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# Generalized Jensen-Hermite-Hadamard Mercer Type Inequalities for Generalized Strongly Convex Functions on Fractal Sets

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**Abstract.** In this paper, we present a variant of discrete Jensen-type inequality for generalized strongly convex functions on a real linear fractal set $\mathbb{R}^{\alpha}$ ( $0 < \alpha \leq 1$). Further, we also demonstrate a generalized Jensen–Mercer type inequality for generalized strongly convex function by employing local fractional calculus. Using this generalized Jensen-Mercer inequality, we establish a Hermite-Hadamard-Mercer type inequalities for generalized strongly convex functions.

## 1. Introduction

Convexity theory has emerged as a useful tool for studying a diverse range of problems in both pure and applied sciences. Convex functions play an important role in mathematical inequalities. Convexity theory has many applications in a variety of fascinating and compelling fields of inquiry, as well as its significant contributions to coding theory, optimization, physics, information theory, engineering, and inequality theory. The Hermite-Hadamard inequality is one of the most well-known inequalities for the class of convex functions. Many articles on convex functions and inequalities have been written by a number of mathematicians for their various classes, utilizing, for example, the most recent publications can be found in the monographs (see, [1–3]).

**Definition 1.1.** *Let $I \subset \mathbb{R}$ be an interval and $c$ be a positive number. A function $\Phi : I \to \mathbb{R}$ is called generalized strongly convex function with modulus $c$ if*

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y) - c\lambda(1-\lambda)(x-y)^2,$$

*for all $x, y \in I$ and $\lambda \in [0, 1]$.*

In this definition, if we take c=0 we get the definition of convexity in the classical sense. Strongly convex functions have been introduced by Polyak [4] In 1966. Every strongly convex function is also convex, but this is not always the case. The convergence of a gradient type approach for minimizing a function has been demonstrated using strongly convex functions. Strongly convex function play a vital role in mathematical

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economics, optimization theory, and approximation theory. A lot of applications and properties can be seen in [5–7].

For different classes of generalized convex functions, a variant of the Jensen inequality is proved, and the class of strongly convex functions is one of them.

**Theorem 1.2.** *[5](**Jensen-type inequality**) Suppose that $\Phi : I \to \mathbb{R}^{\alpha}$ is strongly convex function with modulus c.*

$$\Phi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i \Phi(x_i) - c \sum_{i=1}^{n} \lambda_i (x_i - \bar{x})^2.$$

*for any $x_i \in I$ and $\lambda_i \in [0,1]$ $(i = 1, 2, ..., n)$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$ and $\bar{x} = \sum\limits_{i=1}^{n} \lambda_i x_i$.*

This classical Jensen inequality is one of the most important inequalities in convex analysis with applications in mathematics, economics, statistics, and engineering sciences ( see [8–10]). The Jensen inequality has a lot of interesting studies in the literature for example, the Jensen-Mercer inequality is a new form of the Jensen inequality proposed by Mercer in [11].
In 2003, Mercer presents a variant of Jensen's inequality as:

**Theorem 1.3.** *[11] Suppose that $\Phi$ is a convex function on $[\zeta_1, \zeta_n]$, we have*

$$\Phi\left(\zeta_1 + \zeta_n - \sum_{i=1}^{n} \omega_i\, a_i\right) \leq \Phi(\zeta_1) + \Phi(\zeta_n) - \sum_{i=1}^{n} \omega_i \Phi(a_i)$$

*for all $a_i \in [\zeta_1, \zeta_n]$ and $\omega_i \in [0,1]$, where $i = 1, 2, \ldots, n$ with $\sum\limits_{i=1}^{n} \omega_i = 1$.*

Fractional calculus is a field of mathematics that investigates the various methods of defining real number powers and complex number powers of the differentiation and integration operators. Within the topic of fractional calculus, there are two separate approaches: generalised fractional calculus and local fractional calculus. While both deal with non-integer orders of differentiation and integration, they have different underlying principles and applications.Generalized fractional calculus extends traditional fractional calculus by introducing new operators and functions that can capture a wider range of behaviors. It focuses on developing operators and functions that go beyond the limitations of the classical Riemann-Liouville fractional integral and derivative. Generalized operators may include Caputo, Grunwald-Letnikov, or Riesz fractional derivatives, among others. These operators may have different applications in modeling in dynamics and Chaos and to develop new inequalities (see [12–15]). Local fractional calculus, on the other hand, is a different approach that involves fractional derivatives and integrals that are based solely on local properties of a function, rather than the entire function. A combination of fractional fractal calculus was developed for modeling attractors of chaotic dynamical systems and explore new directions in numerical analysis (see [16–18]). It focuses on analyzing properties of functions within a restricted range. Local fractional calculus is one of the most useful techniques for dealing with fractal and continuously non−differentiable functions. The concept of local fractional calculus has piqued the curiosity of mathematicians, as well as physicists and engineers. Yang [19] stated the theory of local fractional calculus on fractal space. Local fractional calculus is a generalization of differentiation and integration of the functions defined on fractal sets. Fractals help us study and understand important scientific concepts, such as the way bacteria grow, patterns in freezing water (snow flakes) and brain waves. After Mandelbrot [20] published his seminal book, fractals have been found valuable in science as well as engineering. Mo et al. [21] defined the generalized convex function on the fractal space $\mathbb{R}^{\alpha}$ $(0 < \alpha \leq 1)$ of a real numbers and established the generalized Jensen's inequality and generalized Hermite-Hadamard's inequality for a generalized convex function in the concept of local fractional calculus. A huge literature is present on fractal Hermite-Hadamrd, Ostrwoski, and Hermite-Mercer type inequalities (see [23–29]).

## 2. Preliminaries

We recall the theory of local fractional calculus. The concepts and important consequences associated with the local fractional derivative and local fractional integral are as follows:

For $0 < \alpha \leq 1$, we have the following $\alpha$-type set of element set:
$\mathbb{Z}^{\alpha} = \{0^{\alpha}, \pm 1^{\alpha}, \pm 2^{\alpha}, ..., \pm n^{\alpha}, ...\}$.
$\mathbb{Q}^{\alpha} = \{m^{\alpha} = (c/d)^{\alpha} : c, d \in \mathbb{Z}, d \neq 0\}$.
$\mathbb{J}^{\alpha} = \{m^{\alpha} \neq (c/d)^{\alpha} : c, d \in \mathbb{Z}, d \neq 0\}$.
$\mathbb{R}^{\alpha} = \mathbb{Q}^{\alpha} \cup \mathbb{J}^{\alpha}$.
If $r^{\alpha}, s^{\alpha}, t^{\alpha} \in \mathbb{R}^{\alpha}$ then the following operations satisfy
(i) $r^{\alpha} + s^{\alpha} \in \mathbb{R}^{\alpha}$, $r^{\alpha}s^{\alpha} \in \mathbb{R}^{\alpha}$;
(ii) $r^{\alpha} + s^{\alpha} = s^{\alpha} + r^{\alpha} = (r+s)^{\alpha} = (s+r)^{\alpha}$;
(iii) $r^{\alpha} + (s^{\alpha} + t^{\alpha}) = (r+s)^{\alpha} + t^{\alpha}$;
(iv) $r^{\alpha}s^{\alpha} = s^{\alpha}r^{\alpha} = (rs)^{\alpha} = (sr)^{\alpha}$;
(v) $r^{\alpha}(s^{\alpha}t^{\alpha}) = (r^{\alpha}s^{\alpha})t^{\alpha}$;
(vi) $r^{\alpha}(s^{\alpha} + t^{\alpha}) = r^{\alpha}s^{\alpha} + r^{\alpha}t^{\alpha}$;
(vii) $r^{\alpha} + 0^{\alpha} = 0^{\alpha} + r^{\alpha} = r^{\alpha}$, $r^{\alpha}1^{\alpha} = 1^{\alpha}r^{\alpha} = r^{\alpha}$.

To introduce the local fractional calculus on $\mathbb{R}^{\alpha}$, we begin the concept of the local fractional continuity as:

**Definition 2.1.** *[19] A non-differentiable mapping $\Phi : \mathbb{R} \longrightarrow \mathbb{R}^{\alpha}$, $\varkappa \longrightarrow \Phi(\varkappa)$ at $\varkappa_0$ is named local fractional continuous at $\varkappa_0$, if for all $\epsilon > 0$ exists $\delta > 0$ such that*

$$|\Phi(\varkappa) - \Phi(\varkappa_0)| < \epsilon^{\alpha}$$

*holds for $|\varkappa - \varkappa_0| < \delta$, for all $\epsilon, \delta \in \mathbb{R}$. If $\Phi(\varkappa)$ is local fractional continuous on the interval $(\zeta_1, \zeta_n)$ then we write $\Phi(\varkappa) \in C_{\alpha}(\zeta_1, \zeta_n)$.*

**Definition 2.2.** *[19] The local fractional derivative of $\Phi(\varkappa)$ of order $\alpha$ at $\varkappa = \varkappa_0$ is*

$$\Phi^{(\alpha)}(\varkappa_0) = \left.\frac{d^{\alpha}\Phi(\varkappa)}{d\varkappa^{\alpha}}\right|_{\varkappa=\varkappa_0} = \lim_{\varkappa \to \varkappa_0} \frac{\Delta^{\alpha}(\Phi(\varkappa) - \Phi(\varkappa_0))}{(\varkappa - \varkappa_0)^{\alpha}},$$

*where $\Delta^{\alpha}(\Phi(\varkappa) - \Phi(\varkappa_0)) = \Gamma(1+\alpha)(\Phi(\varkappa) - \Phi(\varkappa_0))$.*

**Definition 2.3.** *[19] If $\Phi \in C_{\alpha}[\zeta_1, \zeta_n]$, then the local fractional integral of $\Phi(\varkappa)$ of order $\alpha$ is*

$$\begin{aligned} {}_{\zeta_1}I_{\zeta_n}^{(\alpha)}\Phi(\varkappa) &= \frac{1}{\Gamma(1+\alpha)} \int_{\zeta_1}^{\zeta_n} \Phi(w)(dw)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta w \to 0} \sum_{s=0}^{M-1} \Phi(w_s)(\Delta w_s)^{\alpha}, \end{aligned}$$

*where $\Delta w_s = w_{s+1} - w_s$, $\Delta w = max\,\{\Delta w_s | s = 1, 2, ..., M-1\}$, and $[w_s, w_{s+1}]$, $s = 0, 1, .., M-1$ with $w_0 = \zeta_1 < w_1 < ... < w_i < ... < w_{M-1} < w_M = \zeta_n$ is a partition of $[\zeta_1, \zeta_n]$.*

*Here, it implies that ${}_{\zeta_1}I_{\zeta_n}^{(\alpha)}\Phi(\varkappa) = 0$ if $\zeta_1 = \zeta_n$ and ${}_{\zeta_1}I_{\zeta_n}^{(\alpha)}\Phi(\varkappa) = -\,{}_{\zeta_n}I_{\zeta_1}^{(\alpha)}\Phi(\varkappa)$ if $\zeta_1 < \zeta_n$. If for any $\varkappa \in [\zeta_1, \zeta_n]$, there exists ${}_{\zeta_1}I_{\zeta_n}^{(\alpha)}\Phi(\varkappa)$, which we denoted by $\Phi(\varkappa) \in I_{\varkappa}^{(\alpha)}[\zeta_1, \zeta_n]$.*

Define the Mittag-Leffler function [19] on fractal sets $\mathbb{R}^{\alpha}$ ( $0 < \alpha \leq 1$) is given by

$$E_{\alpha}(\varkappa^{\alpha}) = \sum_{k=0}^{\infty} \frac{\varkappa^{k\alpha}}{\Gamma(1+k\alpha)}, \ \varkappa \in \mathbb{R}.$$

**Lemma 2.4.** *[19]*

$${}_{\zeta_1}I_{\zeta_n}^{(\alpha)}E_\alpha(\varkappa^\alpha) = E_\alpha(\zeta_n^\alpha) - E_\alpha(\zeta_1^\alpha).$$

*and*

$$\frac{d^\alpha E_\alpha(\varsigma\varkappa^\alpha)}{d\varkappa^\alpha} = \varsigma E_\alpha(\varsigma\varkappa^\alpha)$$

*where $\varsigma$ is a constant.*

**Lemma 2.5.** *[19]*

*(i) If $\Phi(\varkappa) = \Phi^{(\alpha)}(\varkappa) \in C_\alpha[\zeta_1, \zeta_n]$,*
*then we have*
${}_{\zeta_1}I_{\zeta_n}^{\alpha}\Phi(\varkappa) = \Phi(\zeta_n) - \Phi(\zeta_1).$

*(ii) If $\Phi(\varkappa), \Phi(\varkappa) \in D_\alpha[\zeta_1, \zeta_n]$ and $\Phi^{(\alpha)}(\varkappa)$, $\Phi^{(\alpha)}(\varkappa) \in C_\alpha[\zeta_1, \zeta_n]$,*
*then we have*
${}_{\zeta_1}I_{\zeta_n}^{\alpha}\Phi(\varkappa)\Phi^{(\alpha)}(\varkappa) = \Phi(\varkappa)\Phi(\varkappa)|_{\zeta_1}^{\zeta_n} - {}_{\zeta_1}I_{\zeta_n}^{\alpha}\Phi^{(\alpha)}(\varkappa)\Phi(\varkappa)$

**Lemma 2.6.** *[19]*

$$\frac{d^\alpha \varkappa^{\tau\alpha}}{d\varkappa^\alpha} = \frac{\Gamma(1+\tau\alpha)}{\Gamma(1+(\tau-1)\alpha)}\varkappa^{(\tau-1)\alpha};$$

*and*

$$\frac{1}{\Gamma(1+\alpha)}\int_{\zeta_1}^{\zeta_n}\varkappa^{\tau\alpha}(d\varkappa)^\alpha = \frac{\Gamma(1+\tau\alpha)}{\Gamma(1+(\tau+1)\alpha)}(\zeta_n^{(\tau+1)\alpha} - \zeta_1^{(\tau+1)\alpha}),$$

*while $\tau \in \mathbb{R}$.*

Rainier et al. [22] propose the idea of a generalized strongly convex function on a fractal set with modulo c.

**Definition 2.7.** *Let $I \subset \mathbb{R}$ be an interval and $c$ be a positive number. A function $\Phi : I \to \mathbb{R}^\alpha$ is called generalized strongly convex function with modulus c if*

$$\Phi(\lambda x + (1-\lambda)y) \le \lambda^\alpha\Phi(x) + (1-\lambda)^\alpha\Phi(y) - c^\alpha\lambda^\alpha(1-\lambda)^\alpha(x-y)^{2\alpha} \tag{1}$$

*for all $x, y \in I$ and $\lambda \in [0, 1]$.*

**Remark 2.8.** *. It is to be noted that*

1. *If $\alpha = 1$ then generalized strongly convex functions derives strongly convex function.*
2. *If $c = 0$ then generalized strongly convex functions derives generalized convex function.*
3. *If $\alpha = 1$ and $c = 0$ then generalized strongly convex functions derives convex function.*

**Theorem 2.9.** *[22] A function $\Phi : I \to \mathbb{R}^\alpha$ is generalized strongly convex with modulus c if and only if the function $g : I \to \mathbb{R}^\alpha$ defined by $g(x) = \Phi(x) - c^\alpha x^{2\alpha}$ is generalized convex.*

**Theorem 2.10.** *[22](**Generalized Jensen-type inequality**) Suppose that $\Phi : I \to \mathbb{R}^\alpha$ is a generalized strongly convex function on I. Then for any $x_i \in I$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{n}\lambda_i = 1$ and $\bar{x} = \sum_{i=1}^{n}\lambda_i x_i$, we have*

$$\Phi\left(\sum_{i=1}^{n}\lambda_i x_i\right) \le \sum_{i=1}^{n}\lambda_i{}^\alpha\Phi(x_i) - c^\alpha\sum_{i=1}^{n}\lambda_i{}^\alpha(x_i - \bar{x})^{2\alpha}. \tag{2}$$

**Theorem 2.11.** *[22] (**Generalized Hermite–Hadamard-type inequality**) If $\Phi : I \to \mathbb{R}^\alpha$ is a generalized strongly convex function on $I = [a,b]$ and $\Phi \in I_x^{(\alpha)}[a,b]$, then*

$$\Phi\left(\frac{a+b}{2}\right) - c^\alpha \Phi\left(\frac{a+b}{2}\right)^{2\alpha} \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\left[{}_aI_b^{(\alpha)}\Phi(x) - c^\alpha \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(b^{3\alpha} - a^{3\alpha})\right]$$
$$\leq \frac{\Phi(a)+\Phi(b)}{2^\alpha} - c^\alpha\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^\alpha}\right).$$

Recently, Butt et al. [31] established the generalized Jensen-Mercer inequality.

**Theorem 2.12.** ***Generalized Jensen–Mercer inequality*** *Let $I = [\zeta_1, \zeta_n] \subseteq \mathbb{R}$ be an interval and $\Phi$ be a generalized convex function, then the following inequality holds*

$$\Phi\left(\zeta_1 + \zeta_n - \sum_{i=1}^{n}\omega_i\, x_i\right) \leq \Phi(\zeta_1) + \Phi(\zeta_n) - \sum_{i=1}^{n}\omega_i^\alpha \Phi(x_i)\,.$$

The principal aim of this paper is to present some results related to the Jensen-Mercer inequality in the framework of generalized strongly convex function on fractal sets. We establish generalized Jensen-Mercer type inequalities and generalized Hermite-Hadamard-Mercer type inequality for generalized strongly convex function in fractal space.

The article is organized as: In Section 3, we drive a variant of Jensen-type inequality for a generalized strongly convex function on a real linear fractal set $\mathbb{R}^\alpha$ ( $0 < \alpha \leq 1$). In order to prove this inequality, we need a main lemma on fractal sets which is presented in this section. We establish generalized Jensen-Mercer type inequality for this class of function. In Section 4, generalized Hermite-Hadmard-Mercer type inequality for a generalized strongly convex function are obtained.

## 3. Generalized Jensen-Mercer type inequality

**Lemma 3.1.** *Let $\Phi : I \to \mathbb{R}^\alpha$ is a generalized strongly convex with modulus c, then*

$$\Phi(\zeta_1 + \zeta_n - \zeta_i) \leq \Phi(\zeta_1) + \Phi(\zeta_n) - \Phi(\zeta_i) - 2^\alpha c^\alpha {\lambda_i}^\alpha (1-\lambda_i)^\alpha (\zeta_1 - \zeta_n)^{2\alpha}, \tag{3}$$

*where $\lambda_i \in [0,1]$, $\zeta_1 = \min\limits_{1\leq i\leq n}\zeta_i$, $\zeta_n = \max\limits_{1\leq i\leq n}\zeta_i$ and $\zeta_i \in I$.*

*Proof.* Let $y_i = \zeta_1 + \zeta_n - \zeta_i$ for $i = 1,2,...,n$, then $\zeta_1 + \zeta_n = \zeta_i + y_i$. We may write

$$\zeta_i = \lambda_i \zeta_1 + (1-\lambda_i)\zeta_n$$

and

$$y_i = (1-\lambda_i)\zeta_1 + \lambda_i \zeta_n,$$

where $0 \leq \lambda_i \leq 1$ and $1 \leq i \leq n$. Hence, applying generalized Jensen inequality, we get

$$\begin{aligned}
\Phi(y_i) &= \Phi((1-\lambda_i)\zeta_1 + \lambda_i\zeta_n) \\
&\leq (1-\lambda_i)^\alpha \Phi(\zeta_1) + \lambda_i^\alpha \Phi(\zeta_n) - c^\alpha \lambda_i^\alpha (1-\lambda_i)^\alpha (\zeta_1 - \zeta_n)^{2\alpha} \\
&= \Phi(\zeta_1) + \Phi(\zeta_n) - (\lambda_i^\alpha \Phi(\zeta_1) + (1-\lambda_i)^\alpha \Phi(\zeta_n)) - c^\alpha \lambda_i^\alpha (1-\lambda_i)^\alpha (\zeta_1 - \zeta_n)^{2\alpha} \\
&\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \Phi(\lambda_i\zeta_1 + (1-\lambda_i)\zeta_n) - 2^\alpha c^\alpha \lambda_i^\alpha (1-\lambda_i)^\alpha (\zeta_1 - \zeta_n)^{2\alpha} \\
&= \Phi(\zeta_1) + \Phi(\zeta_n) - \Phi(\zeta_i) - 2^\alpha c^\alpha \lambda_i^\alpha (1-\lambda_i)^\alpha (\zeta_1 - \zeta_n)^{2\alpha}.
\end{aligned}$$

Therefore, inequality (3) follows from $y_i = \zeta_1 + \zeta_n - \zeta_i$. □

**Theorem 3.2.** *(**Generalized Jensen-Mercer type inequality**) Let $\Phi : I \to \mathbb{R}^{\alpha}$ is a generalized strongly convex with modulus c, then*

$$\Phi\left(\zeta_1+\zeta_n-\sum_{i=1}^{n}\omega_i\zeta_i\right)$$
$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-\sum_{i=1}^{n}\omega_i^{\alpha}\Phi(\zeta_i) \tag{4}$$
$$-c^{\alpha}\left(2^{\alpha}\sum_{i=1}^{n}\omega_i^{\alpha}\lambda_i^{\alpha}(1-\lambda_i)^{\alpha}(\zeta_1-\zeta_n)^{2\alpha}+\sum_{i=1}^{n}\omega_i^{\alpha}(\zeta_i-\bar{\zeta})^{2\alpha}\right),$$

*where $\bar{\zeta}=\sum_{i=1}^{n}\omega_i\zeta_i$, $\sum_{i=1}^{n}\omega_i=1$, $\lambda_i\in[0,1]$, $\zeta_1=\min_{1\leq i\leq n}\zeta_i$, $\zeta_n=\max_{1\leq i\leq n}\zeta_i$ and $\zeta_i\in I$.*

*Proof.* Since $\sum_{i=1}^{n}\omega_i=1$ and by Lemma 3.1, we have

$$\Phi\left(\zeta_1+\zeta_n-\sum_{i=1}^{n}\omega_i\zeta_i\right)$$
$$=\Phi\left(\sum_{i=1}^{n}\omega_i(\zeta_1+\zeta_n-\zeta_i)\right)$$
$$\leq\sum_{i=1}^{n}\omega_i^{\alpha}\Phi(\zeta_1+\zeta_n-\zeta_i)-c^{\alpha}\sum_{i=1}^{n}\omega_i^{\alpha}(\zeta_i-\bar{\zeta})^{2\alpha}$$
$$\leq\Phi(\zeta_1)+\Phi(\zeta_n)-\sum_{i=1}^{n}\omega_i^{\alpha}\Phi(\zeta_i)-\sum_{i=1}^{n}\omega_i^{\alpha}c^{\alpha}2^{\alpha}\lambda_i^{\alpha}(1-\lambda_i)^{\alpha}(\zeta_1-\zeta_n)^{2\alpha}$$
$$-c^{\alpha}\sum_{i=1}^{n}\omega_i^{\alpha}(\zeta_i-\bar{\zeta})^{2\alpha}$$
$$=\Phi(\zeta_1)+\Phi(\zeta_n)-\sum_{i=1}^{n}\omega_i^{\alpha}\Phi(\zeta_i)$$
$$-c^{\alpha}\left(2^{\alpha}\sum_{i=1}^{n}\omega_i^{\alpha}\lambda_i^{\alpha}(1-\lambda_i)^{\alpha}(\zeta_1-\zeta_n)^{2\alpha}+\sum_{i=1}^{n}\omega_i^{\alpha}(\zeta_i-\bar{\zeta})^{2\alpha}\right).$$

This completes the proof. □

**Remark 3.3.** *s*

1. *If $\alpha = 1$ then inequality* (4) *gives Jensen-Mercer inequality for strongly convex function proved by Moradi in [30].*
2. *If $c = 0$ then inequality* (4) *gives generalized Jensen-Mercer inequality for generalized convex function proved by Butt et al. in [31].*
3. *If $\alpha = 1$ and $c = 0$ then inequality* (4) *gives Jensen-Mercer inequality for classical convex function [11].*

**Theorem 3.4.** *Let $\Phi : I \to \mathbb{R}^{\alpha}$ be a generalized strongly convex function on $I = [\zeta_1, \zeta_n]$ with modulus $c$ and $\lambda_i \in [0,1]$. Then*

$$\begin{aligned}
&\Phi(\zeta_1 + \zeta_n - \bar{\zeta}) \\
&\leq \sum_{i=1}^{n} \omega_i^{\alpha} \Phi(\zeta_1 + \zeta_n - ((1-\lambda_i)\zeta_i + \lambda_i \bar{\zeta})) - c^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} (1-\lambda_i)^{2\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \\
&\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \sum_{i=1}^{n} \omega_i^{\alpha} \Phi(\zeta_i) \\
&\quad - c^{\alpha} \left( 2^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} \lambda_i^{\alpha} (1-\lambda_i)^{\alpha} (\zeta_1 - \zeta_n)^{2\alpha} + \sum_{i=1}^{n} \omega_i^{\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \right),
\end{aligned}$$

*for all $\zeta_i \in [\zeta_1, \zeta_n]$ and $\omega_i \in [0,1]$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{n} \omega_i = 1$, where $\bar{\zeta} = \sum_{i=1}^{n} \omega_i \zeta_i$.*

*Proof.* Firstly, since $\Phi$ is generalized strongly convex function, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \omega_i^{\alpha} \Phi(\zeta_1 + \zeta_n - ((1-\lambda_i)\zeta_i + \lambda_i \bar{\zeta})) - c^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} (1-\lambda_i)^{2\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \\
&\geq \Phi\left( \sum_{i=1}^{n} \omega_i (\zeta_1 + \zeta_n - ((1-\lambda_i)\zeta_i + \lambda_i \bar{\zeta}) \right) \\
&= \Phi(\zeta_1 + \zeta_n - \bar{\zeta})
\end{aligned}$$

On the other hand

$$
\begin{aligned}
&\sum_{i=1}^{n} \omega_i^{\alpha} \Phi(\zeta_1 + \zeta_n - ((1-\lambda_i)\zeta_i + \lambda_i \bar{\zeta})) - c^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} (1-\lambda_i)^{2\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \\
&= \sum_{i=1}^{n} \omega_i^{\alpha} \Phi((1-\lambda_i)(\zeta_1 + \zeta_n - \zeta_i) + \lambda_i(\zeta_1 + \zeta_n - \bar{\zeta})) - c^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} (1-\lambda_i)^{2\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \\
&\leq \sum_{i=1}^{n} \omega_i^{\alpha} \Big[ (1-\lambda_i)^{\alpha} \Phi(\zeta_1 + \zeta_n - \zeta_i) + \lambda_i^{\alpha} \Phi(\zeta_1 + \zeta_n - \bar{\zeta})) - c^{\alpha} \lambda_i^{\alpha} (1-\lambda_i)^{\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \Big] \\
&\quad - c^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} (1-\lambda_i)^{2\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \\
&\leq \sum_{i=1}^{n} \omega_i^{\alpha} \Big[ (1-\lambda_i)^{\alpha} [\Phi(\zeta_1) + \Phi(\zeta_n) - \Phi(\zeta_i) - 2^{\alpha} c^{\alpha} \lambda_i^{\alpha} (1-\lambda_i)^{\alpha} (\zeta_1 - \zeta_n)^{2\alpha}] \\
&+ \lambda_i^{\alpha} [\Phi(\zeta_1) + \Phi(\zeta_n) - \sum_{i=1}^{n} \omega_i^{\alpha} \Phi(\zeta_i) + c^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} - 2^{\alpha} c^{\alpha} \lambda_i^{\alpha} (1-\lambda_i)^{\alpha} (\zeta_1 - \zeta_n)^{2\alpha}] \\
&\quad - c^{\alpha} \lambda_i^{\alpha} (1-\lambda_i)^{\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \Big] - c^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} \lambda_i^{\alpha} (1-\lambda_i)^{2\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \\
&\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \sum_{i=1}^{n} \omega_i^{\alpha} \Phi(\zeta_i) \\
&\quad - c^{\alpha} \left( 2^{\alpha} \sum_{i=1}^{n} \omega_i^{\alpha} \lambda_i^{\alpha} (1-\lambda_i)^{\alpha} (\zeta_1 - \zeta_n)^{2\alpha} + \sum_{i=1}^{n} \omega_i^{\alpha} (\zeta_i - \bar{\zeta})^{2\alpha} \right).
\end{aligned}
$$

This completes the proof. □

## 4. Generalized Hermite-Hadamard-Mercer type inequality

**Theorem 4.1.** *If* $\Phi \in I_x^{(\alpha)}[\zeta_1, \zeta_n]$ *and* $\Phi$ *be a generalized strongly convex function on* $[\zeta_1, \zeta_n]$*, then*

$$
\begin{aligned}
&\Phi\left(\zeta_1 + \zeta_n - \frac{a+b}{2}\right) - c^{\alpha} \left[ \left(\zeta_1 + \zeta_n - \frac{a+b}{2}\right)^{2\alpha} - (\zeta_1^{2\alpha} + \zeta_n^{2\alpha}) \right] \\
&\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \left[ {}_aI_b^{(\alpha)} \Phi(x) - c^{\alpha} \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} (b^{3\alpha} - a^{3\alpha}) \right] \\
&\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \Phi\left(\frac{a+b}{2}\right) + c^{\alpha} \left(\frac{a+b}{2}\right)^{2\alpha} \qquad (5)
\end{aligned}
$$

*and*

$$\Phi\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)-c^{\alpha}\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)^{2\alpha}$$
$$\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\left[{}_{\zeta_1+\zeta_n-b}I^{(\alpha)}_{\zeta_1+\zeta_n-a}\Phi(x)-c^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(b^{3\alpha}-a^{3\alpha})\right]$$
$$\leq \frac{\Phi(\zeta_1+\zeta_n-a)+\Phi(\zeta_1+\zeta_n-b)}{2^{\alpha}}-c^{\alpha}\left(\frac{(\zeta_1+\zeta_n-a)^{2\alpha}+(\zeta_1+\zeta_n-b)^{2\alpha}}{2^{\alpha}}\right)$$
$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-\frac{\Phi(a)+\Phi(b)}{2^{\alpha}}-c^{\alpha}\left(\zeta_1^{2\alpha}+\zeta_n^{2\alpha}-\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right), \tag{6}$$

*for all* $a,b\in[\zeta_1,\zeta_n]$.

*Proof.* Suppose that $\Phi\in I_x^{(\alpha)}[\zeta_1,\zeta_n]$ and $\Phi$ is a generalized strongly convex function then by Theorem 2.9, this is equivalent to saying that the function $g:[\zeta_1,\zeta_n]\to\mathbb{R}^{\alpha}$ defined by $g(x)=\Phi(x)-c^{\alpha}x^{2\alpha}$ is generalized convex. By [[32], Theorem 3.1], the above implies that the generalized Hermite-Hadamard-Mercer inequality holds for $g$,

$$g\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)\leq g(\zeta_1)+g(\zeta_n)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{(\alpha)}g(x)$$
$$\leq g(\zeta_1)+g(\zeta_n)-g\left(\frac{a+b}{2}\right) \tag{7}$$

and

$$g\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{\zeta_1+\zeta_n-b}I^{(\alpha)}_{\zeta_1+\zeta_n-a}g(x)$$
$$\leq \frac{g(\zeta_1+\zeta_n-a)+g(\zeta_1+\zeta_n-b)}{2^{\alpha}}$$
$$\leq g(\zeta_1)+g(\zeta_n)-\frac{g(a)+g(b)}{2^{\alpha}}. \tag{8}$$

Equivalently, we have (7)

$$\Phi\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)-c^{\alpha}\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)^{2\alpha}$$
$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-c^{\alpha}(\zeta_1^{2\alpha}+\zeta_n^{2\alpha})-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{(\alpha)}[\Phi(x)-c^{\alpha}x^{2\alpha}]$$
$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-c^{\alpha}(\zeta_1^{2\alpha}+\zeta_n^{2\alpha})-\left[\Phi\left(\frac{a+b}{2}\right)-c^{\alpha}\left(\frac{a+b}{2}\right)^{2\alpha}\right]$$

Consequently,

$$\Phi\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)-c^{\alpha}\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)^{2\alpha}$$

$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-c^{\alpha}(\zeta_1^{2\alpha}+\zeta_n^{2\alpha})-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{(\alpha)}\Phi(x)+c^{\alpha}\frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{(b-a)^{\alpha}\Gamma(1+3\alpha)}(b^{3\alpha}-a^{3\alpha})$$

$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-c^{\alpha}(\zeta_1^{2\alpha}+\zeta_n^{2\alpha})-\left[\Phi\left(\frac{a+b}{2}\right)-c^{\alpha}\left(\frac{a+b}{2}\right)^{2\alpha}\right]$$

Thus, we have

$$\Phi\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)-c^{\alpha}\left[\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)^{2\alpha}-(\zeta_1^{2\alpha}+\zeta_n^{2\alpha})\right]$$

$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\left[{}_aI_b^{(\alpha)}\Phi(x)-c^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(b^{3\alpha}-a^{3\alpha})\right]$$

$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-\left[\Phi\left(\frac{a+b}{2}\right)-c^{\alpha}\left(\frac{a+b}{2}\right)^{2\alpha}\right]$$

And (8), we have

$$\Phi\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)-c^{\alpha}\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)^{2\alpha}$$

$$\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{\zeta_1+\zeta_n-b}I^{(\alpha)}_{\zeta_1+\zeta_n-a}[\Phi(x)-c^{\alpha}x^{2\alpha}]$$

$$\leq \frac{\Phi(\zeta_1+\zeta_n-a)+\Phi(\zeta_1+\zeta_n-b)}{2^{\alpha}}-c^{\alpha}\left(\frac{(\zeta_1+\zeta_n-a)^{2\alpha}+(\zeta_1+\zeta_n-b)^{2\alpha}}{2^{\alpha}}\right)$$

$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-c^{\alpha}(\zeta_1^{2\alpha}+\zeta_n^{2\alpha})-\frac{\Phi(a)+\Phi(b)}{2^{\alpha}}+c^{\alpha}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right)$$

Thus, we have

$$\Phi\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)-c^{\alpha}\left(\zeta_1+\zeta_n-\frac{a+b}{2}\right)^{2\alpha}$$

$$\leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\left[{}_{\zeta_1+\zeta_n-b}I^{(\alpha)}_{\zeta_1+\zeta_n-a}\Phi(x)-c^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(b^{3\alpha}-a^{3\alpha})\right]$$

$$\leq \frac{\Phi(\zeta_1+\zeta_n-a)+\Phi(\zeta_1+\zeta_n-b)}{2^{\alpha}}-c^{\alpha}\left(\frac{(\zeta_1+\zeta_n-a)^{2\alpha}+(\zeta_1+\zeta_n-b)^{2\alpha}}{2^{\alpha}}\right)$$

$$\leq \Phi(\zeta_1)+\Phi(\zeta_n)-\frac{\Phi(a)+\Phi(b)}{2^{\alpha}}-c^{\alpha}\left(\zeta_1^{2\alpha}+\zeta_n^{2\alpha}-\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right)$$

This completes the proof. □

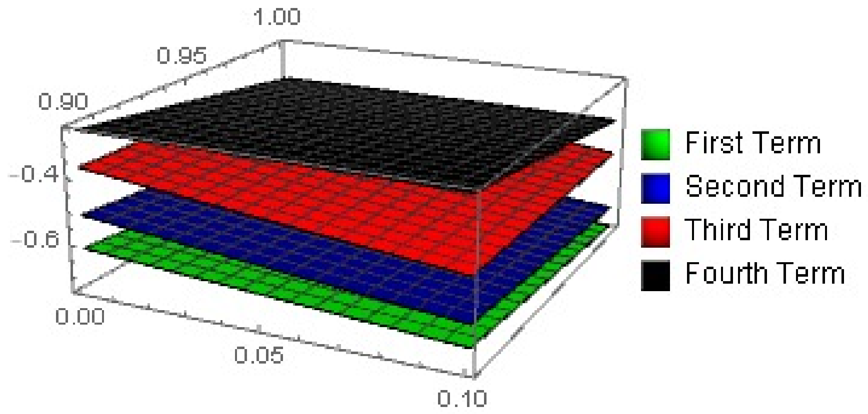


Figure 1: Graphical behaviour of Left, Middle and Right Terms of 9.

**Remark 4.2.** *s*

1. *If $\alpha = 1$ then the inequalities 5 and 6 of Theorem 4.1 reduces to the following inequalities.*

$$\Phi\left(\zeta_1 + \zeta_n - \frac{a+b}{2}\right) - c\left[\left(\zeta_1 + \zeta_n - \frac{a+b}{2}\right)^2 - (\zeta_1^2 + \zeta_n^2)\right]$$
$$\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \frac{1}{(b-a)}\left[{}_aI_b\Phi(x) - \frac{c}{3}(b^3 - a^3)\right]$$
$$\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \Phi\left(\frac{a+b}{2}\right) + c\left(\frac{a+b}{2}\right)^2 \tag{9}$$

*and*

$$\Phi\left(\zeta_1 + \zeta_n - \frac{a+b}{2}\right) - c\left(\zeta_1 + \zeta_n - \frac{a+b}{2}\right)^2$$
$$\leq \frac{1}{(b-a)}\left[{}_{\zeta_1+\zeta_n-b}I_{\zeta_1+\zeta_n-a}\Phi(x) - \frac{c}{3}(b^3 - a^3)\right]$$
$$\leq \frac{\Phi(\zeta_1 + \zeta_n - a) + \Phi(\zeta_1 + \zeta_n - b)}{2} - c\left(\frac{(\zeta_1 + \zeta_n - a)^2 + (\zeta_1 + \zeta_n - b)^2}{2}\right)$$
$$\leq \Phi(\zeta_1) + \Phi(\zeta_n) - \frac{\Phi(a) + \Phi(b)}{2} - c\left(\zeta_1^2 + \zeta_n^2 - \frac{a^2 + b^2}{2}\right), \tag{10}$$

*for all $a, b \in [\zeta_1, \zeta_n]$.*
*Consider the following example for the validity of 9 and 10 inequalities.*

***Example 4.3.*** *The following graphs of strongly convex function $\Phi(x) = x^2 - cosx$ for c=0.2 and $[\zeta_1, \zeta_n] = [0, 1]$ for all $a, b \in [\zeta_1, \zeta_n]$ show the validity of 9 and 10 inequalities.*

2. *If $c = 0$ then the inequalities 5 and 6 of Theorem 4.1 reduce to generalized Hermite-Hadamard-Mercer type inequalities for generalized convex function of Theorem 3.1. proved by Butt et al. in [32].*

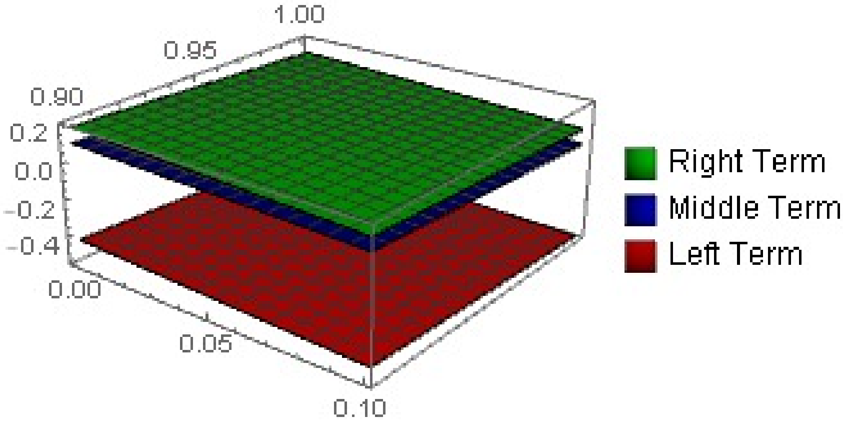


Figure 2: Graphical behaviour of First, Second, Third and Fourth Terms of 10.

3. *If $\alpha = 1$ and $c = 0$ then the inequalities 5 and 6 of Theorem 4.1 reduce to the inequalities proved by Kian and Moslehian in [33], Theorem 2.1.*

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