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Generalized Inequalities for Quasi-Convex Functions via Generalized Riemann-Liouville Fractional Integrals

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Abstract. We establish some new Generalized Hermite-Hadamard-type inequalities involving generalized fractional integrals for quasi-convex functions. Our results are consistent with previous findings in the literature. The analysis used in the proofs is fairly elementary and based on the use of Hölder inequality and the power inequality.

1. Introduction

The H-H inequality shows that the mean value of a continuous convex function is greater than the value of the function at the midpoint of this range and less than the arithmetic mean of its endpoints and it has many applications for real analysis. So, it has been studied by many researchers.

Let us give this unique inequality which is named as H-H inequality in the literature: Let $g : I \longrightarrow \mathbb{R}$ be a convex mapping defined on the interval $I \subseteq \mathbb{R}$ and $\varepsilon, \delta \in I$ with $\varepsilon < \delta$,then

$$g\left(\frac{\varepsilon+\delta}{2}\right) \leq \frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta} g(x)\,dx \leq \frac{g(\varepsilon)+g(\delta)}{2}. \tag{1}$$

In the case where g is concave, the above inequality is reversed.

Later, many researchers used different classes of convex functions to generalize, improve, and extend this inequality. (See [3], [7], [8]-[11], [14]-[19], [21], [23]-[44]).

Some researchers have been proven that studies for the inequality of H-H can be generalized with the help of fractional integrals. So new studies have been carried out in the field of convex functions and inequalities using the concepts of fractional derivatives and fractional integrals. (For interested researchers [1], [3]-[6], [11]-[22], [26], [28] and [32]-[44]).

Let's remind some definitions and inequalities as following:

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$\vartheta : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$$\begin{array}{c} \int_0^1 \frac{\vartheta(l)}{l} dl < \infty, \\ \frac{1}{A_1} \leq \frac{\vartheta(u)}{\vartheta(t)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{u}{t} \leq 2 \\ \frac{\vartheta(t)}{t^2} \leq A_2 \frac{\vartheta(u)}{u^2} \text{ for } u \leq t \\ \left| \frac{\vartheta(u)}{u^2} - \frac{\vartheta(t)}{t^2} \right| \leq A_3 \left| t-u \right| \frac{\vartheta(t)}{t^2} \text{ for } \frac{1}{2} \leq \frac{u}{t} \leq 2 \end{array} \tag{2}$$

where $A_1, A_2, A_3 > 0$ are independent of $t, u > 0$. If $\vartheta(t)\, t^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\vartheta(t)}{t^\beta}$ is decreasing for some $\beta \geq 0$, then ϑ satisfies (2).

In [32], Sarıkaya and Ertuğral defined new left-sided and right-sided generalized fractional integral operators which are useful in the proofs of our main results, respectively, as following:

Definition 1.1. *Let $g \in L[\varepsilon,\delta]$. The generalized fractional integrals ${}_{\varepsilon^+}I_\vartheta g$ and ${}_{\delta^-}I_\vartheta g$ with $\varepsilon \geq 0$ are defined by*

$$ {}_{\varepsilon^+}I_\vartheta g(x) = \int_\varepsilon^x \frac{\vartheta(x-l)}{x-l} g(l)\, dl, x > \varepsilon \tag{3}$$

$$ {}_{\delta^-}I_\vartheta g(x) = \int_x^\delta \frac{\vartheta(x-l)}{x-l} g(l)\, dl, x < \delta \tag{4}$$

where $\vartheta : [0,\infty) \longrightarrow [0,\infty)$ a function which satisfies $\int_0^1 \frac{\vartheta(l)}{l} dl < \infty$.

The above generalized fractional integrals produce different kinds of fractional integrals as R-L, k−R-L, Katugampola, conformable, Hadamard, etc... You can find the different cases of the above integral operators (3) and (4) in the study [32]. (For interested researchers [5], [11], [18]-[21], [26], [34]-[40].)

In [32], Ertuğral and Sarıkaya achieved the basic H-H inequality with the help of generalized fractional integrals in (3) and (4) as follows:

Theorem 1.2. *Let $g : [\varepsilon,\delta] \longrightarrow \mathbb{R}$ be a convex function on (ε,δ) with $\varepsilon < \delta$,then the following inequalities for generalized fractional integral hold:*

$$ g\left(\frac{\varepsilon+\delta}{2}\right) \leq \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_\vartheta g(\delta) + {}_{\delta^-} I_\vartheta g(\varepsilon) \right] \leq \frac{g(\varepsilon)+g(\delta)}{2}, \tag{5}$$

where $\Lambda(1) = \int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} dl$ and $\Lambda(1) \neq 0$.

The following lemma is used to obtain some inequalities that is trapezoid inequalities for generalized fractional integrals as in [32]:

Lemma 1.3. *Let $g : [\varepsilon,\delta] \to \mathbb{R}$ be a differentiable mapping on (ε,δ) with $\varepsilon < \delta$. If $g' \in L[\varepsilon,\delta]$, then the following equality for generalized fractional integrals holds:*

$$\begin{aligned} & \frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+}I_\vartheta g(\delta) + {}_{\delta^-} I_\vartheta g(\varepsilon) \right] \\ = & \frac{\delta-\varepsilon}{2\Lambda(1)} \int_0^1 \left[\Lambda(1-l) - \Lambda(l) \right] g'(l\varepsilon + (1-l)\delta)\, dl, \end{aligned} \tag{6}$$

where $\Lambda(1) = \int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} dl$ and $\Lambda(1) \neq 0$.

The following theorem is an inequality for generalized fractional integrals via the right side of the H-H inequality obtained by using Lemma 1.3:

Theorem 1.4. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|$ is convex on $[\varepsilon, \delta]$, then the following inequality for generalized fractional integrals hold:*

$$\begin{aligned} & \left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{2\Lambda(1)} \left[{}_{\varepsilon^+} I_{\vartheta} g(\delta) + {}_{\delta^-} I_{\vartheta} g(\varepsilon) \right] \right| \\ \leq \; & \frac{(\delta - \varepsilon)}{\Lambda(1)} \int_0^1 l \left| [\Lambda(1-l) - \Lambda(l)] \right| dl \frac{[g(\varepsilon) + g(\delta)]}{2}. \end{aligned} \tag{7}$$

You can find some results for this and other generalized fractional integrals in [3], [12] and [42]-[44].

Now, let's remind some inequalities that we encountered in the results obtained in our study. Firstly, we give the basic H-H inequality via fractional integrals which is proved by Sarıkaya et al. in [34]:

Theorem 1.5. *Let $g : [\varepsilon, \delta] \longrightarrow \mathbb{R}$ be a positive function with $0 \leq \varepsilon < \delta$ and $g \in L_1[\varepsilon, \delta]$. If g is a convex function on $[\varepsilon, \delta]$, then the following inequalities fractional integrals hold:*

$$g\left(\frac{\varepsilon + \delta}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\delta - \varepsilon)^{\alpha}} \left[J_{\varepsilon^+}^{\alpha} g(\delta) + J_{\delta^-}^{\alpha} g(\varepsilon) \right] \leq \frac{g(\varepsilon) + g(\delta)}{2} \tag{8}$$

with $\alpha > 0$.

Since the results which are obtained in this study by using quasi-convex functions, let us remind the definition of quasi-convex functions [30]:

Definition 1.6. *The function $g : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be quasi-convex if for every $x, y \in I$ and $\omega \in [0, 1]$ we have*

$$g(\omega x + (1 - \omega) y) \leq \max\{g(x), g(y)\}. \tag{9}$$

Quasi-convexity is a weaker condition than classical convexity. Cause of this situation, you can say every convex function is quasi-convex but there are quasi-convex functions that are not convex (See [16]).

The classical H-H inequality for quasi-convex functions was obtained by Dragomir and Pearce in [8] as follows:

Theorem 1.7. *Let $g : I \to \mathbb{R}$ be a quasi-convex map on I and nonnegative, and suppose $\varepsilon, \delta \in I \subseteq \mathbb{R}$ with $\varepsilon < \delta$ and $g \in L_1[\varepsilon, \delta]$. Then we have the inequality*

$$\frac{1}{\delta - \varepsilon} \int_{\varepsilon}^{\delta} g(x) \, dx \leq \max\{g(\varepsilon), g(\delta)\}. \tag{10}$$

The following theorems which are H-H type inequalities for via quasi-convex function was obatained by Ion in [16] as follows:

Theorem 1.8. *Assume $\varepsilon, \delta \in \mathbb{R}$ with $\varepsilon < \delta$ and $g : [\varepsilon, \delta] \to \mathbb{R}$ is a differentiable function on (ε, δ). If $|g'|$ is quasi-convex on $[\varepsilon, \delta]$, then the following inequality holds true*

$$\left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{\delta - \varepsilon} \int_{\varepsilon}^{\delta} g(x) \, dx \right| \leq \frac{(\delta - \varepsilon) \sup\left\{ \left| g'(\varepsilon) \right|, \left| g'(\delta) \right| \right\}}{4}. \tag{11}$$

Theorem 1.9. *Assume $\varepsilon, \delta \in \mathbb{R}$ with $\varepsilon < \delta$ and $g : [\varepsilon, \delta] \to \mathbb{R}$ is a differentiable function on (ε, δ). Assume $p \in \mathbb{R}$ with $p > 1$. If $|g'|^{p\backslash(p-1)}$ is quasi-convex on $[\varepsilon, \delta]$ then the following inequality holds true*

$$\begin{aligned} & \left| \frac{g(\varepsilon) + g(\delta)}{2} - \frac{1}{\delta - \varepsilon} \int_{\varepsilon}^{\delta} g(x) \, dx \right| \\ \leq \; & \frac{(\delta - \varepsilon)}{2(p + 1)^{1\backslash p}} \left[\sup\left\{ \left| g'(\varepsilon) \right|^{p\backslash(p-1)}, \left| g'(\delta) \right|^{p\backslash(p-1)} \right\} \right]^{(p-1)\backslash p}. \end{aligned} \tag{12}$$

The following theorems that are H-H type inequalities for via quasi-convex function was obatained Alomari et al. in [2] as follows:

Theorem 1.10. *Let $g : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varepsilon, \delta \in I^{\circ}$ with $\varepsilon < \delta$. If $\left|g'\right|^{q}$ is quasi-convex on $[\varepsilon, \delta]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(\varepsilon)+f(\delta)}{2}-\frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta} g(x)\,dx\right| \leq \frac{(\delta-\varepsilon)}{4}\left(\sup\left\{\left|g'(\varepsilon)\right|^{q},\left|g'(\delta)\right|^{q}\right\}\right)^{\frac{1}{q}}. \tag{13}$$

Theorem 1.11. *Let $g : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varepsilon, \delta \in I^{\circ}$ with $\varepsilon < \delta$. If $\left|g'\right|$ is quasi-convex on $[\varepsilon, \delta]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta} g(x)\,dx - g\left(\frac{\varepsilon+\delta}{2}\right)\right| \qquad (14)\\ \leq\ & \frac{\delta-\varepsilon}{8}\left[\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\delta)\right|\right\}+\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\varepsilon)\right|\right\}\right]. \end{aligned}$$

Theorem 1.12. *Let $g : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\varepsilon < \delta$, . If $\left|g'\right|^{p\backslash(p-1)}$ is quasi-convex on $[\varepsilon, \delta]$, $p > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta} g(x)\,dx - g\left(\frac{\varepsilon+\delta}{2}\right)\right| \qquad (15)\\ \leq\ & \frac{(\delta-\varepsilon)}{4(p+1)^{1\backslash p}}\left[\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{p\backslash(p-1)},\left|g'(\delta)\right|^{p\backslash(p-1)}\right\}\right)^{(p-1)\backslash p}\right.\\ &\left.+\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{p\backslash(p-1)},\left|g'(\varepsilon)\right|^{p\backslash(p-1)}\right\}\right)^{(p-1)\backslash p}\right]. \end{aligned}$$

Theorem 1.13. *Let $g : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\varepsilon, \delta \in I^{\circ}$ with $\varepsilon < \delta$. If $\left|g'\right|^{q}$ is quasi-convex on $[\varepsilon, \delta]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{\delta-\varepsilon}\int_{\varepsilon}^{\delta} g(x)\,dx - g\left(\frac{\varepsilon+\delta}{2}\right)\right| \qquad (16)\\ \leq\ & \frac{\delta-\varepsilon}{8}\left[\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{q},\left|g'(\delta)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{q},\left|g'(\varepsilon)\right|^{q}\right\}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

In [26], Özdemir and Çetin established some fractional inequalities for differentiable quasi-convex mappings which are connected with H-H inequality as following:

Theorem 1.14. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a positive function with $0 \leq \varepsilon < \delta$ and $g \in L_1[\varepsilon, \delta]$. If g is a quasi-convex function on $[\varepsilon, \delta]$, then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(\delta-\varepsilon)^{\alpha}}\left[J_{\varepsilon+}^{\alpha} g(\delta)+J_{\delta-}^{\alpha} g(\varepsilon)\right] \leq \max\{g(\varepsilon), g(\delta)\} \tag{17}$$

with $\alpha > 0$.

Theorem 1.15. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $\left|g'\right|$ is quasi-convex on $[\varepsilon, \delta]$ and $\alpha > 0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta-\varepsilon)^{\alpha}}\left[J_{\varepsilon+}^{\alpha} g(\delta) + J_{\delta-}^{\alpha} g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\max\left\{\left|g'(\varepsilon)\right|, \left|g'(\delta)\right|\right\}. \end{aligned} \tag{18}$$

Theorem 1.16. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$such that $g' \in L_1[\varepsilon, \delta]$. If $\left|g'\right|^q$ is quasi-convex on $[\varepsilon, \delta]$, and $p > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta-\varepsilon)^{\alpha}}\left[J_{\varepsilon+}^{\alpha} g(\delta) + J_{\delta-}^{\alpha} g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{2(\alpha p+1)^{\frac{1}{p}}}\left(\max\left\{\left|g'(\varepsilon)\right|^q, \left|g'(\delta)\right|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{19}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0, 1]$.

Here we remind a previous basic inequality for generalized fractional integral inequality and a lemma that produces left sided H-H type inequalities related this basic inequality [3].

Theorem 1.17. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a function with $\varepsilon < \delta$ and $g \in L_1[\varepsilon, \delta]$. If g is a convex function on $[\varepsilon, \delta]$, then we have the following inequalities for generalized fractional integral operators:*

$$g\left(\frac{\varepsilon+\delta}{2}\right) \leq \frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{+}}I_{\vartheta} g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^{-}}I_{\vartheta} g(\varepsilon)\right] \leq \frac{g(\varepsilon)+g(\delta)}{2} \tag{20}$$

where the mapping $\Psi : [0, 1] \to \mathbb{R}$ is defined by

$$\Psi(x) = \int_0^x \frac{\vartheta\left(\left(\frac{\delta-\varepsilon}{2}\right) l\right)}{l} dl. \tag{21}$$

Lemma 1.18. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be differentiable function on (ε, δ) with $\varepsilon < \delta$. If $g' \in L[\varepsilon, \delta]$, then we have the following identity for generalized fractional integral operators:*

$$\begin{aligned} &\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^{+}}I_{\vartheta} g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^{-}}I_{\vartheta} g(\varepsilon)\right] - g\left(\frac{\varepsilon+\delta}{2}\right) \\ =\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left[\int_0^1 \Psi(l)\, g'\left(\frac{l\varepsilon}{2} + \frac{(2-l)\delta}{2}\right) dl - \int_0^1 \Psi(l)\, g'\left(\frac{(2-l)\varepsilon}{2} + \frac{l\delta}{2}\right) dl\right] \end{aligned} \tag{22}$$

where the mapping $\Psi(l)$ is defined as in Theorem 1.16.

The following results for quasi-convex functions with the help of $k-$Riemann-Liouville fractioal integral operators obtained by Hussain et al. in [15].

Theorem 1.19. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be positive function and $g \in L_1[\varepsilon, \delta]$. If g is quasi-convex on $[\varepsilon, \delta]$, the subsequent inequality for $k-$fractional integrals is valid:*

$$\frac{\Gamma_k(\alpha+k)}{2(\delta-\varepsilon)^{\frac{\alpha}{k}}}\left[{}_kJ_{\varepsilon^+}^{\alpha} g(\delta) + {}_kJ_{\delta^-}^{\alpha} g(\varepsilon)\right] \leq \max\{g(\varepsilon), g(\delta)\} \tag{23}$$

with $\frac{\alpha}{k} > 0$.

Theorem 1.20. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a differentiable function on (ε, δ) such that $g' \in L_1[\varepsilon, \delta]$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $q > 1$, the subsequent inequality for k–fractional integrals is valid:*

$$\begin{aligned} &\left|\frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\delta - \varepsilon)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{\varepsilon^+} g(\delta) + {}_kJ^{\alpha}_{\delta^-} g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta - \varepsilon}{2\left(\frac{\alpha}{k}p + 1\right)^{\frac{1}{p}}}\left(\max\left\{|g'(\varepsilon)|^q, |g'(\delta)|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{24}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{\alpha}{k} \in [0, 1]$.

Theorem 1.21. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a differentiable function on (ε, δ) such that $g' \in L_1[\varepsilon, \delta]$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $q \geq 1$, the subsequent inequality for k–fractional integrals is valid:*

$$\begin{aligned} &\left|\frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\delta - \varepsilon)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{\varepsilon^+} g(\delta) + {}_kJ^{\alpha}_{\delta^-} g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta - \varepsilon}{\left(\frac{\alpha}{k} + 1\right)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|g'(\varepsilon)|^q, |g'(\delta)|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{25}$$

with $\frac{\alpha}{k} \in [0, 1]$.

Corollary 1.22. *In Theorem 1.5 of [1], if we take $g(x) = 1$, we get the inequality:*

$$\begin{aligned} &\left|\frac{g(\varepsilon) + g(\delta)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\delta - \varepsilon)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{\varepsilon^+} g(\delta) + {}_kJ^{\alpha}_{\delta^-} g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta - \varepsilon}{\left(\frac{\alpha}{k} + 1\right)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|g'(\varepsilon)|, |g'(\delta)|\right\}\right). \end{aligned} \tag{26}$$

with $\frac{\alpha}{k} \in [0, 1]$.

By using the above results we build new inequalities related to left-sided and right-sided H-H-type generalized fractional integral inequalities via quasi-convex functions by using elementary analysis such as Hölder inequality, properties of modulus, power mean inequality.

2. Main Results

The point of this study is to generalize the inequalities for quasi-convex functions found in the literature with the help of a new fractional integral operator. Throughout this study, for brevity, we use

$$\Lambda(\mu) = \int_0^{\mu} \frac{\vartheta((\delta - \varepsilon) l)}{l} dl \text{ and } \Lambda(1) \neq 0. \tag{27}$$

Firstly, let us obtain H-H inequality for the quasi-convex functions by using this new fractional integral operator given in (3) and (4).

Theorem 2.1. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be a positive function with $0 \leq \varepsilon < \delta$ and $g \in L_1[\varepsilon, \delta]$. If g is a quasi-convex function on $[\varepsilon, \delta]$, then we have the following inequality for generalized fractional integral operators:*

$$\frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_{\vartheta} g(\delta) + {}_{\delta^-}I_{\vartheta} g(\varepsilon)\right] \leq \max\{g(\varepsilon), g(\delta)\}. \tag{28}$$

Proof. Since g is quasi-convex on $[\varepsilon, \delta]$, we have

$$g(\varepsilon l + (1-l)\delta) \le \max\{g(\varepsilon), g(\delta)\} \tag{29}$$

and

$$g((1-l)\varepsilon + l\delta) \le \max\{g(\varepsilon), g(\delta)\}. \tag{30}$$

By adding the inequalities (29) and (30), we obtain

$$\frac{1}{2}\left[g(l\varepsilon + (1-l)\delta) + g((1-l)\varepsilon + l\delta)\right] \le \max\{g(\varepsilon), g(\delta)\}. \tag{31}$$

Multiplying both sides of (31) by $\frac{\vartheta((\delta-\varepsilon)l)}{l}$, then integrating the resulting inequality with respect to l over $(0, 1]$, we get

$$\begin{aligned}&\frac{1}{2}\left[\int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} g(l\varepsilon + (1-l)\delta)\,dl + \int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l} g((1-l)\varepsilon + l\delta)\,dl\right]\\ \le\ & \max\{g(\varepsilon), g(\delta)\}\int_0^1 \frac{\vartheta((\delta-\varepsilon)l)}{l}\,dl.\end{aligned}$$

Then by using the definition of generalized fractional integral operators, we get the inequality in (28). So the proof is completed. □

Corollary 2.2. *If we choose $\vartheta(l) = l$ in Theorem 2.1, the inequality (28) reduces to the inequality (10).*

Corollary 2.3. *If we choose $\vartheta(l) = \frac{l^\alpha}{\Gamma(\alpha)}$ in Theorem 2.1, the inequality (28) reduces to the inequality (17).*

Corollary 2.4. *If we choose $\vartheta(l) = \frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.1, the inequality (28) reduces to the inequality (23).*

Remark 2.5. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.1.*

Now, by using a lemma in the literature we present new generalized inequalities for quasi-convex functions via generalized fractional integral operators.

Theorem 2.6. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|$ is quasi-convex on $[\varepsilon, \delta]$ and $g \in L_1[\varepsilon, \delta]$, $\alpha > 0$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}&\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_\vartheta g(\delta) + {}_{\delta^-}I_\vartheta g(\varepsilon)\right]\right|\\ \le\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\max\left\{|g'(\varepsilon)|, |g'(\delta)|\right\}\int_0^1 |\Lambda(1-l) - \Lambda(l)|\,dl\end{aligned} \tag{32}$$

$\Lambda(\mu)$ is as in (27).

Proof. Using Lemma 1.3, the properties of modulus and the quasi-convexity of $|g'|$,we get

$$\begin{aligned}&\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_\vartheta g(\delta) + {}_{\delta^-}I_\vartheta g(\varepsilon)\right]\right|\\ \le\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\int_0^1 |\Lambda(1-l) - \Lambda(l)|\left|g'(l\varepsilon + (1-l)\delta)\right|dl\\ \le\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\int_0^1 |\Lambda(1-l) - \Lambda(l)|\max\left\{|g'(\varepsilon)|, |g'(\delta)|\right\}dl\end{aligned}$$

The proof of inequality (32) is completed. □

Corollary 2.7. *If we choose $\vartheta(l) = l$ in Theorem 2.6, the inequality (32) reduces to the inequality (11).*

Corollary 2.8. *If we choose $\vartheta(l) = \frac{l^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.6, the inequality (32) reduces to the inequality (18).*

Corollary 2.9. *If we choose $\vartheta(l) = \frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.6, the inequality (32) reduces to the inequality (26).*

Corollary 2.10. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.6.*

Theorem 2.11. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $g' \in L[\varepsilon, \delta]$, $p > 1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned} &\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_{\vartheta}g(\delta) + {}_{\delta^-}I_{\vartheta}g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\left[\max\left\{|g'(\varepsilon)|^q, |g'(\delta)|^q\right\}\right]^{\frac{1}{q}}\left(\int_0^1 |\Lambda(1-l)-\Lambda(l)|^p\, dl\right)^{\frac{1}{p}} \end{aligned} \tag{33}$$

$\Lambda(\mu)$ is as in (27).

Proof. Using Lemma 1.3, properties of modulus and Hölder inequality, we have

$$\begin{aligned} &\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_{\vartheta}g(\delta) + {}_{\delta^-}I_{\vartheta}g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\int_0^1 |\Lambda(1-l)-\Lambda(l)|\left|g'(l\varepsilon+(1-l)\delta)\right| dl \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-l)-\Lambda(l)|^p\, dl\right)^{\frac{1}{p}}\left(\int_0^1 \left|g'(l\varepsilon+(1-l)\delta)\right|^q dl\right)^{\frac{1}{q}}. \end{aligned}$$

Since the quasi-convexity of $|g'|^q$ on $[\varepsilon, \delta]$, we get

$$\begin{aligned} &\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_{\vartheta}g(\delta) + {}_{\delta^-}I_{\vartheta}g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-l)-\Lambda(l)|^p\, dl\right)^{\frac{1}{p}}\left[\max\left\{|g'(\varepsilon)|^q, |g'(\delta)|^q\right\}\right]^{\frac{1}{q}}. \end{aligned}$$

So the proof is completed. □

Corollary 2.12. *If we choose $\vartheta(l) = l$ in Theorem 2.11, the inequality (32) reduces to the inequality (12).*

Corollary 2.13. *If we choose $\vartheta(l) = \frac{l^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.11, the inequality (32) reduces to the inequality (19).*

Corollary 2.14. *If we choose $\vartheta(l) = \frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.11, the inequality (32) reduces to the inequality (24).*

Corollary 2.15. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.11.*

Theorem 2.16. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$, be a differentiable mapping on (ε, δ) with $\varepsilon < \delta$. If $|g'|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $g' \in L[\varepsilon, \delta]$, $q \geq 1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned} &\left|\frac{g(\varepsilon)+g(\delta)}{2} - \frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_{\vartheta}g(\delta) + {}_{\delta^-}I_{\vartheta}g(\varepsilon)\right]\right| \\ \leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-l)-\Lambda(l)|\, dl\right)\left[\max\left\{|g'(\varepsilon)|^q, |g'(\delta)|^q\right\}\right]^{\frac{1}{q}} \end{aligned} \tag{34}$$

$\Lambda(\mu)$ is as in (27).

Proof. Using Lemma 1.3 and power-mean integral inequality, we have

$$\begin{aligned}
&\left|\frac{g(\varepsilon)+g(\delta)}{2}-\frac{1}{2\Lambda(1)}\left[{}_{\varepsilon^+}I_\vartheta g(\delta)+{}_{\delta^-}I_\vartheta g(\varepsilon)\right]\right| \\
\leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\int_0^1 |\Lambda(1-l)-\Lambda(l)|\left|g'(l\varepsilon+(1-l)\delta)\right| dl \\
\leq\ & \frac{\delta-\varepsilon}{2\Lambda(1)}\left(\int_0^1 |\Lambda(1-l)-\Lambda(l)|\, dl\right)^{1-\frac{1}{q}}\left(\int_0^1 |\Lambda(1-l)-\Lambda(l)|\left|g'(l\varepsilon+(1-l)\delta)\right|^q dl\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|g'\right|^q$ is quasi-convex on $[\varepsilon,\delta]$, we have desired result. So, the proof is completed. □

Corollary 2.17. *If we choose $\vartheta(l)=l$ in Theorem 2.16, the inequality (34) reduces to the inequality 13.*

Corollary 2.18. *If we choose $\vartheta(l)=\frac{l^\alpha}{\Gamma(\alpha)}$ in Theorem 2.16, the inequality (34) reduces to the inequality (18).*

Corollary 2.19. *If we choose $\vartheta(l)=\frac{l^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.16, the inequality (34) reduces to the inequality (25).*

Corollary 2.20. *Other results for different fractional integral operators as Katugampola, conformable, Hadamard, etc... can also be found by changing the operator $\vartheta(l)$ in Theorem 2.16.*

Now we give some new inequalities for generalized fractional integral operators with Lemma 1.18 obtained by Budak et al. in [3].

Theorem 2.21. *Let $g:[\varepsilon,\delta]\to\mathbb{R}$ be differentiable function on (ε,δ) with $\varepsilon<\delta$. If $\left|g'\right|$ is quasi-convex on $[\varepsilon,\delta]$ and $g'\in L[\varepsilon,\delta]$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}
&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right| \qquad (35)\\
\leq\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_0^1 |\Psi(l)|\, dl\right)\left[\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\delta)\right|\right\}\right. \\
&\left.+\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\varepsilon)\right|\right\}\right]
\end{aligned}$$

where $\Psi(1)$is as in (21).

Proof. Using Lemma 1.18 and the quasi-convexity of $\left|g'\right|$ on $[\varepsilon,\delta]$,we get

$$\begin{aligned}
&\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta)+{}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right]-g\left(\frac{\varepsilon+\delta}{2}\right)\right| \\
\leq\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left[\int_0^1 |\Psi(l)|\left|g'\left(\frac{l}{2}\varepsilon+\frac{2-l}{2}\delta\right)\right| dl+\int_0^1 |\Psi(l)|\left|g'\left(\frac{2-l}{2}\varepsilon+\frac{l}{2}\delta\right)\right| dl\right] \\
\leq\ & \frac{\delta-\varepsilon}{4\Psi(1)}\int_0^1 |\Psi(l)|\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\delta)\right|\right\} dl \\
&+\int_0^1 |\Psi(l)|\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|,\left|g'(\varepsilon)\right|\right\} dl
\end{aligned}$$

By making the necessary arrangements the desired result is achieved. □

Corollary 2.22. *If we choose $\vartheta(l)=l$ in Theorem 2.16, we get the inequality (14).*

Theorem 2.23. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be differentiable function on (ε, δ) with $\varepsilon < \delta$. If $\left|g'\right|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $g' \in L[\varepsilon, \delta]$, $p > 1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned} &\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right] - g\left(\frac{\varepsilon+\delta}{2}\right)\right| \\ \le\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_0^1 |\Psi(l)|^p\,dl\right)^{\frac{1}{p}}\left[\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{\frac{p}{p-1}}, \left|g'(\delta)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\left.+\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{\frac{p}{p-1}}, \left|g'(\varepsilon)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right] \end{aligned}$$

where $\Psi(1)$is as in (21).

Proof. Using Lemma 1.18 and Hölder inequality, we get

$$\begin{aligned} &\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right] - g\left(\frac{\varepsilon+\delta}{2}\right)\right| \\ \le\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_0^1 |\Psi(l)|^p\,dl\right)^{\frac{1}{p}}\left[\left(\int_0^1 \left|g'\left(\frac{l}{2}\varepsilon + \frac{2-l}{2}\delta\right)\right|^{\frac{p}{p-1}} dl\right)^{\frac{p-1}{p}}\right. . \end{aligned}$$

If we use the quasi-convexity of $\left|g'\right|$ on $[\varepsilon, \delta]$ last inequality, the desired result is achieved. □

Corollary 2.24. *If we choose $\vartheta(l) = l$ in Theorem 2.23, we get the inequality in (15).*

Theorem 2.25. *Let $g : [\varepsilon, \delta] \to \mathbb{R}$ be differentiable function on (ε, δ) with $\varepsilon < \delta$. If $\left|g'\right|^q$ is quasi-convex on $[\varepsilon, \delta]$ and $g' \in L[\varepsilon, \delta]$, $q \ge 1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned} &\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right] - g\left(\frac{\varepsilon+\delta}{2}\right)\right| \\ \le\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_0^1 |\Psi(l)|\,dl\right)\left[\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{q}, \left|g'(\delta)\right|^{q}\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|g'\left(\frac{\varepsilon+\delta}{2}\right)\right|^{q}, \left|g'(\varepsilon)\right|^{q}\right\}\right)^{\frac{1}{q}}\right] \end{aligned}$$

where $\Psi(1)$is as in (21).

Proof. Using Lemma 1.18 and power-mean inequality, we get

$$\begin{aligned} &\left|\frac{1}{2\Psi(1)}\left[{}_{\left(\frac{\varepsilon+\delta}{2}\right)^+}I_\vartheta g(\delta) + {}_{\left(\frac{\varepsilon+\delta}{2}\right)^-}I_\vartheta g(\varepsilon)\right] - g\left(\frac{\varepsilon+\delta}{2}\right)\right| \\ \le\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left[\int_0^1 |\Psi(l)|\left|g'\left(\frac{l}{2}\varepsilon + \frac{2-l}{2}\delta\right)\right| dl\right. \\ &\left.+\int_0^1 |\Psi(l)|\left|g'\left(\frac{2-l}{2}\varepsilon + \frac{l}{2}\delta\right)\right| dl\right] \\ \le\ & \frac{\delta-\varepsilon}{4\Psi(1)}\left(\int_0^1 |\Psi(l)|\,dl\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 |\Psi(l)|\left|g'\left(\frac{l}{2}\varepsilon + \frac{2-l}{2}\delta\right)\right|^q dl\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1 |\Psi(l)|\left|g'\left(\frac{2-l}{2}\varepsilon + \frac{l}{2}\delta\right)\right|^q dl\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{36}$$

If we use the quasi-convexity of $|g'|$ in (36), the desired result is achieved. □

Corollary 2.26. *If we choose $\vartheta(l) = l$ in Theorem 2.25, we get the inequality in (16).*

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References

[1] Ali, A., Gulshan, G., Hussain, R., Latif, A. Muddassar, M. and Park, J., Generalized Inequalities of the type of Hermite-Hadamard-Fejer with Quasi-Convex Functions by way of k-Fractional Derivatives, J. Computational Analysis and applications, 22(7) (2017), 1208-1219.

[2] Alomari, M., Darus, M., Dragomir, S.S., Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are quasi-convex. RGMIA Res. Rep. Coll., 12: Supplement, Article 14 2009, 1-11.

[3] Budak, H., Ertuğral, F., Sarıkaya, M.Z., New Generalization of Hermite-Hadamard Type Inequalities via Generalized Fractional Integrals. ResearchGate Article (2017).
https://www.researchgate.net/publication/321760465.

[4] Belarbi S., Dahmani Z., On some new fractional integral inequalities, J. Ineq. Pure Appl. Math., 10(3) (2009), Art. 86

[5] Carter, M., Brunt, B.V.: The Lebesgue-Stieljies Integral: A Practical Introduction. New York, Springer (2000).

[6] Dahmani Z., New inequalities in fractional integrals, Int. J. Nonlinear Sci., 9(4) (2010), 493-497.

[7] Dragomir, S.S, Agarwal, R.P.: Two Inequalities for Di erentiable Mappings and Applications To Special Meansof Real Numbers and to Trapezoidal Formula. Appl. Math. Lett. 11 (1999), 91-95.

[8] Dragomir, S.S., Pearce, C.E.M., Quasi-Convex Functions and Hadamard's Inequality. B. Aust. Math. Soc. 57 (1998), 377-385.

[9] Dragomir, S.S., Pearce, C.E.M., Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA, Monographs. Victoria University, 2000.

[10] Dragomir, S.S., Pecaric, J., and Persson, L.E., Some Inequalities of Hadamard Type. Soochow J. of Math. 21 (1995), 335-341.

[11] Ertuğral, F. Sarıkaya, M.Z., Budak, H., On Hermite-Hadamard Type Inequalities Associated With The Generalized Fractional Integrals. ResearchGate Article (2019).
https://www.researchgate.net/publication/334634529.

[12] Gidergelmez, H.F., Akkurt, A., Yıldırım, H., Hermite-Hadamard Type Inequalities for Generalized Fractional Integrals via Strongly Convex Functions. KJM. 7 (2019), 268-273.

[13] Gorenflo R., Mainardi F., Fractional Calculus: Integral and Differential Equations of Fractional Order, Springer Verlag, Wien., (1997), 223-276.

[14] Hadamard, J.: ėtude sur les propri ėtės des fonctions entières en particulier dűne fonction considėrėe par Riemann. Journal de mathėmatiques pures et appliquėes, 4e sėrie. 9 (1893), 171-216.

[15] Hussain, R., Ali, A., Latif, A. and Gulshan, G., Some $k-$Fractional associates of Hermite-Hadamard's Inequality for Quasi-Convex Functions and Applications to Special Means, Fractional Differential Calculus, 7(2) (2017),301-309.

[16] Ion, D. A., Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Sci. Ser., 34 (2007), 82–87.

[17] Iscan I., New general integral inequalities for quasi-geometrically convex functions via fractional integrals, J. Inequal. Appl., (491) (2013), 1-15.

[18] I scan, I.: Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals. Stud. Univ. Babes -Bolyai Math. 60 (2015), 355-366.

[19] Khan, M.A., Khurshid, Y., Ali, T.: Hermite{Hadamard Inequality for Fractional Integrals via -Convex Functions. Acta Math. Univ. Comen. 86 (2017), 153-164.

[20] K lbas, A.A., Srivastava, H.M., Trujillo J.J.: Theory and Applications of Fractional Di erential Equations. Elsevier, Amsterdam (2006).

[21] Kunt, M., Karap nar, D., Turhan, S., İscan, İ.: The Left Riemann-Liouville Fractional Hermite-Hadamard Type Inequalities for Convex Functions. Math. Slovaca. 69 (2019), 773-784.

[22] Miller S., Ross B., An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley. Soons. USA., 1993.

[23] Mitrinovi c, D.S.: Analytic Inequalities. Springer, Berlin (1970).

[24] Mitrinovi c, D.S., Pe cari c, J.E., Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer, Dordrecht (1993).

[25] Niculescu, C.P., Persson, L.E.: Convex Functions and Their Applications: A Contemporary Approach. Springer, New York (2006).

[26] Özdemir, M. E., Yıldız, Ç., Annals of the University of Craiova, Mathematics and Computer Science., 40 (2), (2013),167-173.

[27] Pearce C. E. M., Quasi-convexity, fractional programming and extrenal traffic congestion, in "Frontiers in Global Optimization", Kluwer, Dordrecht, "Nonlinear Optimization and its Applications", 74 (2004), 403-409.

[28] Pearce C. E. M. and Rubinov A. M., P- functions, quasi-convex functions and Hadamard type inequalities, J. Math. Anal. Applic., 240 (1999), 92-104.

[29] Pe cari c, J.E., Proschan, F.,Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Boston, Academic Press (1992).

[30] Ponstein, J., Seven types of convexity, SIAM Review 9 (1967), 115-119.

[31] Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York (1973).
[32] Sar kaya, M.Z., Ertuğral, F.: On The Generalized Hermite-Hadamard Inequalities. Annals of University of Craiova, Math. Comp. Sci. Ser.,47 (1) (2020),193-213.
[33] Sarıkaya M.Z., Ogunmez H., On new inequalities via Riemann-Liouville fractional integration, Abst. Appl. Anal., Art.ID 428983, (2012), 10 pages. http://dx.doi.org./10.1155/2012/428983.
[34] Sar kaya, M.Z., Set, E., Yaldiz, H., Ba sak, N.: Hermite-Hadamard's Inequalities for Fractional Integrals and Related Fractional Inequalities. Math. Comput. Modell. 57 (2013), 2403-2407.
[35] Sarikaya, M.Z. and Yildirim, H., On generalization of the Riesz potential, Indian Jour. of Math. and Mathematical Sci. 3 (2007), no. 2, 231-235.
[36] Set, E. and Çelik, B, Fractional Hermite-Hadamard Type Inequalities for Quasi-convex functions, Ordu Univ. J. Sci. Tech. 6, 1 (2016), 137–149.
[37] Set, E., Karata s S.S., Khan, M.A.: Hermite{Hadamard Type Inequalities Obtained via Fractional Integrals for Di erentiable m-Convex and (;m)-Convex Functions. Int. J. Anal (2016). https://doi.org/10.1155/2016/4765691.
[38] Set, E., Sar kaya, M.Z., Ozdemir, M.E., Yıldırım, H.: The Hermite-Hadamard's Inequality for Some Convex Functions via Fractional Integrals and Related Results. JAMSI. 10 (2014), 69-83.
[39] Shi, D.-P., Xi, B.-Y., Qi, F.: Hermite-Hadamard Type Inequalities for Reimann-Liouville Fractional Integrals of (alpha;m)-Convex Functions. Fractional Di er. Calc. 4 (2014), 31-43.
[40] Tun c, M.: On New Inequalities for h-Convex Functions via Riemann-Liouville Fractional Integration. Filomat, 27(2013), 559-565.
[41] Varosanec, S.: On h-convexity. J. Math. Anal. Appl. 326 (2007), 303-311.
[42] Yaldiz, H., Set, E.: Re nements Hermite-Hadamard-Fejer Type Inequalities For Generalized Fractional Integrals.ResearchGate Article (2018). https://www.researchgate.net/publication/323357856
[43] Yıldırım, M.E., Sarıkaya, M.Z., Yldırım, H.: The Generalized Hermite-Hadamard-Fejer Type Inequalities For Generalized Fractional Integrals, ResearchGate Article (2018) https://www.researchgate.net/publication/322592667.
[44] Zhao, D.M., Ali, A., Kashuri, A., Budak, H.: Generalized Fractional Integral Inequalities of Hermite-Hadamard Type for Harmonically Convex Functions. Adv. Di er. Equ. 1 (2020),1-14.