Informatics. 5, 2013, 229–239.
[17] Maji P.K., Biswas R., Roy A.R., Fuzzy soft sets. Journal of Fuzzy Mathematics 9(3), 2001, 589–602.
[18] Maji P.K., Biswas R., Roy A.R., Intuitionistic fuzzy soft sets. Journal of fuzzy mathematics 9(3), 2001, 677–692.
[19] Maji P.K., Biswas R., Roy A. Soft set theory. Comput. Math. Appl. 45(4-5), 2003, 555–562.
[20] Molodtsov D., Soft set theory-first results. Comput. Math. Appl. 37(4-5), 1999, 19–31.
[21] Molodtsov D., Leonov V.Y., Kovkov D.V., Soft sets technique and its application. Nechetkie Sistemy i Myagkie Vychisleniya. 9, 2006, 8–39.
[22] Olgun M., Unver M. & Yardımcı S. Pythagorean fuzzy topological spaces. Complex Intell. Syst. 5, 2019, 177—183, https://doi.org/10.1007/s40747-019-0095-2.
[23] Peng X., Yang Y., Some results for pythagorean fuzzy sets. International Journal of Intelligent Systems. 30(11), 2015, 1133–1160.
[24] Peng X., Yang Y., Song J., Jiang Y., Pythagoren fuzzy soft set and its application. Computer Engineering. 41(7), 2015, 224–229.
[25] Peng X., New operations for interval-valued pythagorean fuzzy set. Scientia Iranica E. 26(2), 2019, 1049–1076.
[26] Roy A.R., Maji P.K., A fuzzy soft set theoretic approach to decision making problems. J. Comput. Appl. Math. 203, 2007, 412418.

[27] Riaz M., Naeem K., Aslam M., Afzal D., Ahmahdi F. and Jamal S.S, Multi-criteria Group Decision Making with Pythagorean Fuzzy Soft Topology. Journal of Intelligent and Fuzzy Systems. 2020, 1–18, DOI:10.3233/JIFS-190854.

[28] Riaz M., Naeem K., & Afzal, D. Pythagorean m-polar fuzzy soft sets with TOPSIS method for MCGDM. Punjab University Journal of Mathematics. 52(3), 2020, 21–46.

[29] Xiao Z., Gong K., Zou Y., A combined forecasting approach based on fuzzy soft sets. J. Comput. Appl. Math. 228, 2009, 326333.

[30] Yager R.R. Pythagorean fuzzy subsets. Proc. Joint IFSA World Congress NAFIPS Annual Meet., 1, Edmonton, Canada, 57–61, 2013.

[31] Yager R.R. and Abbasov A.M. Pythagorean membership grades, complex numbers and decision making. Int. J. Intell. Syst. 28(5), 2014, 436–452 .

[32] Yolcu A., and Ozturk T.Y., Some New Results on Pythagorean Fuzzy Soft Topological Spaces, TWMS J. App. Eng. Math. Accepted.

[33] Zadeh, L.A. Fuzzy Sets. Information and Control. 8, 1965, 338–353.

[34] Zou Y., Xiao Z., Data analysis approaches of soft sets under incomplete information. Knowledge-Based Syst. 21, 2008, 941–945.