TURKISH JOURNAL OF SCIENCE
VOLUME 8, ISSUE 2, 93–101
ISSN: 2587–0971

https://www.tjoscience.com

Coefficient Estimates and Fekete-Sezegö Problem for Certain Subclass of Analytic and Univalent Functions Associated with Cosine and Sine Functions of Complex Order

Nizami Mustafa[a], Veysel Nezir[a]

[a] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey*

Abstract. The main focus of this paper is to give some coefficient estimates for the analytic and univalent functions on the open unit disk in the complex plane that are subordinated to cosine and sine functions with complex order. For the defined here subclass of analytic and univalent functions $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau,\beta)$, $\tau \in \mathbb{C} - \{0\}$ and $\beta \in [0,1]$ with the quantity

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right]\right\}$$

subordinated to $\cos z + \sin z$, we obtain the coefficient estimates for the initial two coefficients and examine the Fekete-Szegö problem for the mentioned class.

1. Introduction and preliminaries

In this section, we give some basic information that we will use in proof of the main results and to discuss the studies known in the literature related to our subject.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be open unit disk in the complex plane $\mathbb{C}$ and $H(U)$ denote the class of all analytic functions in U. By A, we will denote the class of $f \in H(U)$ functions given by the following series expansion, which satisfying the conditions $f(0) = 0$ and $f'(0) - 1 = 0$

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n,\ a_n \in \mathbb{C}. \tag{1}$$

As is known that the subclass of univalent functions of A is denoted by S, in the literature. This class was first time introduced by Köebe [12] and has become the core ingredient of advanced research in this field. Bieberbach [4] published a paper in which the famous coefficient hypothesis was proposed. This conjecture states that if $f \in S$ and has the series form (1), then $|a_n| \le n$ for all $n \ge 2$. Many researchers worked hard to solve this problem. But for the first time this long-lasting conjecture solved by de-Branges [5] in 1985.

Corresponding author: NM mail address: nizamimustafa@gmail.com ORCID:0000-0002-2758-0274, VN ORCID:0000-0001-9640-8526

Received: 21 July 2023; Accepted: 03 September 2023; Published: 30 September 2023

Keywords. Starlike function, convex function, cosine function, sine function, coefficient estimate, Fekete-Szegö problem, complex order

2010 *Mathematics Subject Classification.* 30C45, 30C50, 30C55, 30C80

Cited this article as: Mustafa N, Nezir V, Coefficient Estimates and Fekete-Sezegö Problem for Certain Subclass of Analytic and Univalent Functions Associated with Cosine and Sine Functions of Complex Order, Turkish Journal of Science, 2023, 8(2), 93-101.

It is well-know that a univalent function $f \in S$ is called a starlike and convex function, if this function maps open unit disk U onto the star shaped and convex shaped domain of the complex plane, respectively. The set of all starlike and convex functions in U, which satisfies the following conditions are denoted by S^* and C, respectively

$Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in U;$

that is,

$S^* = \left\{f \in S : \ Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U\right\}$ and$C = \left\{f \in S : \ Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in U\right\}.$

Some of the important and well-investigated subclass of S include the classes $S^*(\alpha)$ and $C(\alpha)$ given below, which are called of the starlike functions and convex functions of order α ($\alpha \in [0,1)$), respectively

$$S^*(\alpha) = \left\{f \in S : \ Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in U\right\}, C(\alpha) = \left\{f \in S : \ Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha,\ z \in U\right\}.$$

It is well-known that an analytical function ω satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ is called Schwartz function. Let's $f, g \in H(U)$, then it is said that f is subordinate to g and denoted by $f \prec g$, if there exists a Schwartz function ω , such that $f(z) = g(\omega(z))$.

In 1992, Ma and Minda [14] using subordination terminology presented unified version of the classes $S^*(\varphi)$ and $C(\varphi)$ as follows

$$(S^* \vee C)(\varphi) = \left\{f \in S : \ (1-\beta)\frac{zf'(z)}{f(z)} + \beta\frac{(zf'(z))'}{f'(z)} \prec \varphi(z), z \in U\right\}, \beta \in [0,1],$$

where $\varphi(z)$ is a univalent function with $\varphi(0) = 1$, $\varphi'(0) > 0$ and the region $\varphi(U)$ is star-shaped about the point $\varphi(0) = 1$ and symmetric with respect to real axis. Such a function has a series expansion of the following form

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} b_n z^n,\ b_1 > 0.$$

In the past few years, numerous subclasses of the collection S have been introduced as special choices of the classes $S^*(\varphi)$ and $C(\varphi)$ (see for example [1–3, 6, 7, 9–11, 13, 15, 17–32]).

Finding bounds for the function coefficients in a given collection is one of the most fundamental problems in geometric function theory, since it impacts geometric features.

The first order of Hankel determinant of the function $f \in S$ defined by

$$H_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and is known as the Fekete-Szegö functional. The functional $H_{2,1}(f, \mu) = a_3 - \mu a_2^2$ is known as the generalized Fekete-Szegö functional, where μ is a complex or real number [8]. Estimating the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö probblem in the theory of analytic functions.

Now by using the definition of subordination, we indroduce a new subclass of analytic and univalent functions associated with sine cosie and sine functions of comlpex order.

Definition 1.1. *For $\tau \in \mathbb{C} - \{0\}$ and $\beta \in [0,1]$ a function $f \in S$ is said to be in the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$, if the folloing condition is satisfied*

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} + \beta\left\{1 + \frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)} - 1\right]\right\} \prec \cos z + \sin z, z \in U;$$

that is,

$$\left(S^{*}_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau,\beta) = \left\{ \begin{array}{l} f \in S: \ (1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)}-1\right]\right\} \prec \\ \cos z+\sin z, z \in U \end{array} \right\}.$$

Remark 1.2. *Taking $\beta = 0$ and $\beta = 1$ in Definition 1.1, we have the classes*

$$S^{*}_{\cos,\sin}(\tau) = \left\{f \in S: \ 1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right] \prec \cos z+\sin z, z \in U\right\}$$

and

$$C_{\cos,\sin}(\tau) = \left\{f \in S: \ 1+\frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)}-1\right] \prec \cos z+\sinh z, z \in U\right\},$$

respectively.

Remark 1.3. *Taking $\tau = 1$ in Definition 1.1, we have the class*

$$\left(S^{*}_{\cos,\sin} \vee C_{\cos,\sin}\right)(\beta) = \left\{f \in S: \ (1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{\left(zf'(z)\right)'}{f'(z)} \prec \cos z+\sin z, z \in U\right\}.$$

Remark 1.4. *Taking $\tau = 1$, $\beta = 0$ and $\beta = 1$ in Definition 1.1, we have the classes*

$$S^{*}_{\cos,\sin} = \left\{f \in S: \ \frac{zf'(z)}{f(z)} \prec \cos z+\sin z, z \in U\right\}$$

and

$$C_{\cos,\sin} = \left\{f \in S: \ \frac{\left(zf'(z)\right)'}{f'(z)} \prec \cos z+\sin z, z \in U\right\},$$

respectively.

Remark 1.5. *It is useful to remind that the classes $S^{*}_{\cos,\sin}$, $C_{\cos,\sin}$, $S^{*}_{\cos,\sin}(\tau)$ and $C_{\cos,\sin}(\tau)$ are reviewed in [17], [19], [18] and [20], respectively.*

Let P be the class of analytic functions in U satisfied the conditions $p(0) = 1$ and $Re(p(z)) > 0$, $z \in U$, which from the subordination principle easily can written

$$\mathrm{P} = \left\{p \in A: \ p(z) \prec \frac{1+z}{1-z}, z \in U\right\},$$

where $p(z)$ has the series expansion of the form

$$p(z) = 1+p_1 z+p_2 z^2+p_3 z^3+\cdots = 1+\sum_{n=1}^{\infty} p_n z^n, \ z \in U. \tag{2}$$

The class P defined above is knowwn as the class Caratheodory functions [16].

Now, let us present some necessary lemmas known in the literature for the proof of our main results.

Lemma 1.6 ([8]). *Let the function $p(z)$ belong in the class* P*. Then,*
$\left|p_n\right| \leq 2$ for each $n \in \mathrm{N}$ and $\left|p_n-\lambda p_k p_{n-k}\right| \leq 2$ for $n,k \in \mathrm{N}$, $n > k$ and *$\lambda \in [0,1]$.*
The equalities holds for

$$p(z) = \frac{1+z}{1-z}.$$

Lemma 1.7 ([8]). *Let the an analytic function $p(z)$ be of the form (1.), then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)y$$

for $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

2. Coefficient estimates for the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$

In this section, we give upper bound estimates for initial two coefficients for the function belonging to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$. The following theorem is on these estimates.

Theorem 2.1. *Let the function $f \in A$ given by* (1) *belong to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$. Then,*

$$|a_2| \leq \frac{|\tau|}{1+\beta} \text{ and } |a_3| \leq \frac{|\tau|}{2(1+2\beta)} \begin{cases} 1 & \text{if } \left|2(1+3\beta)\tau - (1+\beta)^2\right| \leq 2(1+\beta)^2, \\ \frac{\left|2(1+3\beta)\tau-(1+\beta)^2\right|}{2(1+\beta)^2} & \text{if } \left|2(1+3\beta)\tau - (1+\beta)^2\right| \geq 2(1+\beta)^2. \end{cases}$$

Proof. Let $f \in \left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$, $\tau \in \mathbb{C} - \{0\}$ and $\beta \in [0, 1]$, then there exsists a Schwartz function ω, such that

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} + \beta\left\{1 + \frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)} - 1\right]\right\} = \cos\omega(z) + \sin\omega(z).$$

We express the Schwartz function ω in terms of the Caratheodory function $p \in \mathrm{P}$ as follows

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots.$$

It follows from that

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots. \tag{3}$$

From the series expansion (1) of the function $f(z)$, we can write

$$\begin{aligned} &(1-\beta)\left\{1 + \tfrac{1}{\tau}\left[\tfrac{zf'(z)}{f(z)} - 1\right]\right\} + \beta\left\{1 + \tfrac{1}{\tau}\left[\tfrac{(zf'(z))'}{f'(z)} - 1\right]\right\} = \\ &1 + \tfrac{1}{\tau}\left\{(1+\beta)a_2 z + \left[2(1+2\beta)a_3 - (1+3\beta)a_2^2\right]z^2 + \cdots\right\}. \end{aligned} \tag{4}$$

Since

$$\cos z + \sin z = 1 + z - \frac{z^2}{2!} + \cdots, \tag{5}$$

using equality (3), we have

$$\sin\omega(z) + \cos\omega(z) = 1 + \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{3}{4}p_1^2\right)z^2 + \cdots. \tag{6}$$

By equalizing (4) and (6), then comparing the coefficients of the same degree terms on the right and left sides, we obtain the following equalities for three initial coefficients of the function $f(z)$

$$a_2 = \frac{\tau p_1}{2(1+\beta)}, \tag{7}$$

$$a_3 = \frac{\tau}{8(1+2\beta)}\left\{2p_2 + \left[\frac{(1+3\beta)\tau}{(1+\beta)^2} - \frac{3}{2}\right]p_1^2\right\}. \tag{8}$$

Using Lemma 1.6, from the equalities (7) we can easily see that

$$|a_2| \leq \frac{|\tau|}{1+\beta}.$$

Using Lemma 1.7, from the second equality of (7) we can write

$$a_3 = \frac{\tau}{8(1+2\beta)}\left\{\left[\frac{(1+3\beta)\tau}{(1+\beta)^2} - \frac{1}{2}\right]p_1^2 + \left(4-p_1^2\right)x\right\},$$

for $x \in \mathbb{C}$ with $|x| \leq 1$. Applying triangle inequality, from this equality we obtain

$$|a_3| \leq \frac{|\tau|}{8(1+2\beta)}\left[\left|\frac{(1+3\beta)\tau}{(1+\beta)^2} - \frac{1}{2}\right|t^2 + \left(4-t^2\right)\xi\right],$$

where $\xi = |x|$ and $t = \left|p_1\right|$. If we maximize the function $\varphi : [0,1] \to \mathbb{R}$ defined as

$$\varphi(\xi) = \left|\frac{(1+3\beta)\tau}{(1+\beta)^2} - \frac{1}{2}\right|t^2 + \left(4-t^2\right)\xi,\ \xi \in [0,1],$$

we write

$$|a_3| \leq \frac{|\tau|}{8(1+2\beta)}\left\{\left[\left|\frac{(1+3\beta)\tau}{(1+\beta)^2} - \frac{1}{2}\right| - 1\right]t^2 + 4\right\},\ t \in [0,2].$$

From this, immediately obtained the desired estimate for $|a_3|$.

Thus, the proof of theorem is completed. □

In the case $\tau \in \mathbb{R} - \{0\}$, Theorem 2.1 is given as below.

Theorem 2.2. *Let the function $f \in A$ given by* (1) *belong to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau,\beta), \tau \in \mathbb{R} - \{0\}$. Then,* $|a_2| \leq \frac{\tau}{1+\beta}\begin{cases} -1 & \text{if } \tau < 0, \\ 1 & \text{if } \tau > 0. \end{cases}$ *and*

$$|a_3| \leq \frac{\tau}{2(1+2\beta)}\begin{cases} \frac{2(1+3\beta)\tau-(1+\beta)^2}{2(1+\beta)^2} & \text{if } \tau \leq -\frac{(1+\beta)^2}{2(1+3\beta)}, \\ -1 & \text{if } -\frac{(1+\beta)^2}{2(1+3\beta)} \leq \tau < 0, \\ 1 & \text{if } 0 < \tau \leq \frac{3(1+\beta)^2}{2(1+3\beta)}, \\ \frac{2(1+3\beta)\tau-(1+\beta)^2}{2(1+\beta)^2} & \text{if } \frac{3(1+\beta)^2}{2(1+3\beta)} \leq \tau. \end{cases}$$

Taking $\tau = 1$ in Theorem 2.1, we obtain the following estimates for the initial two coefficients for the function belonging to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\beta)$.

Theorem 2.3. *Let the function $f \in A$ given by* (1) *belong to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\beta)$. Then,* $|a_2| \leq \frac{1}{1+\beta}$ *and* $|a_3| \leq \frac{1}{2(1+2\beta)}$.

Setting $\beta = 0$ and $\beta = 1$ in Theorem 2.1, we obtain the following results obtained in [18] and [20], respectiely.

Corollary 2.4. *If the function $f \in A$ given by* (1) *belong to the class $S^*_{\cos,\sin}(\tau)$, then*
$|a_2| \le |\tau|$ *and* $|a_3| \le \frac{|\tau|}{2}\begin{cases} 1 & \text{if } |2\tau - 1| \le 2, \\ \frac{|2\tau-1|}{2} & \text{if } |2\tau - 1| \ge 2. \end{cases}$

Corollary 2.5. *If the function $f \in A$ given by* (1) *belong to the class $C_{\cos,\sin}(\tau)$, then*
$|a_2| \le \frac{|\tau|}{2}$ *and* $|a_3| \le \frac{|\tau|}{6}\begin{cases} 1 & \text{if } |2\tau - 1| \le 2, \\ \frac{|2\tau-1|}{2} & \text{if } |2\tau - 1| \ge 2. \end{cases}$

Setting $\beta = 0$ and $\beta = 1$ in Theorem 2.2 or $\tau = 1$ in Corollary 2.4 and Corollary 2.5, we obtain the following results for $|a_2|$ and $|a_3|$ obtained in [17] and [19], respctively.

Corollary 2.6. *If the function $f \in A$ given by* (1) *belong to the class $S^*_{\cos,\sin}$, then*
$|a_2| \le 1$ *and* $|a_3| \le \frac{1}{2}$.

Corollary 2.7. *If the function $f \in A$ given by* (1) *belong to the class $C_{\cos,\sin}$, then*
$|a_2| \le \frac{1}{2}$ *and* $|a_3| \le \frac{1}{6}$.

3. The Fekete-Szegö problem for the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$

In this section, we examine the Fekete-Szegö problem for the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$. The following theorem is on the Fekete-Szegö inequality for the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$.

Theorem 3.1. *Let the function $f \in A$ given by* (1) *belong to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$, $\tau \in \mathbb{C} - \{0\}$ and $\mu \in \mathbb{C}$ or $\mu \in \mathbb{R}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{2(1+2\beta)}\begin{cases} 1 & \text{if } \gamma(\tau,\mu,\beta) \le \frac{1}{2(1+2\beta)}, \\ 2(1+2\beta)\gamma & \text{if } \gamma(\tau,\mu,\beta) \ge \frac{1}{2(1+2\beta)}, \end{cases}$$

where

$$\gamma(\tau,\mu,\beta) = \left|\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{4(1+2\beta)}\right|.$$

Proof. Let $f \in \left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau, \beta)$,,$\tau \in \mathbb{C} - \{0\}$, $\beta \in [0, 1]$ and $\mu \in \mathbb{C}$. Then, from the equalities (7) and (8), we can write

$$a_3 - \mu a_2^2 = \frac{\tau}{4}\left\{\frac{p_2}{1+2\beta} + \left\{\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{3}{4(1+2\beta)}\right\}p_1^2\right\}.$$

Using Lemma 1.7, last equality we can write as follows

$$a_3 - \mu a_2^2 = \frac{\tau}{4}\left\{\left\{\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{4(1+2\beta)}\right\}p_1^2 + \frac{4-p_1^2}{2(1+2\beta)}x\right\} \tag{9}$$

for $x \in \mathbb{C}$with $|x| \le 1$. From here, applying triangle inequality we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{4}\left\{\left|\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{4(1+2\beta)}\right|t^2 + \frac{4-t^2}{2(1+2\beta)}\xi\right\}$$

with $t = |p_1| \in [0,2]$ and $\xi = |x|$. By maximizing the function

$$\psi(\xi) = \left|\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{4(1+2\beta)}\right| t^2 + \frac{4-t^2}{2(1+2\beta)}\xi, \xi \in [0,1],$$

we obtain the followwing inequality

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{4}\left\{\left[\left|\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{4(1+2\beta)}\right| - \frac{1}{2(1+2\beta)}\right] t^2 + \frac{2}{1+2\beta}\right\}, t \in [0,2]. \quad (10)$$

From this obbtained the desired result of theorem.

Thus, the proof of theorem is completed. □

In the cases $\tau \in \mathbb{R} - \{0\}$ and $\mu \in \mathbb{R}$, Theorem 3.1 is given as below.

Theorem 3.2. *Let the function $f \in A$ given by* (1) *belong to the class* $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau,\beta)$, $\tau \in \mathbb{R} - \{0\}$ *and* $\mu \in \mathbb{R}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{2(1+2\beta)}\begin{cases} -2(1+2\beta)\gamma \text{ if } \tau < 0 \text{ and } \mu \le \mu_1(\tau,\beta), \\ 1 \text{ if } \tau < 0 \text{ and } \mu \in [\mu_1(\tau,\beta), \mu_2(\tau,\beta)], \\ 2(1+2\beta)\gamma \text{ if } \tau < 0 \text{ and } \mu_2(\tau,\beta) \le \mu, \\ 2(1+2\beta)\gamma \text{ if } \tau > 0 \text{ and } \mu \le \mu_2(\tau,\beta), \\ 1 \text{ if } \tau > 0 \text{ and } \mu \in [\mu_2(\tau,\beta), \mu_1(\tau,\beta)], \\ -2(1+2\beta)\gamma \text{ if } \tau > 0 \text{ and } \mu_1(\tau,\beta) \le \mu. \end{cases}$$

where

$$\gamma(\tau,\mu,\beta) = \left|\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{4(1+2\beta)}\right|,$$

$$\mu_1(\tau,\beta) = \frac{2\tau(1+3\beta) + (1+\beta)^2}{4\tau(1+2\beta)}, \mu_2(\tau,\beta) = \frac{2\tau(1+3\beta) - 3(1+\beta)^2}{4\tau(1+2\beta)}.$$

Taking $\tau = 1$ in Theorem 3.1, we obtain the following estimates for the Fekete-Szegö functional for the function belonging to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\beta)$.

Theorem 3.3. *Let the function $f \in A$ given by* (1) *belong to the class* $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\beta)$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2(1+2\beta)}\begin{cases} 1 & \text{if } \left|\frac{1+4\beta-\beta^2}{4(1+2\beta)} - \mu\right| \le \frac{(1+\beta)^2}{2(1+2\beta)}, \\ 2(1+2\beta)\gamma(\mu,\beta) & \text{if } \left|\frac{1+4\beta-\beta^2}{4(1+2\beta)} - \mu\right| \ge \frac{(1+\beta)^2}{2(1+2\beta)}, \end{cases}$$

where

$$\gamma(\mu,\beta) = \left|\frac{1}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{4(1+2\beta)}\right|.$$

Setting $\beta = 0$ and $\beta = 1$ in Theorem 3.1, we obtain the following results obtained in [18] and [20], respectively.

Theorem 3.4. *Let the function $f \in A$ given by* (1) *belong to the class* $S^*_{\cos,\sin}(\tau)$ *and* $\mu \in \mathbb{C}$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{4}\begin{cases} 2 & \text{if } \left|2(1-2\mu)\tau - 1\right| \le 2, \\ \left|2(1-2\mu)\tau - 1\right| & \text{if } \left|2(1-2\mu)\tau - 1\right| \ge 2. \end{cases}$$

Theorem 3.5. *If the function $f \in A$ given by* (1) *belong to the class $C_{\cos,\sin}(\tau)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{12}\begin{cases} 2 & \text{if } \left|(2-3\mu)\tau - 1\right| \le 2, \\ \left|(2-3\mu)\tau - 1\right| & \text{if } \left|(2-3\mu)\tau - 1\right| \ge 2. \end{cases}$$

Setting $\beta = 0$, $\beta = 1$ and $\tau = 1$ in Theorem 3.1 or $\tau = 1$ in Theorem 3.4 and Theorem 3.5 , we obtain the following results obtained in [17] and [19], respectively.

Corollary 3.6. *If the function $f \in A$ given by* (1) *belong to the class $S^*_{\cos,\sin}$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{4}\begin{cases} 2 & \text{if } \left|1-4\mu\right| \le 2, \\ \left|1-4\mu\right| & \text{if } \left|1-4\mu\right| \ge 2. \end{cases}$$

Corollary 3.7. *If the function $f \in A$ given by* (1) *belong to the class $C_{\cos,\sin}$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{12}\begin{cases} 2 & \text{if } \left|1-3\mu\right| \le 2, \\ \left|1-3\mu\right| & \text{if } \left|1-3\mu\right| \ge 2. \end{cases}$$

Taking $\mu = 0$ and $\mu = 1$ in Theorem 3.1, we get the second result of Theorem 2.1 and the following result for the first order Haankel determinant, respectively.

Corollary 3.8. *If the function $f \in A$ given by* (1) *belong to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau,\beta)$,then*

$$\left|a_3 - a_2^2\right| \le \frac{|\tau|}{2(1+2\beta)}\begin{cases} 1 & \text{if } \left|1+\beta+2\tau\right| \le 2(1+\beta), \\ \frac{|1+\beta+2\tau|}{2(1+\beta)} & \text{if } \left|1+\beta+2\tau\right| \ge 2(1+\beta). \end{cases}$$

Setting $\tau \in \mathbb{R} - \{0\}$ in Corollary 3.8, we obtain the following result for the first order Haankel determinant.

Corollary 3.9. *If the function $f \in A$ given by* (1) *belong to the class $\left(S^*_{\cos,\sin} \vee C_{\cos,\sin}\right)(\tau,\beta)$,then*

$$\left|a_3 - a_2^2\right| \le \frac{|\tau|}{2(1+2\beta)}\begin{cases} -(1+\beta+2\tau) & \text{if } \tau \le -\frac{3(1+\beta)}{2}, \\ 1 & \text{if } -\frac{3(1+\beta)}{2} \le \tau < 0, \\ 1 & \text{if } 0 < \tau \le \frac{1+\beta}{2}, \\ 1+\beta+2\tau & \text{if } \frac{1+\beta}{2} \le \tau. \end{cases}$$

Taking $\beta = 0$ and $\beta = 1$ in the Corollary 3.8, we get the following resualts obtained in [18] and [20], repectively.

Corollary 3.10. *If the function $f \in A$ given by* (1) *belong to the class $S^*_{\cos,\sin}(\tau)$then,*

$$\left|a_3 - a_2^2\right| \le \frac{|\tau|}{4}\begin{cases} 2 & \text{if } |1+2\tau| \le 2, \\ |1+2\tau| & \text{if } |1+2\tau| \ge 2. \end{cases}$$

Corollary 3.11. *If the function $f \in A$ given by* (1) *belong to the class $C_{\cos,\sin}(\tau)$then,*

$$\left|a_3 - a_2^2\right| \le \frac{|\tau|}{12}\begin{cases} 2 & \text{if } |1+\tau| \le 2, \\ |1+\tau| & \text{if } |1+\tau| \ge 2. \end{cases}$$

Setting $\beta = 0$, $\beta = 1$ and $\tau = 1$ in Corollary 3.8 or $\tau = 1$ in Corollary 3.9 and Corollary 3.10, we obtain the following results obtained in [17] and [19], respctively.

Corollary 3.12. *If the function $f \in A$ given by* (1) *belong to the class $S^*_{\cos,\sin}$then,*

$$\left|a_3 - a_2^2\right| \le \frac{3}{4}\,.$$

Corollary 3.13. *If the function $f \in A$ given by* (1) *belong to the class $C_{\cos,\sin}$, then,*

$$\left|a_3 - a_2^2\right| \le \frac{1}{6}.$$

References

[1] Alotaibi, A., Arif, M., Alghamdi, MA., Hussain, S. Starlikness associated with cosine hyperbolic function. Mathematics 2020, 8, p. 1118.

[2] Arif, M., Ahmad, K., Liu, J-L., Sokol, J. A new class of analytic functions associated with Salagean operatör. J. Funct. Spaces 2019, 2019, 5157394.

[3] Bano, K., Raza, M. Starlike functions associated with cosine function. Bull. Iran. Math. Soc. 2020, 47, pp. 1513-1532.

[4] Bieberbach, L. Über die Koeffizienten derjenigen Potenzreihen.welche eine schlichte Abbidung des Einbeiskreises vermitteln. Sitzungsberichte Preuss. Akad. Der Wiss. 1916, 138, pp. 940-955.

[5] De Branges, L. A proof of the Bieberbach conjecture. Acta Math. 1985, 154, pp. 137-152.

[6] Brannan, DA., Kirwan, WE. On some classes of bounded univalent functions. J. Lond. Math. Soc. 1969, 2, pp. 431-443.

[7] Cho, NE., Kumar, V., Kumar, SS., Ravichandran, V. Radius problems for starlike functions associated with the sine function. Bull. Iran. Math. Soc. 2019, 45, pp. 213-232.

[8] Duren, PL. Univalent Functions. In Grundlehren der Mathematischen Wissenschaften, New York, Berlin, Heidelberg and Tokyo, Springer-Verlag, 1983, Volume 259.

[9] Janowski, W. Extremal problems for a family of functions with positive real part and for some related families. Ann. Pol. Math. 1970, 23, pp. 159-177.

[10] Kazımoğlu, S. Fekete-Szegö inequality for certain subclasses of analytic functions defined by the combination of differential and integral operators. Journal of the Institute of Science and Technology. 13(3), 2023, 2093–2104.

[11] Kazımoğlu, S., Deniz, E., Cotîrlă, L-I. Certain Ssubclasses of analytic and bi-Univalent functions governed by the gegenauer polynomials linked with q-derivative. Symmetry. 15(6), 2023, 1192.

[12] Köebe, P. Über die Uniformisierrung der algebraishen Kurven, durch automorpher Funktionen mit imaginarer Substitutions gruppe. Nachr. Akad. Wiss. Göttingen Math.-Phys. 1909, pp. 68-76.

[13] Kumar, SS., Arora, K. Starlike functions associated with a petal shaped domain. ArXiv 2020, arXiv: 2010.10072.

[14] Ma, WC., Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, Li, Z., Ren, F. , Yang, L., Zhang, S., Eds.; Conference Proceedings and Lecture Notes in Analysis; Internotianal Press: Cambridge, MA, USA, 1994, Volume 1, pp. 157-169.

[15] Mendiratta, R., Nagpal, S., Ravichandran, V. On a subclass of strongly starlike functions assiociated ith exponential function. Bull. Malays. Math. Soc. 2015, 38, pp. 365-386.

[16] Miller, SS. Differential inequalities and Caratheodory functions. Bull. Am. Math. Soc. 1975, 81, pp. 79-81.

[17] Mustafa, N., Demir, HA. Coefficient estimates for certain subclass of analytic and univalent functions with associated with sine and cosine functions. 4th International Black Sea Congress on Modern Scientific Research on June 6-8, 2023, pp. 2555-2563, Rize Turkey.

[18] Mustafa, N., Demir, HA. Coefficient estimates for certain subclass of analytic and univalent functions with associated with sine and cosine functions with complex order. Journal of Scientific and Engineering Research, 2023, 10(6): pp.131-140.

[19] Mustafa, N., Demir, HA. Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine and cosine functions. 4th International Black Sea Congress on Modern Scientific Research on June 6-8, 2023, pp. 2564-2572, Rize Turkey.

[20] Mustafa, N., Demir, HA. The Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine and cosine functions with complex order. Eastern Anatolian Journal of Science, 2023 (submitted).

[21] Mustafa, N., Nezir, V. Coefficient estimates and Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function. 4. International Istanbul Current Scientific Research Congress on May 1-2, 2023, pp.475-481, Istanbul, Türkiye.

[22] Mustafa, N., Nezir, V., Kankılıç, A. Coefficient estimates for certain subclass of analytic and univalent functions associated with sine hyperbolic function. 13th International Istanbul Scientific Research Congress on Life, Engineering and Applied Sciences on April 29-30, 2023, pp.234-241, Istanbul, Turkey.

[23] Mustafa, N., Nezir, V., Kankılıç, A. Coefficient estimates for certain subclass of analytic and univalent functions associated with sine hyperbolic function with complex order. Journal of Scientific and Engineering Research, 2023, 10(6): pp.18-25.

[24] Mustafa, N., Nezir, V., Kankılıç, A. The Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function. 13th International Istanbul Scientific Research Congress on Life, Engineering and Applied Sciences on April 29-30, 2023, pp. 242-249, Istanbul, Turkey.

[25] Mustafa, N., Nezir, V., Kankılıç, A. The Fekete-Szegö problem for certain subclass of analytic and univalent functions associted with sine hyperbolic function with complex order. Eastern Anatolian Journal of Science, 2023, 9(1), pp. 1-6.

[26] Sharma, K., Jain, NK., Ravichandran, V. Starlike function associated witha cardioid.Afr. Math. 2016, 27, pp. 923-939.

[27] Shi,. L., Srivastava, HM. Arif, M. , Hussain, S., Khan, H. An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function. Symmetry 2019, 11, p.598.

[28] Sokol, J. A certain class of starlike functions. Comput. Math. Appl. 2011, 62, pp. 611-619.

[29] Sokol, J., Stankiewcz, J. Radius of conexity of some subclasses of strongy starlike functions. Zeszyty Nauk. Politech. Rzeszowskiej Mat, 19, 1996, 101-105.

[30] Srivastava, HM., Khan, B., Khan, N., Tahir, M., Ahmad, S., Khan, N. Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function. Bull. Sci. Math. 167, 2021, 102942.

[31] Topkaya, S., Deniz, E. The general subclasses of the analytic functions and their various properties. Asian Research Journal of Mathematics. 7(4), 2017, 1-12.

[32] Ullah, K., Zainab, S., Arif, M., Darus, M., Shutayi, M. Radius Problems for Starlike Functions Associated with the TanHyperbolic Function. J. Funct. Spaces 2021, 2021, 9967640.