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Boundary Value Problems for Differential Equations Involving the Generalized Caputo-Fabrizio Fractional Derivative in λ-Metric Space

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Abstract. In this paper, by using the fixed point results of α-φ-Geraghty type mappings, the existence and uniqueness results for solutions to differential equations involving the generalized Caputo-Fabrizio derivative are investigated in λ-metric spaces. As application, an illustrative example is given to show the applicability of our theoretical results.

1. Introduction

In recent years, fractional calculus has attracted the attention of many researchers from various disciplines (physics, biology, chemistry, applied sciences,...). Indeed, The use of fractional derivatives has been observed to be beneficial for modeling many problems in engineering sciences (see, for example, [1, 2, 17, 21, 33, 35, 36].

Various there are several notions about fractional derivatives in the literature. Caputo and Riemann-Liouville introduced the basic notions (see for example [10, 27]), which imply the singular kernel $k(t,s) = \frac{(t-s)^{-q}}{\Gamma(1-q)}$, $0 < q < 1$. These derivatives play an important role in modeling phenomena in physics. However, as introduced by Fabrizio and Caputo [8], some phenomena related to material heterogeneities cannot be well modeled using fractional Caputo derivatives or Riemann-Liouville. Therefore, Fabrizio and Caputo [8] proposed a new fractional derivative with non-singular kernel $k(t,s) = e^{\frac{-q(t-s)}{1-q}}$, $0 < q < 1$. Later fractional derivative of Caputo-Fabrizio was used by many researchers to model several problems in engineering sciences (see [3, 4, 7, 11, 18, 26, 29, 30, 37]). Additionally, other fractional order derivatives with non-singular kernels have been introduced by some researchers (more details see [9, 10, 19, 20, 25, 32]).

In 1993, Czerwik proposed the notion of λ-metric (see [14, 15]). Following these initial works, the existence of a fixed point for the different operators in the definition of λ-metric spaces has been widely studied (see [12, 16, 22–24, 28, 31]).

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In this paper, we study the existence-uniqueness of solutions for problems of generalized fractional order differential equations of the Caputo-Fabrizio in λ-metric space.

$$\begin{cases} \left(D^q_{0,d,c}z\right)(\xi) = f\left(\xi, z(\xi), (D^q_{0,d,c}z)(\xi)\right), & \xi \in J = [0, \Lambda] \quad d > 0, c \geq 0 \\ z(0) = z_0 \end{cases} \tag{1}$$

Where $\Lambda > 0$, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is derivative function, $D^q_{0,d,c}$ is generalized Caputo-Fabrizio fractional derivative with $q \in (0, 1)$.

This work is arranged as follows. In the second section, we recall the notions of fractional calculus and the λ-metric space. The third Section is concerned to prove the main result. Finally, We provide an example illustrating the main result.

2. Preliminaries

We start with definition of λ-metric spaces, which was introduced by Afshari, Aydi and Karapinar [5, 6].

Definition 2.1. *Let Υ be a nonempty set, $\lambda \in \mathbb{R}^{*+}$ and $M : \Upsilon \times \Upsilon \to [0, \infty)$ such that for all $\varsigma, \gamma, \epsilon \in \Upsilon$*

(i) $M(\varsigma, \gamma) = 0 \Leftrightarrow \varsigma = \gamma$;

(ii) $M(\varsigma, \gamma) = M(\gamma, \varsigma)$;

(iii) $M(\varsigma, \gamma) \leq \lambda[M(\varsigma, \epsilon) + M(\epsilon, \gamma)]$.

Then, the triple (Υ, M, λ) is called a λ-metric space.

Example 2.2. *[5, 6] let $M : [0, 1] \times [0, 1] \to [0, \infty)$ by defined by*

$$M(\gamma, \epsilon) = \mid \gamma^2 - \epsilon^2 \mid, \text{ for all } \gamma, \epsilon \in [0, 1].$$

It is clear that the triple (Υ, M, λ) is a λ-metric space with $\lambda \geq 2$, but it is easy to see that the pair $([0, 1], M)$ is not a metric space.

Example 2.3. *[5, 6] let $\Upsilon = C(\mathbb{R})$ and $M : \Upsilon \times \Upsilon \to \mathbb{R}^+$ defined by*

$$M(\vartheta, \eta) = \| \left(\vartheta - \eta\right)^2 \|_{L^\infty(\mathbb{R})}, \text{ for all } \vartheta, \eta \in C(\mathbb{R}).$$

Then, the triple $(C(\mathbb{R}), M, 2)$ is a λ-metric space.

In 2012, B. Samet and Erdal Karapinar [22] originated the concept of α-admissibility presented in [30].

Definition 2.4. *[30] Let $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ be a self-mapping and $\alpha : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ be a function. We say that $\mathcal{P}$ is a α-admissible if*

$$\alpha(\vartheta, \eta) \geq 1 \Longrightarrow \alpha(\mathcal{P}\vartheta, \mathcal{P}\eta) \geq 1 \text{ for all } \vartheta, \eta \in \Upsilon.$$

Example 2.5. *[29] Let $\Upsilon = \mathbb{R}^*_+$. Define $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ and $\alpha : \Upsilon \times \Upsilon \longrightarrow \mathbb{R}^+$ as follows $\mathcal{P}\vartheta = \ln(\vartheta)$ for all $\vartheta \in Y$, and*

$$\alpha(\vartheta, \eta) = \begin{cases} 0 & \text{if } \vartheta < \eta, \\ 2 & \text{if } \vartheta \geq \eta. \end{cases}$$

Then, $\mathcal{P}$ is α-admissible.

Example 2.6. *[30] We define the mappings $\mathcal{P} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $\alpha : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, as follows $\mathcal{P}\vartheta = \sqrt{\vartheta}$ for all $\vartheta \in \mathbb{R}^+$, and*

$$\alpha(\vartheta, \eta) = \begin{cases} 0 & \text{if } \vartheta < \eta, \\ e^{\vartheta - \eta} & \text{if } \vartheta \geq \eta. \end{cases}$$

Then, $\mathcal{P}$ is α-admissible.

Definition 2.7. *[6] Let (Υ, M, λ) be a λ-metric space and $\alpha : \Upsilon \times \Upsilon \longrightarrow \mathbb{R}^+$ be a function. We say that Y is α-regular if*

$$(\theta_n)_{n\in\mathbb{N}} \subset \Upsilon \text{ suth that, } \alpha(\theta_n, \theta_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} \theta_n = \theta,$$

there exists a subsequence $(\theta_{n_k})_{k\in\mathbb{N}}$ of $(\theta_n)_{n\in\mathbb{N}}$, such that

$$\alpha(\theta_{n_k}, \theta) \geq 1, \quad \forall k \in \mathbb{N}.$$

We denote by Ψ the set of all increasing functions $\mu : \mathbb{R}^+ \to [0, \frac{1}{c^2})$, $\quad c \geq 1$ and Φ the set of all continuous and nondecreasing functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\varphi(ct) \leq c\varphi(t) \leq ct \quad \text{for } c > 1.$$

Definition 2.8. *[5] Let (Υ, M, λ) be a λ-metrice space. An operator $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ is a generalized α-φ-Geraghty contraction, if there exists $\alpha : \Upsilon \times \Upsilon \longrightarrow [0, \infty)$ such that*

$$\alpha(\vartheta, \eta)\varphi\Big(\lambda^3\, M(\mathcal{P}\vartheta, \mathcal{P}\eta)\Big) \leq \mu\Big(\varphi(M(\vartheta, \eta))\Big)\varphi(M(\vartheta, \eta)), \quad \forall \vartheta, \eta \in \Upsilon,$$

where $\mu \in \Psi$ and $\varphi \in \Phi$.

Theorem 2.9. *[5] Let (Υ, M, λ) be a λ-metrice space, and $\mathcal{P} : \Upsilon \longrightarrow \Upsilon$ be a generalized α-φ-Geraghty contraction. Assume that*

1) *$\mathcal{P}$ is α-admissible;*
2) *there exists $\theta_0 \in \Upsilon$ such that $\alpha(\theta_0, \mathcal{P}\theta_0) \geq 1$;*
3) *either $\mathcal{P}$ is continous or Υ is α-regular.*

Then $\mathcal{P}$ has a fixed point. Moreover, if

4) *for all fixed point ϑ, η of $\mathcal{P}$, either*

$$\alpha(\vartheta, \eta) \geq 1 \text{ or } \alpha(\eta, \vartheta) \geq 1,$$

then $\mathcal{P}$ has a uniques fixed point.

Now, we introduce definitions of generalized Caputo-Fabrizio fractional derivatives which are used throughout this paper.

Definition 2.10. *[8] Let $d > 0, c \geq 0, 0 < q < 1, m \in \mathbb{N} \bigcup \{0\}$ and $f \in C^{m+1}(\mathbb{R}^+)$. The fractional derivative of order $q + m$ of f with respect to Kernel function $K_{d,c}$, where*

$$K_{d,c}(\xi) = (\frac{d^2 + c^2}{d})e^{-d\xi}cos(c\xi), \;\; \xi \geq 0$$

is defined by

$$D^{q+m}_{0,d,c}\Big(f\Big)(\xi) = \Big(\frac{1}{1-q}\Big)\Big(\frac{d^2+c^2}{d}\Big)\int_0^{\xi} e^{\frac{-dq(\xi-\tau)}{1-q}} cos\Big(\frac{cq(\xi-\tau)}{1-q}\Big)f^{m+1}(\tau)d\tau.$$

Definition 2.11. *[8] Let $h \in C[0, T]$. The fractional integral of h is given by*

$$\Big(I^q_{0,d,c}h\Big)(\xi) = \eta_q h(\xi) + q\int_0^{\xi} g(\tau)d\tau - \delta_q \frac{c^2}{d^2+c^2}\int_0^{\xi} e^{\frac{-dq(\xi-\tau)}{1-q}} h(\tau)d\tau$$

where $\eta_q = \frac{d(1-q)}{d^2+c^2}$ and $\delta_q = \frac{c^2 q}{d^2+c^2}$.

3. Main Result

Let $(C^1(\Lambda), \| \cdot \|)$ be the Banach space of all continuous functions on J, where $\| z \| = \sup_{\xi\in\Lambda} | z(\xi) |$ and $M : C^1(\Lambda) \times C^1(\Lambda) \to \mathbb{R}_+^*$ be defined by

$$M(y,z) = \sup_{\xi\in\Lambda} \Big(y(\xi) - z(\xi) \Big)^2.$$

Then $(C^1(\Lambda), M, 2)$ is a complete $\lambda-$metrice space with $\lambda = 2$.

In this paper, we make use of the following assumptions:

(A_1**)** There exists a function $\nu : C^1(\Lambda) \times C^1(\Lambda) \to]0,\infty)$ and $\xi_0 \in C^1(\Lambda)$ such that

$$\nu\Big(\xi_0(t), \theta_h + \eta_q h(t) + q\int_0^t h(\tau)d\tau + \delta_q \int_0^t \exp\{\frac{-aq(t-\tau)}{1-q}\}h(\tau)d\tau\Big) \geq 0,$$

$h \in C^1(\Lambda)$, with $h(t) = f(t, \xi_0(t), h(t))$ and $\theta_h = x_0 + \eta_q h(0)$.

(A_2**)** There exists $\varphi \in \Phi$ and $\sigma : C^1(\Lambda) \times C^1(\Lambda) \to \mathbb{R}^{*+}$ and $\chi : \Lambda \to]0,1[$ such that for each $z, y, z_1, y_1 \in C^1(\Lambda)$ and $\tau \in \Lambda$

$$| f(\tau,z,y) - f(\tau,z_1,y_1) | \leq \sigma(z,y) \mid z - z_1 \mid + \chi(\tau) \mid y - y_1 |,$$

with

$$\| 2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + (q+\delta_q)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} d\tau \|_\infty^2 \leq \frac{1}{4}\varphi\Big(\| (z-y)^2 \|_\infty \Big),$$

where $\chi_s = \sup_{\tau\in J} | \chi(\tau) |$.

(A_3**)** For each $t \in \Lambda$ and $z, y \in C^1(\Lambda)$, we have

$$\nu\Big(z(t) \quad , \quad y(t)\Big) \geq 0 \quad \Rightarrow \quad \nu\Big(A_g, A_h\Big) \geq 0,$$

where ν is defined in assumption **(**A_1**)** and

$$A_h = \theta_g + \eta_q g(t) + q\int_0^t g(\tau)d\tau + \delta_q \int_0^t \exp\{\frac{-aq(t-\tau)}{1-q}\}g(\tau)d\tau,$$
$$A_g = \theta_h + \eta_q h(t) + q\int_0^t h(\tau)d\tau + \delta_q \int_0^t \exp\{\frac{-aq(t-\tau)}{1-q}\}h(\tau)d\tau,$$

and $h, g \in C^1(\Lambda)$, with $h(\tau) = f(\tau, y(\tau), h(\tau),\ g(\tau) = f(\tau, z(\tau), g(\tau)))$ and $\theta_h = u_0 + \eta_q h(0),\ \theta_g = u_0 + \eta_q g(0)$.

(A_4**)** If $(p_n)_{n\in\mathbb{N}} \subset C^1(\Lambda)$ such that $\lim_{n\to\infty} p_n = p$ and $\nu(p_n, p_{n+1}) \geq 0$, then $\nu(p_n, p) \geq 0$.

(A_5**)** If u, v two fiexd solutions of problem **(1)**, either

$$\nu(u,v) \geq 0 \quad \text{or} \quad \nu(v,u) \geq 0.$$

Lemma 3.1. *Let $g \in C^1[0,T]$. A function $x \in C^1[0,T]$ is solution of problem*

$$\begin{cases} \left(D^q_{0,d,c} z\right)(t) = g(t), \quad \forall t \in \Lambda = [0,T] \quad 0 < q < 1, \quad d > 0, c \geq 0 \\ z(0) = z_0 \end{cases} \tag{2}$$

if and only if z satisfies the following equation

$$z(t) = z_0 - \frac{d(1-q)}{d^2+c^2} g(0) + \left(I^q_{0,d;c} g(.)\right)(t) \quad t \in [0,T] \tag{3}$$

Proof. Let $z \in C^1[0,T]$ be a solution of **(2)**. One has

$$\left(D^q_{0,a,b} x\right)'(t) \quad = \quad g'(t) \quad , \quad \forall t \in [0,T].$$

By Definition **2.10**, we obtain

$$\begin{aligned} \left(D^q_{0,d,c} z\right)'(t) &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right) \\ &\quad \left\{z'(t) + \int_0^t \frac{d}{dt}\left(e^{\frac{-dq(t-s)}{1-q}} \cos\left(\frac{cq(t-s)}{1-q}\right)\right) z'(t) ds\right\} \\ &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right) z'(t) \\ &- \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right)\left(\frac{qd}{1-q}\right) \int_0^t e^{\frac{-dq(t-s)}{1-q}} cos\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds \\ &- \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right)\left(\frac{qc}{1-q}\right) \int_0^t e^{\frac{-dq(t-s)}{1-q}} sin\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds \\ &= \left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right) z'(t) \\ &- \left(\frac{qd}{1-q}\right) g(t, z(t)) - \left(\frac{qc}{1-q}\right)\left(\frac{1}{1-q}\right)\left(\frac{d^2+c^2}{d}\right)\gamma(t), \end{aligned} \tag{4}$$

where

$$\gamma(t) = \int_0^t e^{\frac{-dq(t-s)}{1-q}} sin\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds.$$

On the other hand

$$\gamma'(t) = \int_0^t \frac{d}{dt}\left(e^{\frac{-dq(t-s)}{1-q}} sin\left(\frac{cq(t-s)}{1-q}\right)\right) z'(t) ds.$$

Then,

$$\begin{aligned} \gamma(t)' &= \frac{-dq}{1-q} \int_0^t e^{\frac{-dq(t-s)}{1-q}} \sin\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds \\ &+ \frac{cq}{1-q} \int_0^t e^{\frac{-dq(t-s)}{1-q}} \cos\left(\frac{cq(t-s)}{1-q}\right) z'(t) ds, \\ &= \frac{-dq}{1-q}\gamma(t) + \frac{dcq}{d^2+c^2} g(t). \end{aligned} \tag{5}$$

Using that $\gamma(0) = 0$ and integrating the equality **(5)**, we get

$$\gamma(t) = \frac{dcq}{d^2+c^2} \int_0^t e^{\frac{-dq(t-s)}{1-q}} g(s) ds.$$

Hence by **(4)**, we deduce that

$$\begin{aligned}\Big(D^{q}_{0,d,c}z\Big)'(t) &= \Big(\frac{1}{1-q}\Big)\Big(\frac{d^2+c^2}{d}\Big)\\ &\quad\Big\{z'(t)-\Big(\frac{qd}{1-q}\Big)g(t)-\Big(\frac{qc}{1-q}\Big)^2\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\Big\}\end{aligned}$$

By using

$$\Big(D^{q}_{0,d,c}x\Big)'(t)=g'(t)\ ,\qquad t\in[0,T].$$

We obtain that

$$\begin{aligned}g'(t) &= \Big(\frac{1}{1-q}\Big)\Big(\frac{d^2+c^2}{d}\Big)\\ &\quad\Big\{z'(t)-\Big(\frac{qd}{1-q}\Big)g(t)-\Big(\frac{qc}{1-q}\Big)^2\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\Big\}.\end{aligned}$$

Then,

$$\begin{aligned}z'(t) &= \frac{d(1-q)}{d^2+c^2}g'(t)\\ &+ \frac{qd^2}{d^2+c^2}g(t)+\frac{dc^2q^2}{(d^2+c^2)(1-q)}\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds. \qquad (6)\end{aligned}$$

Using that $z(0)=z_0$ and integrating the **(6)**, we have

$$\begin{aligned}z(t)-z_0 &= \frac{qc^2}{d^2+c^2}\int_0^t g(\tau)d\tau+\frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)\\ &+ \frac{dc^2q^2}{(d^2+c^2)(1-q)}\int_0^t\int_0^\tau e^{\frac{-dq(\tau-s)}{1-q}}g(s)dsd\tau.\end{aligned}$$

From Fubini's theorem, we have

$$\begin{aligned}\int_0^t\int_0^\tau e^{\frac{-dq(\tau-s)}{1-q}}g(s)dsd\tau &= \int_0^t e^{\frac{dqs}{1-q}}g(s)\Big(\int_s^t e^{\frac{-dq\tau}{1-q}}d\tau\Big)ds\\ &= \Big(\frac{1-q}{dq}\Big)\int_0^t g(s)ds-\Big(\frac{1-q}{dq}\Big)\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\end{aligned}$$

Then,

$$\begin{aligned}z(t)-z_0 &= \frac{qd^2}{d^2+c^2}\int_0^t g(\tau)d\tau+\frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)\\ &+ \frac{dc^2q^2}{(d^2+c^2)(1-q)}\Big(\Big(\frac{1-q}{dq}\Big)\int_0^t g(s)ds-\Big(\frac{1-q}{dq}\Big)\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\Big)\\ &= \frac{qd^2}{d^2+c^2}\int_0^t g(\tau)d\tau+\frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)+\frac{c^2q}{d^2+c^2}\int_0^t g(s)ds\\ &- \frac{c^2q}{d^2+c^2}\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\\ &= \frac{d(1-q)}{d^2+c^2}g(t)-\frac{d(1-q)}{d^2+c^2}g(0)+q\Big(\int_0^t g(s)ds-\frac{c^2}{d^2+c^2}\int_0^t e^{\frac{-dq(t-s)}{1-q}}g(s)ds\Big)\end{aligned}$$

So, we get **(3)**.

Conversely, if z satisfies **(3)**, then $\left(D^q_{0,d,c}z\right)(t) = g(t), \qquad \forall t \in \Lambda = [0,T]$ and $z(0) = z_0$. $\square$

We can deduce the following result

Lemma 3.2. *A function x is a solution of problem 1, if and only if x satisfies the following integral equation*

$$z(t) = \theta_g + \left(I^q_{0,a;b}g(.)\right)(t) \quad t \in \Lambda = [0,T]$$

with $g(t) = f(t, z(t), g(t)$ *and* $\theta_g = z_0 + \eta_q g(0) = z_0 - \frac{a(1-q)}{a^2+b^2}g(0)$.

Theorem 3.3. *Under assumptions* (A_1)*-*(A_5)*, the problem***(1)** *has a unique solution.*

Proof. consider the maping $Q : C^1(\Lambda) \to C^1(\Lambda)$ with

$$\begin{aligned} Q : C^1(\Lambda) &\to C^1(\Lambda) \\ x &\mapsto Qz(t) = \theta_h + \eta_q h(t) + q\int_0^t h(s)ds + \delta_q \int_0^t \exp\{\frac{-dq(t-s)}{1-q}\}h(s)ds, \end{aligned}$$

where $h \in C^1(\Lambda)$, such that $h(t) = f(t, z(t), h(t))$ and $\theta_h = z_0 + \eta_q h(0)$.

Using Lemma **3.2**, the problem reduces to finding a fixed point of the map Q.

Let $\alpha : C^1(\Lambda) \times C^1(\Lambda) \to [0, \infty)$ be the function defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \nu(x(t), y(t)) \geq 0 \quad t \in J, \\ 0 & \quad otherwise. \end{cases}$$

We have to prove that Q is a generalized α-φ-Geraghty operator:

Lets $x, y \in C^1(\Lambda)$ and $t \in \Lambda$, we have

$$\begin{aligned} Qz(t) - Qy(t) = \theta_g - \theta_h + \eta_q[g(t) - h(t)] + q\int_0^t g(s) - h(s)ds \\ + \delta_q \int_0^t \exp\{\frac{-dq(t-s)}{1-q}\}g(s) - h(s)ds, \end{aligned}$$

where $h, g \in C^1(\Lambda)$, such that $h(t) = f(t, y(t), h(t)),\ g(t) = f(t, z(t), g(t))$ and

$$\begin{aligned} \theta_g &= u_0 + \eta_q g(0), \\ \theta_h &= u_0 + \eta_q h(0). \end{aligned}$$

Then

$$\begin{aligned} \mid Qz(t) - Qy(t) \mid &\leq \mid \theta_g - \theta_h \mid + \eta_q \mid g(t) - h(t) \mid + q\int_0^t \mid g(s) - h(s) \mid ds \\ &\quad + \delta_q \int_0^t \exp\{\frac{-dq(t-s)}{1-q}\} \mid g(s) - h(s) \mid ds \\ &\leq +\eta_q \mid g(0) - h(0) \mid + \eta_q \mid g(t) - h(t) \mid \\ &\quad + \int_0^t \left(q + \delta_q \exp\{\frac{-dq(t-s)}{1-q}\}\right) \mid g(s) - h(s) \mid ds. \end{aligned}$$

By (A_2), we get

$$\begin{aligned} \mid g(t)-h(t) \mid &= \mid f(t,z(t),g(t)) - f(t,y(t),h(t)) \mid \\ &\leq \sigma(x,y) \mid z(t)-y(t) \mid + \chi(t) \mid g(t)-h(t) \mid \\ &\leq \sigma(z,y) \mid \Big(z(t)-y(t)\Big)^2 \mid^{1/2} + \chi(t) \mid g(t)-h(t) \mid . \end{aligned}$$

Thus,

$$\| g-h \|_\infty \leq \frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} .$$

Next, we have

$$\begin{aligned} \mid Qz(t)-Qy(t) \mid &\leq 2\eta_q \frac{\sigma(x,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} \\ &+ \int_0^t \Big(q+\delta_q \exp\{\frac{-dq(t-s)}{1-q}\}\Big)\frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} ds \\ &\leq 2\eta_q \frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} \\ &+ \int_0^t \Big(q+\delta_q\Big)\frac{\sigma(z,y)}{1-\chi_s} \| \Big(z-y\Big)^2 \|_\infty^{1/2} ds \\ &\leq \| \Big(z-y\Big)^2 \|_\infty^{1/2} \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \\ &\leq \| \Big(z-y\Big)^2 \|_\infty^{1/2} \| \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \|_\infty . \end{aligned}$$

So,

$$\begin{aligned} \mid Qz(t)-Qy(t) \mid^2 &\leq \| \Big(z-y\Big)^2 \|_\infty \| \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \|_\infty^2 \\ &\leq \| \Big(z-y\Big)^2 \|_\infty \| \Big[2\eta_q \frac{\sigma(z,y)}{1-\chi_s} + \Big(q+\delta_q\Big)\int_0^t \frac{\sigma(z,y)}{1-\chi_s} ds\Big] \|_\infty^2 . \end{aligned}$$

This implies

$$\begin{aligned} \mid Qz(t)-Qy(t) \mid^2 &\leq \| \Big(z-y\Big)^2 \|_\infty \frac{1}{4}\varphi(\| \Big(z-y\Big)^2 \|_\infty) \\ &\leq \frac{1}{4}M(z,y)\varphi(M(z,y)). \end{aligned}$$

Then,

$$M(Qz,Qy) \leq \frac{1}{4}M(z,y)\varphi(M(z,y)).$$

And thus,

$$2^3 M(Qz(t),Qy(t)) \leq \frac{1}{32}M(z,y)\varphi(M(z,y)).$$

Since $\varphi \in \Phi$, we have

$$\begin{aligned} \alpha(z,y)\varphi\Big(2^3 M(Qz(t),Qy(t)\Big) &\leq \alpha(z,y)\varphi\Big(\frac{1}{32}M(z,y)\varphi(M(z,y))\Big) \\ &\leq \varphi\Big(\frac{1}{32}M(z,y)\Big)\varphi(M(z,y)) \\ &\leq \frac{1}{32}\varphi\Big(M(z,y)\Big)\varphi(M(z,y)). \end{aligned}$$

Hence,

$$\alpha(z,y)\varphi\Big(c^3 M(Qz(t),Qy(t)\Big) \leq \mu\Big(\varphi\Big(M(z,y)\Big)\Big)\varphi(M(z,y)) + L\psi(N(z,y)),$$

where $\mu(t) = \frac{t}{32}$, $\varphi \in \Phi$, $L = 0$ and $c = 2$.

So, Q is generalized α-φ-Geraghty operator.

Lets $z, y \in C^1(\Lambda)$ such that $\alpha(z,y) \geq 1$.
Thus, for each $t \in \Lambda$, we have

$$\nu(z(t), y(t)) \geq 0.$$

By **(**A_3**)**, then

$$\nu(Qz(t), Qy(t)) \geq 0,$$

this implies that

$$\alpha(Qz, Qy) \geq 1.$$

Hence, Q is a α-admissible.

From **(**A_1**)**, there exist $\xi_0 \in C^1(\Lambda)$ such that such that

$$\nu\Big(\xi_0(t), \theta_h + \eta_q h(t) + q\int_0^t h(s)ds + \delta_q \int_0^t \exp\{\frac{-dq(t-s)}{1-q}\}h(s)ds\Big) \geq 0,$$

this implies that

$$\nu(\xi_0, Q\xi_0) \geq 0.$$

Thus,

$$\alpha(\xi_0, Q\xi_0) \geq 1.$$

So, there exist $\xi_0 \in C^1(\Lambda)$ such that

$$\alpha(\xi_0, Q\xi_0) \geq 1.$$

Finally, if $(p_n)_{n\in\mathbb{N}} \subset C^1(\Lambda)$ such that $\lim_{n\to\infty} p_n = p$ and $\alpha(p_n, p_{n+1}) \geq 1$, which gives

$$\nu(p_n, p_{n+1}) \geq 0.$$

Then, from **(**A_4**)** we have $\nu(p_n, p) \geq 0$.
And thus,

$$\nu(p_n, p) \geq 0.$$

This implies that

$$\alpha(p_n, p) \geq 1.$$

Therefore, by applying Theorem **2.9**, we conclude that if Q has a fixed point in $C^1(\Lambda)$, then it is a solution of the fractional problem **(1)**.
Moreover, **(**A_5**)**, if u and v are two fixed points of Q, then either

$$\nu(u,v) \geq 0 \text{ or } \nu(v,u) \geq 0.$$

This implies that either

$$\alpha(u,v) \geq 1 \text{ or } \alpha(v,u) \geq 1.$$

From an application of Theorem **2.9**, then the problem **(1)** has the uniqueness solution. □

4. Example

We consider the following Caputo-Fabrizio fractional problem.

$$\begin{cases} ({}^{CF}\mathcal{D}^{q}_{0,1,0}\, z)(t) &= g\Big(t, z(t), ({}^{CF}\mathcal{D}^{q}_{0,1,0}\, z)(t)\Big); \quad t \in \Lambda := [0,1], \\ z(0) = 0, \end{cases} \tag{7}$$

Where ${}^{CF}\mathcal{D}^{q}_{0,1,0}$ is Generalized of Caputo-Fabrizio fractional derivative of order $q \in (0,1)$ and $g : \Lambda\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function definied by the following expression

$$g(t,z,y) = \frac{1}{5}\Big[\frac{1+z+\sin(z)}{2+z} - \frac{e^{-t}}{1+y}\Big].$$

Let $(C^1(\Lambda), M, 2)$ is a complete $\lambda-$metrice space with $c = 2$, such that

$$\begin{aligned} d : C^1(\Lambda)\times C^1(\Lambda) &\to \mathbb{R}_+ \\ (z,y) &\mapsto M(z,y) = \sup_{t\in\Lambda}\Big(z(t)-y(t)\Big)^2 \\ &=\| \Big(z-y\Big)^2 \|_\infty . \end{aligned}$$

Lets $z, y, v, u \in C^1(\Lambda)$ and $t \in \Lambda$, we have

$$\begin{aligned} g\Big(t,z(t),u(t)\Big) - g\Big(t,y(t),v(t)\Big) &= \frac{1}{5}\Big[\frac{1+z(t)+\sin(x(t))}{2+z(t)} - \frac{e^{-t}}{1+u(t)}\Big] \\ &- \frac{1}{5}\Big[\frac{1+y(t)+\sin(y(t))}{2+y(t)} - \frac{e^{-t}}{1+v(t)}\Big] \\ &= \frac{1}{5}\Big[\frac{z(t)-y(t)}{(1+z(t))(1+y(t))} \\ &+ \frac{(2+y(t))\sin(z(t)) - (2+z(t))\sin(y(t))}{(1+z(t))(1+y(t))}\Big] \\ &+ \frac{e^{-t}}{5}\frac{u(t)-v(t)}{(1+u(t))(1+v(t))}. \end{aligned}$$

And thus,

$$\begin{aligned} \mid g\Big(t,z(t),u(t)\Big) - g\Big(t,y(t),v(t)\Big) \mid &\le \frac{1}{5} \mid z(t)-y(t) \mid \\ &+ \mid (2+y(t))\sin(z(t)) - (2+z(t))\sin(y(t)) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t)-v(t) \mid . \end{aligned}$$

Case-1: if $y(t) \le z(t)$, we get

$$\begin{aligned} \mid g\Big(t,z(t),u(t)\Big) - g\Big(t,y(t),v(t)\Big) \mid &\le \mid z(t)-y(t) \mid \\ &+ \mid (2+z(t))\Big(\sin(z(t)) - \sin(y(t))\Big) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t)-v(t) \mid \\ &\le \mid z(t)-y(t) \mid +2(2+ \mid x(t) \mid) \\ &\quad \mid \cos\Big(\frac{z(t)+y(t)}{2}\Big) \| \sin\Big(\frac{z(t)-y(t)}{2}\Big) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t)-v(t) \mid . \end{aligned}$$

Since $\sin z \leq x$ for all $z \geq 0$, then

$$\begin{aligned} \mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid &\leq \mid z(t) - y(t) \mid + (2+ \mid z(t) \mid) \mid z(t) - y(t) \mid \\ &+ \frac{e^{-t}}{5} \mid u(t) - v(t) \mid \\ &\leq \Big(3+ \parallel z \parallel_\infty \Big) \parallel z - y \parallel_\infty + \frac{e^{-t}}{5} \parallel u - v \parallel_\infty . \end{aligned}$$

Case-2: if $y(t) > z(t)$, we obtain

$$\mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid \leq \Big(3+ \parallel y \parallel_\infty \Big) \parallel z - y \parallel_\infty + \frac{e^{-t}}{5} \parallel u - v \parallel_\infty .$$

So,

$$\begin{aligned} \mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid &\leq \min\{3+ \parallel y \parallel_\infty, 3+ \parallel z \parallel_\infty\} \parallel z - y \parallel_\infty \\ &+ \frac{e^{-t}}{5} \parallel u - v \parallel_\infty . \end{aligned}$$

Then hypothesis **(A_2)** is satisfied

$$\mid g\Big(t, z(t), u(t)\Big) - g\Big(t, y(t), v(t)\Big) \mid \leq \sigma(z, y) \mid z(t) - y(t) \mid + \chi(t) \mid u(t) - v(t) \mid,$$

where

$$\begin{aligned} \sigma(z, y) &= \min\{3+ \parallel y \parallel_\infty, 3+ \parallel z \parallel_\infty\}, \\ \chi(t) &= \frac{1}{5} e^{-t}. \end{aligned}$$

We define the function $\alpha : C(\Lambda) \times C(\Lambda) \to \mathbb{R}_+^*$ by

$$\alpha(z, y) = \begin{cases} 1 & \text{if } \varrho(z(t), y(t)) \geq 0 \quad t \in \Lambda, \\ 0 & \qquad \textit{otherwise}, \end{cases}$$

and

$$\begin{aligned} \varrho : C(\Lambda) \times C(\Lambda) &\to \mathbb{R} \\ (z, y) &\mapsto \varrho(z, y) = \parallel z - y \parallel_\infty . \end{aligned}$$

Thus, hypothesis **(A_3)** is satisfied with

$$\xi_0(t) = z(0).$$

Moreover **(A_4)** holds from the definitions of the ϱ.

Finally, by Theorem **3.3**, we get the existence of solutions and the uniqueness of problem **(7)**.

Conclusion

This paper presents contributions to the study of differential equations involving the generalized Caputo-Fabrizio fractional derivative in the λ-Metric Space, using fixed point theory of α-φ-Geraghty type. Furthermore, we have concluded this study with an illustrative example of our theoretical results

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Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest

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