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The Complex-type Cyclic-Pell Sequence and its Applications

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Abstract. In this paper, we define the complex-type cyclic-Pell sequence and then, we give miscellaneous properties of this sequence by using matrix method. Also, we study the complex-type cyclic-Pell sequence modulo m. In addition, we describe the complex-type cyclic-Pell sequence in a 2-generator group and we investigate that in finite groups in detail. Finally, we obtain the lengths of the periods of the complex-type cyclic-Pell sequences in dihedral groups D_2, D_3, D_4, D_6, D_8, D_{16} and D_{32} with respect to the generating pair (x, y).

1. Introduction

The well-known the Pell sequence $\{P_n\}$ is defined by the following recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2}$$

for $n \geq 2$ and with initial conditions $P_0 = 0$ and $P_1 = 1$.

The complex Fibonacci sequence $\{F_n^*\}$ is defined [21] by the following equation: for $n \geq 0$

$$F_n^* = F_n + iF_{n+1}$$

where $i = \sqrt{-1}$ is the imaginary unit and F_n is the n^{th} Fibonacci number (cf. [5, 22]).

Suppose that $\left\{c_j\right\}_{j=0}^{k-1}$, $(k \geq 2)$ is a sequence of real numbers such that $c_{k-1} \neq 0$. The k-generalized Fibonacci sequence $\{a_n\}_{n=0}^{+\infty}$ is defined as

$$a_{n+k} = c_{k-1}a_{n+k-1} + c_{k-2}a_{n+k-2} + \cdots + c_0 a_n$$

for $n \geq 0$ and where $a_0, a_1, \ldots, a_{k-1}$ are specified by the initial conditions.

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In [23], Kalman gave a number of closed-form formulas for the generalized sequence using the companion matrix as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Also, he proved that

$$(A_k)^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In the literature, many interesting properties and applications of the recurrence sequences relevant to this paper have been studied by many authors; see for example, [3, 7–9, 14, 15, 28, 29]. Especially, in [18] and [17], the authors defined the new sequences using the quaternions and complex numbers and then they gave miscellaneous properties and many applications of the sequences defined. In the first part of this paper, we define the complex-type cyclic-Pell sequence and then, we give miscellaneous properties of this sequence by the aid of the matrix method.

We recall that when a sequence is composed only of repetitions of a fixed subsequence A sequence is periodic if after a certain points it consists only of repetitions of a fixed subsequence. We refer to the number of members in the shortest repeating subsequence as the period of the sequence. For instance, when a sequence with the terms $x, y, z, t, y, z, t, y, z, t, \ldots$ is considered, one would say it is periodic after the initial term k and it has period 3. Also, the first r terms in a sequence form a repeating subsequence, then it is said to be simply periodic with period r. For instance, when a sequence with the terms $x, y, z, t, x, y, z, t, x, y, z, t, \ldots$ is considered, one would say it is simply periodic with period 4.

The study of the linear recurrence sequences modulo m began with the earlier work of Wall [30] where the periods of the ordinary Fibonacci sequences modulo m were investigated. Recently, the theory extended to some special linear recurrence sequences by several authors; see for example, [20, 26].

For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_u = a_{u+1}$, $0 \le u \le n-1$, $x_{n+u} = \prod_{v=1}^{n} x_{u+v-1}$, $u \ge 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted as $F_A(G)$ in [11].

A k-nacci (k-step Fibonacci) sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases}$$

We also require that the initial elements of the sequence $x_0, x_1, x_2, \ldots, x_{j-1}$ generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k\left(G; x_0, x_1, x_2, \ldots, x_{j-1}\right)$ in [25].

Note also that the orbit of a k-generated group is a k-nacci sequence.

From [17], we use the following definition as our preliminary information.

Definition 1.1. *Let G be a k-generated group. For a generating k-tuple $(x_1, x_2, \ldots, x_k)$, the complex-type k-Finonacci orbit is defined by $a_i = x_{i+1}$, $(0 \le i \le k-1)$,*

$$a_{n+k} = (a_n)^{i^k} (a_{n+1})^{i^{k-1}} \ldots (a_{n+k-1})^{i}, \; n \ge 0$$

where the following conditions are achieved for any $x, y \in G$ and any integer u:

(i). *Let e be the identity of G and consider $z = a + ib$, where a, b are integers, then*
$* \; x^z \equiv x^{a(\text{mod } |x|)+ib(\text{mod } |x|)} = x^{a(\text{mod } |x|)}x^{ib(\text{mod } |x|)} = x^{ib(\text{mod } |x|)}x^{a(\text{mod } |x|)} = x^{ib(\text{mod } |x|)+a(\text{mod } |x|)}$,
$* \; x^{ia} = \left(x^i\right)^a = (x^a)^i$,
$* \; e^u = e$,
$* \; x^{0+i0} = e$.

(ii). *Given $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, b_1, a_2 and b_2 are integers,* $y^{-z_2}x^{-z_1} = \left(x^{z_1}y^{z_2}\right)^{-1}$.

(iii). *If $yx \neq xy$, then $y^i x^i \neq x^i y^i$.*

(iv). $y^i x^i = (xy)^i$ *and* $x^{-1}y^{-1} = \left(x^i y^i\right)^i$,

(v). $y^i x = xy^i$ *and so* $x^i y^{-1} = \left(xy^i\right)^i$ *and* $x^{-1}y^i = \left(x^i y\right)^i$.

The study of the recurrence sequences in groups began with the earlier work of Wall [30]. In the mid-eighties, Wilcox studied the Fibonacci sequences in abelian groups in [31]. In [12], the theory was expanded to some finite simple groups by Campbell et al.. There, they defined the Fibonacci length of the Fibonacci orbit and the basic Fibonacci length of the basic Fibonacci orbit in a 2-generator group. The concept of Fibonacci length for more than two generators has also been considered; see, for example, [10, 11]. In [25], Knox signified that a k-nacci (k-step Fibonacci) sequence in a finite group is periodic. Recently, the theory has been extended to some special linear recurrence sequences by several authors; see for example, [1, 2, 4, 13, 16, 19, 24, 27]. Deveci and Shannon [17] defined the complex-type k-Fibonacci orbit of a k-generator group. They proved that the complex-type k-Fibonacci orbit of a k-generator group is periodic if the group is finite. In the second part of this paper, we redefine the complex-type cyclic-Pell sequence by means of the elements of 2-generator groups which is called the complex-type cyclic-Pell orbit. Then we examine the sequence in finite groups in detail. Finally, we obtain the lengths of the periods of the complex-type cyclic-Pell orbits of the dihedral group D_n for some $n \geq 2$ as applications of the results obtained.

2. The Complex-type Cyclic-Pell Sequence

Now we define the complex-type cyclic-Pell sequence by the following homogeneous linear recurrence relation for $n \geq 1$

$$p_{n+2}^{(c,i)} = \begin{cases} 2p_{n+1}^{(c,i)} + p_n^{(c,i)} & n \equiv 0 \,(\text{mod } 4) \\ i\left(2p_{n+1}^{(c,i)} + p_n^{(c,i)}\right) & n \equiv 1 \,(\text{mod } 4) \\ -2p_{n+1}^{(c,i)} - p_n^{(c,i)} & n \equiv 2 \,(\text{mod } 4) \\ -i\left(2p_{n+1}^{(c,i)} + p_n^{(c,i)}\right) & n \equiv 3 \,(\text{mod } 4) \end{cases}$$

where $p_1^{(c,i)} = 0$, $p_2^{(c,i)} = 1$ and $i = \sqrt{-1}$.

Letting

$$M = \begin{bmatrix} -13 & -6-2i \\ -6+2i & -3 \end{bmatrix}. \tag{1}$$

and by using an induction method on n, we find the relationship between the elements of the sequence $\left\{p_n^{(c,i)}\right\}$ and the matrix M as follows:

$$(M)^n = \begin{bmatrix} p_{4n+2}^{(c,i)} & \overline{p_{4n+1}^{(c,i)}} \\ p_{4n+1}^{(c,i)} & \operatorname{Re}\left(p_{4n}^{(c,i)}\right) - \operatorname{Im}\left(p_{4n+1}^{(c,i)}\right) \end{bmatrix}.$$

In [6], Bicknell defined the generating matrix of the Pell numbers, P-matrix as follows:

$$N = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the matrices M and N, we have the following useful result.

Proposition 2.1. *For $n \geq 0$*

$$\det(M)^n = (-1)^n \cdot \det(N)^{4n}.$$

Proof. It is well-known that the *nth* powers of the matrix N is as follows:

$$(N)^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \tag{2}$$

for $n \geq 0$. Since $\det M = \det(N)^4$ and from the (1) and (2), we have conclusion. □

We use the above definitions and define the matrices:

$$B_1 = \begin{bmatrix} 2i & i \\ 1 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -2i & -i \\ 1 & 0 \end{bmatrix}$$

and

$$B_4 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $M = B_4 B_3 B_2 B_1$. Using the above identities, we define the following matrix:

$$E^n = B_u B_{u-1} \dots B_1 M^k$$

where $n = 4k + u$ such that $u, k \in \mathbb{N}$. So we get

$$E^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{n+1}^{(c,i)} \\ p_n^{(c,i)} \end{bmatrix} \tag{3}$$

for $n = 4k + u$ such that $u, k \in \mathbb{N}$.

Now we investigate the Simpson formulas of the complex-type cyclic-Pell sequence.

If $n = 4k + 1$ ($k \in \mathbb{N}$), then

$$E^n = B_1 M^k = \begin{bmatrix} p_{n+2}^{(c,i)} & -2\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+3}^{(c,i)}\right) + \mathrm{Im} p_{n+2}^{(c,i)}\right] \\ p_{n+1}^{(c,i)} & \overline{p_n^{(c,i)}} \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left(\overline{p_n^{(c,i)}}\right) - \left(p_{n+1}^{(c,i)}\right)\left(-2\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+3}^{(c,i)}\right) + \mathrm{Im} p_{n+2}^{(c,i)}\right]\right) = (-1)^{k+1} \cdot i.$$

If $n = 4k + 2$ ($k \in \mathbb{N}$), then

$$E^n = B_2 B_1 M^k = \begin{bmatrix} p_{n+2}^{(c,i)} & \overline{p_{n+1}^{(c,i)}} \\ p_{n+1}^{(c,i)} & -2\mathrm{Re}\left(p_{n+1}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + \mathrm{Im}\left(p_{n+1}^{(c,i)}\right)\right] \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left(-2\mathrm{Re}\left(p_{n+1}^{(c,i)}\right) + i \cdot \left[\mathrm{Re}\left(p_{n+2}^{(c,i)}\right) + \mathrm{Im}\left(p_{n+1}^{(c,i)}\right)\right]\right) - \left(p_{n+1}^{(c,i)}\right)\left(\overline{p_{n+1}^{(c,i)}}\right) = (-1)^{k+1} \cdot i.$$

If $n = 4k + 3$ ($k \in \mathbb{N}$), then

$$E^n = B_3 B_2 B_1 M^k = \begin{bmatrix} p_{n+2}^{(c,i)} & \mathrm{Re}\left(p_{n+1}^{(c,i)}\right) - \mathrm{Im}\left(p_n^{(c,i)}\right) \\ p_{n+1}^{(c,i)} & \overline{p_n^{(c,i)}} \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left(\overline{p_n^{(c,i)}}\right) - \left(p_{n+1}^{(c,i)}\right)\left[\mathrm{Re}\left(p_{n+1}^{(c,i)}\right) - \mathrm{Im}\left(p_n^{(c,i)}\right)\right] = (-1)^k .$$

If $n = 4k + 4$ ($k \in \mathbb{N}$), then

$$E^n = M^{k+1} = \begin{bmatrix} p_{n+2}^{(c,i)} & \overline{p_{n+1}^{(c,i)}} \\ p_{n+1}^{(c,i)} & \mathrm{Re}\left(p_n^{(c,i)}\right) - \mathrm{Im}\left(p_{n-1}^{(c,i)}\right) \end{bmatrix}.$$

So we get

$$\left(p_{n+2}^{(c,i)}\right)\left[\mathrm{Re}\left(p_n^{(c,i)}\right) - \mathrm{Im}\left(p_{n-1}^{(c,i)}\right)\right] - \left(p_{n+1}^{(c,i)}\right)\left(\overline{p_{n+1}^{(c,i)}}\right) = (-1)^{k+1} .$$

3. The Complex-type Cyclic-Pell Sequence in Groups

If we reduce the sequence $\left\{p_n^{(c,i)}\right\}$ modulo m, taking least nonnegative residues, then we get the following recurrence sequence:

$$\left\{p_n^{(c,i)}(m)\right\} = \left\{p_1^{(c,i)}(m),\ p_2^{(c,i)}(m),\ \ldots,\ p_j^{(c,i)}(m),\ \ldots\right\}$$

where $p_j^{(c,i)}(m)$ is used to mean the nth element of the complex-type cyclic-Pell sequence when read modulo m. We note here that the recurrence relations in the sequences $\left\{p_n^{(c,i)}(m)\right\}$ and $\left\{p_n^{(c,i)}\right\}$ are the same.

Theorem 3.1. *The sequence $\left\{p_n^{(c,i)}(m)\right\}$ is periodic and the length of its period is divisible by* 4.

Proof. Consider the set

$$\begin{aligned} R \ = \ & \{(z_1, z_2) \mid z_k\text{'s are complex numbers } a_k + ib_k \text{ where} \\ & a_k \text{ and } b_k \text{ are integers such that } 0 \le a_k, b_k \le m-1 \text{ and } k \in \{1,2\}\}. \end{aligned}$$

Let $|R|$ be the cardinality of the set R. Since the set R is finite, there are $|R|$ distinct 2-tuples of the complex-type cyclic-Pell sequence modulo m. Thus, it is clear that at least one of these 2-tuples appears twice in the sequence $\left\{p_n^{(c,i)}(m)\right\}$. Let $p_u^{(c,i)}(m) \equiv p_v^{(c,i)}(m)$ and $p_{u+1}^{(c,i)}(m) \equiv p_{v+1}^{(c,i)}(m)$. If $v - u \equiv 0 \pmod 4$, then we get $p_{u+2}^{(c,i)}(m) \equiv p_{v+2}^{(c,i)}(m)$, $p_{u+3}^{(c,i)}(m) \equiv p_{v+3}^{(c,i)}(m)$, ... So, it is easy to see that the subsequence following this 2-tuple repeats; that is, $\left\{p_n^{(c,i)}(m)\right\}$ is a periodic sequence and the length of its period must be divided by 4. □

We denote the lengths of periods of the sequence $\left\{p_n^{(c,i)}(m)\right\}$ by $h_{p_n^{(c,i)}}(m)$. It is easy to see from the equation (3), $h_{p_n^{(c,i)}}(m)$ is the smallest positive integer α such that $E^\alpha \equiv I \pmod m$.

Given an integer matrix $A = \left[a_{ij}\right]$, $A \pmod m$ means that all entries of A are modulo m, that is, $A \pmod m = \left(a_{ij} \pmod m\right)$. Let us consider the set $\langle A\rangle_m = \{(A)^n \pmod m \mid n \ge 0\}$. If $(\det A, m) = 1$, then the set $\langle A\rangle_m$ is a cyclic group; if $(\det A, m) \ne 1$, then the set $\langle A\rangle_m$ is a semigroup. Since $\det M = -1$, the set $\langle M\rangle_m$ is a cyclic group for every positive integer $m \ge 2$. From (3), it is easy to see that $h_{p_n^{(c,i)}}(m) = 2\left|\langle M\rangle_m\right|$.

Theorem 3.2. *Let ε be a prime. If s is the smallest positive integer such that $\left|\langle M\rangle_{\varepsilon^{s+1}}\right| \ne \left|\langle M\rangle_{\varepsilon^s}\right|$, then $\left|\langle M\rangle_{\varepsilon^{s+1}}\right| = \varepsilon\left|\langle M\rangle_{\varepsilon^s}\right|$.*

Proof. Suppose that α is a positive integer and $|\langle M\rangle_m|$ is denoted by $l_{p_n^{(c,i)}}(m)$. Let I be 2×2 identity matrix and $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{\alpha+1})} \equiv I\left(\bmod \varepsilon^{\alpha+1}\right)$. Then we can derive $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{\alpha+1})} \equiv I(\bmod \varepsilon^{\alpha})$, which means that $l_{p_n^{(c,i)}}(\varepsilon^{\alpha})$ divides $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right)$. Moreover, we may write $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{s})} = I + \left(m_{i,j}^{(\alpha)} \cdot \varepsilon^{s}\right)$, by the binomial theorem. Hence, we obtain:

$$(M)^{l_{f_n^{(c,i)}}(\varepsilon^{\alpha})\cdot\varepsilon} = \left(I + \left(m_{i,j}^{(\alpha)} \cdot \varepsilon^{\alpha}\right)\right)^{\varepsilon} = \sum_{n=0}^{\varepsilon} \binom{\varepsilon}{i} \left(m_{i,j}^{(\alpha)} \cdot \varepsilon^{\alpha}\right)^{n} \equiv I\left(\bmod \varepsilon^{\alpha+1}\right).$$

Then we have $(M)^{l_{p_n^{(c,i)}}(\varepsilon^{\alpha})\cdot\varepsilon} \equiv I\left(\bmod \varepsilon^{\alpha+1}\right)$, which implies that $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right)$ divides $l_{p_n^{(c,i)}}(\varepsilon^{s}) \cdot \varepsilon$. According to these results, it is seen that $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right) = l_{p_n^{(c,i)}}(\varepsilon^{\alpha})$ or $l_{p_n^{(c,i)}}\left(\varepsilon^{\alpha+1}\right) = l_{p_n^{(c,i)}}(\varepsilon^{\alpha}) \cdot \varepsilon$, and the latter holds if and only if there is a $m_{i,j}^{(\alpha)}$ which is not divisible by ε. Due to fact that we assume s is the smallest positive integer such that $l_{p_n^{(c,i)}}\left(\varepsilon^{s+1}\right) \neq l_{p_n^{(c,i)}}(\varepsilon^{s})$, there is an $m_{i,j}^{(t)}$ which is not divisible by ε. This shows that $l_{p_n^{(c,i)}}\left(\varepsilon^{s+1}\right) = l_{p_n^{(c,i)}}(\varepsilon^{s}) \cdot \varepsilon$. So we have the conclusion. □

Theorem 3.3. *Let m_1 and m_2 be positive integers with $m_1, m_2 \geq 2$, then $\left|\langle M\rangle_{\mathrm{lcm}[m_1,m_2]}\right| = \mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$.*

Proof. Let $|\langle M\rangle_m|$ is denoted by $l_{p_n^{(c,i)}}(m)$ and let $\mathrm{lcm}[m_1, m_2] = m$. Clearly, $(M)^{l_{p_n^{(c,i)}}(m_1)} \equiv I(\bmod m_1)$ and $(M)^{l_{p_n^{(c,i)}}(m_2)} \equiv I(\bmod m_2)$. Using the least common multiple operation this implies that$(M)^{l_{p_n^{(c,i)}}(m)} \equiv I(\bmod m_1)$ and $(M)^{l_{p_n^{(c,i)}}(m)} \equiv I(\bmod m_2)$. So we get $\left|\langle M\rangle_{m_1}\right| \mid |\langle M\rangle_m|$ and $\left|\langle M\rangle_{m_2}\right| \mid |\langle M\rangle_m|$, which means that $\mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$ divides $\left|\langle M\rangle_{\mathrm{lcm}[m_1,m_2]}\right|$. Now we consider as $\mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right] = \rho$. Then we can write $M^{\rho} \equiv I(\bmod m_1)$ and $M^{\rho} \equiv I(\bmod m_2)$, which yields that $M^{\rho} \equiv I(\bmod m)$. Thus, it is seen that $\mathrm{lcm}\left[\left|\langle M\rangle_{m_1}\right|, \left|\langle M\rangle_{m_2}\right|\right]$ is divisible by $\left|\langle M\rangle_{\mathrm{lcm}[m_1,m_2]}\right|$. So we have the conclusion. □

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times \cdots \times G}_{j \text{ times}}$ such that $\left(x_1, x_2, \ldots, x_j\right) \in X$ if and only if G is generated by $x_1, x_2, \ldots, x_j$. $\left(x_1, x_2, \ldots, x_j\right)$ is said to be a generating j-tuple for G.

Definition 3.4. *Let G be a 2-generator group and let (x_1, x_2) be a generating 2-tuple of G. Then, we define the complex-type cyclic-Pell orbit by*

$$c_1 = x_1,\ c_2 = x_2, c_n = \begin{cases} (c_{n-2})(c_{n-1})^{2} & \text{for } n \equiv 0 \,(\bmod 4) \\ (c_{n-2})^{i}(c_{n-1})^{2i} & \text{for } n \equiv 1 \,(\bmod 4) \\ (c_{n-2})^{-1}(c_{n-1})^{-2} & \text{for } n \equiv 2 \,(\bmod 4) \\ (c_{n-2})^{-i}(c_{n-1})^{-2i} & \text{for } n \equiv 3 \,(\bmod 4) \end{cases}, \quad (n > 2).$$

Let the notation $P_{(x_1,x_2)}^{(i,c)}(G)$ denote the complex-type cyclic-Pell orbit of G for generating 2-tuple (x_1, x_2).

Theorem 3.5. *If G is finite, then the complex-type cyclic-Pell orbit of G is a periodic sequence and the length of its period is divisible by 4.*

Proof. Consider the set

$$\begin{aligned} W \quad = \quad & \left\{\left((w_1)^{a_1(\bmod |w_1|)+ib_1(\bmod |w_1|)}, (w_2)^{a_2(\bmod |w_2|)+ib_2(\bmod |w_2|)}\right) : \right. \\ & \left. i = \sqrt{-1}, w_1, w_2 \in G \text{ and } a_1, a_2, b_1, b_2 \in Z\right\}. \end{aligned}$$

Since the group G is finite, W is a finite set. Then for any $u \geq 0$, there exists $v > u$ such that $c_u = c_v$ and $c_{u+1} = c_{v+1}$. If $v - u \equiv 0\,(\bmod 4)$, then we get $c_{u+2} = c_{v+2}$, $c_{u+3} = c_{v+3}$, ... Because of the repeating, for all generating pairs, the sequence $P_{(x_1,x_2)}^{(i,c)}(G)$ is periodic and the length of its period must be divided by 4. □

We denote the length of the period of the orbit $P_{(x_1,x_2)}^{(i,c)}(G)$ by $LP_{(x_1,x_2)}^{(i,c)}(G)$. From the definition of the orbit $P_{(x_1,x_2)}^{(i,c)}$ it is clear that the length of the period of this sequence in a finite group depends on the chosen generating set and the order in which the assignments of x_1, x_2 are made.

We will now address the lengths of the periods of the orbits $P_{(x,y)}^{(i,c)}(D_2)$, $P_{(x,y)}^{(i,c)}(D_3)$, $P_{(x,y)}^{(i,c)}(D_4)$, $P_{(x,y)}^{(i,c)}(D_6)$, $P_{(x,y)}^{(i,c)}(D_8)$, $P_{(x,y)}^{(i,c)}(D_{16})$ and $P_{(x,y)}^{(i,c)}(D_{32})$. The dihedral group D_n of order $2n$ is defined as follows:

$$D_n = \langle x, y \mid x^n = y^2 = (xy)^2 = e \rangle$$

for every $n \geq 2$. Note that $|x| = n$, $|y| = 2$, $xy = yx^{-1}$ and $yx = x^{-1}y$. By direct calculation, we obtain the orbit $P_{(x,y)}^{(i,c)}(D_n)$ as follows:

$$\begin{aligned}
c_1 &= x,\ c_2 = y,\ c_3 = x^{i},\\
c_4 &= x^{-2i}y,\ c_5 = x^{-3},\ c_6 = x^{6-2i}y,\\
c_7 &= x^{4-3i},\ c_8 = x^{14+8i}y,\ c_9 = x^{13-4i},\\
c_{10} &= x^{-12}y,\ c_{11} = x^{4+13i},\ c_{12} = x^{-4-26i}y,\\
c_{13} &= x^{-39-4i},\ c_{14} = x^{74-34i}y,\ c_{15} = x^{72-39i},\\
c_{16} &= x^{218+112i}y,\ c_{17} = x^{185-72i},\ c_{18} = x^{-152-32i}y,\\
c_{19} &= x^{136+185i},\ c_{20} = x^{120-338i}y,\ c_{21} = x^{-491-136i},\\
c_{22} &= x^{1102-610i}y,\ c_{23} = x^{1356-491i},\ c_{24} = x^{3814+1592i}y,\\
c_{25} &= x^{2693-1356i},\ c_{26} = x^{-1572-1120i}y,\ c_{27} = x^{3596+2693i},\\
c_{28} &= x^{5620-4266i}y,\ c_{29} = x^{-5839-3596i},\ c_{30} = x^{17298-11458i}y,\\
c_{31} &= x^{26512-5839i},\ c_{32} = x^{70322+23136i}y,\ c_{33} = x^{40433-26512i},\\
c_{34} &= x^{-10544-29888i}y,\ c_{35} = x^{86288+40433i},\ c_{36} = x^{162032-50978i}y,\\
c_{37} &= x^{-61523-86288i},\ c_{38} = x^{285078-223554i}y,\ c_{39} = x^{533396-61523i},\\
c_{40} &= x^{1351870+346600i}y,\ c_{41} = x^{631677-533396i},\ c_{42} = x^{88516-720192i}y,\\
c_{43} &= x^{1973780+631677i},\ c_{44} = x^{4036076-543162i}y,\ c_{45} = x^{-454647-1973780i},\\
c_{46} &= x^{4945370-4490722i}y,\ c_{47} = x^{10955224-454647i},\ c_{48} = x^{26855818+5400016i}y,\\
c_{49} &= x^{10345385-10955224i},\ c_{50} = x^{6165048-16510432i}y,\ c_{51} = x^{43976088+10345385i},\\
c_{52} &= x^{94117224-4180338i}y,\ c_{53} = x^{1984709-43976088i},\ c_{54} = x^{90147806-92132514i}y,\\
c_{55} &= x^{228241116+1984709i},\ c_{56} = x^{546630038+88163096i}y,\ c_{57} = x^{178310901-228241116i},\\
c_{58} &= x^{190008236-368319136i}y,\ c_{59} = x^{964879388+178310901i},\ c_{60} = x^{2119767012+11697334i}y,\\
c_{61} &= x^{201705569-964879388i},\ c_{62} = x^{1716355874-1918061442i}y,\ c_{63} = x^{4801002272+201705569i},\\
c_{64} &= x^{11318360418+1514650304i}y,\ c_{65} = x^{3231006177-4801002272i},\ c_{66} = x^{4856348064-8087354240i}y,\\
c_{67} &= x^{20975710752+3231006177i},\ c_{68} = x^{46807769568+1625341886i}y,\ c_{69} = x^{6481689949-20975710752i},\\
c_{70} &= x^{33844389670-40326079618i}y,\ c_{71} = x^{101627869988+6481689949i},\ c_{72} = x^{237100129646+27362699720i}y,\\
c_{73} &= x^{61207089389-101627869988i},\ c_{74} = x^{114685950868-175893040256i}y,\ c_{75} = x^{453413950500+61207089389i},\\
c_{76} &= x^{1021513851868+53478861478i}y,\ c_{77} = x^{168164812345-453413950500i},\ c_{78} = x^{685184227178-853349039522i}y,\\
c_{79} &= x^{2160112029544+168164812345i},\ c_{80} = x^{5005408286266+5170194148321i}y,\ c_{81} = x^{1202203642009-2160112029544i},\\
c_{82} &= x^{2601001002248-3803204644256i}y,\ c_{83} = x^{9766521318056+1202203642009i},\ c_{84} = x^{22134043638360+1398797360238i}y,\\
c_{85} &= x^{3999798362485-9766521318056i},\ c_{86} = x^{14134446913390-18134245275874i}y,\ c_{87} = x^{46035011869804+3999798362485},\\
c_{88} &= x^{106204470652998+10134648550904i}y,\ c_{89} = x^{24269095464293-46035011869804i},\ c_{90} = x^{57666279724412-81935375188704i}y,\\
c_{91} &= x^{209905762247212+24269095464293i},\ c_{92} = x^{477477804218836+33397184260118i}y,\ c_{93} = x^{91063463984529-209905762247212i},
\end{aligned}$$

$$
\begin{aligned}
c_{94} &= x^{295350876249778-386414340234306i}y,\ c_{95} = x^{982734442715824+91063463984529i},\ c_{96} = x^{2260819761681426+204287412265248i}y,\\
c_{97} &= x^{499638288515025-982734442715824i},\ c_{98} = x^{1261543184651376-1761181473166400i}y,\ c_{99} = x^{4505097389048624+499638288515025i},\\
c_{100} &= x^{1027173796274862 4+761904896136350i}y,\ c_{101} = x^{2023448080787725-4505097389048624i},\ c_{102} = x^{6224841801173174-8248289881960898i}y,\\
c_{103} &= x^{21001677152970420+2023448080787725i},\ c_{104} = x^{48228196107114014+4201393720385448i}y,\ c_{105} = x^{10426235521558621-21001677152970420i},\\
c_{106} &= x^{27375725063996772-37801960585555392i}y,\ c_{107} = x^{96605598324081204+10426235521558621i},\ c_{108} = x^{220\,586\,921\,712\,159\,180+16\,949\,489\,542\,438\,150i}y.
\end{aligned}
$$

Using the above information, the orbits $P^{(i,c)}_{(x,y)}(D_2)$, $P^{(i,c)}_{(x,y)}(D_3)$, $P^{(i,c)}_{(x,y)}(D_4)$, $P^{(i,c)}_{(x,y)}(D_6)$, $P^{(i,c)}_{(x,y)}(D_8)$, $P^{(i,c)}_{(x,y)}(D_{16})$ and $P^{(i,c)}_{(x,y)}(D_{32})$ become, respectively:

$$
\begin{aligned}
c_5 &= x^{-3} = x = c_1,\ c_6 = x^{6-2i}y = y = c_2,\\
c_7 &= x^{4-3i} = x^i = c_3,\ c_8 = x^{14+8i}y = y = c_4, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
c_{105} &= x^{10426235521558621-21001677152970420i} = x = c_1,\ c_{106} = x^{27375725063996772-37801960585555392i}y = y = c_2,\\
c_{107} &= x^{96605598324081204+10426235521558621i} = x^i = c_3,\ c_{108} = x^{220\,586\,921\,712\,159\,180+16\,949\,489\,542\,438\,150i}y = y = c_4, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
c_9 &= x^{13-4i} = x = c_1,\ c_{10} = x^{-12}y = y = c_2,\\
c_{11} &= x^{4+13i} = x^i = c_3,\ c_{12} = x^{-4-26i}y = y = c_4, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
c_{105} &= x^{10426235521558621-21001677152970420i} = x = c_1,\ c_{106} = x^{27375725063996772-37801960585555392i}y = y = c_2,\\
c_{107} &= x^{96605598324081204+10426235521558621i} = x^i = c_3,\ c_{108} = x^{220\,586\,921\,712\,159\,180+16\,949\,489\,542\,438\,150i}y = y = c_4, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
c_{17} &= x^{185-72i} = x = c_1,\ c_{18} = x^{-152-32i}y = y = c_2,\\
c_{19} &= x^{136+185i} = x^i = c_3,\ c_{20} = x^{120-338i}y = y = c_4, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
c_{33} &= x^{40433-26512i} = x = c_1,\ c_{34} = x^{-10544-29888i}y = y = c_2,\\
c_{35} &= x^{86288+40433i} = x^i = c_3,\ c_{36} = x^{162032-50978i}y = y = c_4, \ldots,
\end{aligned}
$$

and

$$
\begin{aligned}
c_{65} &= x^{3231006177-4801002272i} = x = c_1,\ c_{66} = x^{4856348064-8087354240i}y = y = c_2,\\
c_{67} &= x^{20975710752+3231006177i} = x^i = c_3,\ c_{68} = x^{46807769568+1625341886i}y = y = c_4, \ldots.
\end{aligned}
$$

So we get $LP^{(i,c)}_{(x,y)}(D_2) = 4$, $LP^{(i,c)}_{(x,y)}(D_3) = 104$, $LP^{(i,c)}_{(x,y)}(D_4) = 8$, $LP^{(i,c)}_{(x,y)}(D_6) = 104$, $LP^{(i,c)}_{(x,y)}(D_8) = 16$, $LP^{(i,c)}_{(x,y)}(D_{16}) = 32$ and $LP^{(i,c)}_{(x,y)}(D_{32}) = 64$.

Corollary 3.6. *For $n = 2^k$ such that $k \geq 2$, the length of the period of the complex-type cyclic-Pell orbit $LP^{(i,c)}_{(x,y)}(D_n)$ is $2n$.*

Proof. From the orbit $P^{(i,c)}_{(x,y)}(D_n)$, we can deduce the following:

$$
\begin{aligned}
c_1 &= x,\ c_2 = y, \ldots,\\
c_9 &= x^{13-4i},\ c_{10} = x^{-12}y, \ldots,\\
c_{17} &= x^{185-72i},\ c_{18} = x^{-152-32i}y, \ldots,\\
c_{8u+1} &= x^{4u\lambda_1+1-4u\lambda_2 i},\ c_{8u+2} = x^{-4u\lambda_3-4u\lambda_4 i}y, \ldots,
\end{aligned}
$$

where $\gcd(\beta_1, \beta_2) = 1$. So we need an $u \in \mathbb{N}$ such that $4u = \tau n$ for $\tau \in \mathbb{N}$. If $n = 2^k$ such that $k \geq 2$, then $u = \frac{n}{4}$, and we obtain $LP^{(i,c)}_{(x,y)}(D_n) = 8\frac{n}{4} = 2n$. $\square$

4. Conclusion

In Section 2, we defined the complex-type cyclic-Pell sequence and then, we obtained the relationships among the elements of the sequence and the generating matrix of the sequence. Also, we gave the Simpson formula of the complex-type cyclic-Pell sequence. In Section 3, we studied the complex-type cyclic-Pell sequence modulo m. Furthermore, we got the cyclic groups generated by reducing the multiplicative orders of the generating matrices and the auxiliary equations of these sequences modulo m and then, we investigated the orders of these cyclic groups. Moreover, using the terms of 2-generator groups which is called the complex-type cyclic-Pell orbit, we redefined the complex-type cyclic-Pell sequence. Also, the sequence in finite groups was examined in detail. Finally, for some $n \geq 2$ as applications of the results obtained, we got the lengths of the periods of the complex-type cyclic-Pell orbits of the dihedral group D_n and we reached the length of the period of the complex-type cyclic-Pell orbit $LP^{(i,c)}_{(x,y)}(D_n)$ for $n = 2^k$ when $k \geq 2$.

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