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Exponential $s-$Convex Functions in the First Sense on the Co-ordinates and Some Novel Integral Inequalities

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Abstract. This article describes new classes of convexity, namely exponential $s-$convex functions in the first sense on the co-ordinates. Then, some new integral inequalities are proved by using some classical inequalities and properties of exponential $s-$convex functions in the first sense on the co-ordinates.

1. Introduction

The concept of convexity, which has an important place in inequality theory, has been used by many researchers and has been used extensively, especially in the field of inequality theory. The definition of the convex functions can be given as follow.

Definition 1.1. *(See [3]) Let I be on interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex, if*

$$f\left(tx + (1-t)\, y\right) \leq t f\left(x\right) + (1-t)\, f\left(y\right)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The main purpose of the studies on different types of convexity is to optimize the bounds and generalize some known classical inequalities. An important class of convex functions, the definition of which has been given with the motivation of this main purpose, is exponentially convex functions, and the definition is given as follows.

Definition 1.2. *(See [4]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be exponential convex function, if*

$$f\left((1-t)\, x + ty\right) \leq (1-t)\frac{f(x)}{e^{\alpha x}} + t\frac{f(y)}{e^{\alpha y}}$$

for all $x, y \in I, \alpha \in \mathbb{R}$ and $t \in [0,1]$.

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If the above inequality holds in the reversed sense, then f is said to be exponentially concave function. Note that if $\alpha = 0$, then the class of exponentially convex functions reduce to class of classical convex function. However, the converse is not true.

In [1], Dragomir mentioned an expansion of the concept of convex function, which is used in many inequalities in the field of inequality theory and has applications in different fields of mathematics, especially convex programming.

Definition 1.3. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Expressing convex functions in coordinates brought up the question that it is possible for Hermite-Hadamard inequality to expand into coordinates. The answer to this motivating question has been found in Dragomir's paper (see [1]) and has taken its place in the literature as the expansion of Hermite-Hadamard inequality to a rectangle from the plane $\mathbb{R}^2$ stated below.

Theorem 1.4. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned} \tag{1}$$

The above inequalities are sharp.

In this study, exponentially convex functions on the co-ordinates have been introduced and a fundamental integral inequality of Hadamard-type has been proved for exponentially convex functions on the co-ordinates.

Definition 1.5. *(See [2]) A function $F : R^+ \to R$, is said to be $s-$convex in the first sense if*

$$F(\beta_1 u_1 + \beta_2 u_2) \leq \beta_1^s F(u_1) + \beta_2^s F(u_2)$$

for all $\beta_1, \beta_2 \geq 0$, $u_1, u_2 \geq 0$ with $\beta_1^s + \beta_2^s = 1$ and for some fixed $s \in (0,1]$. We denote this class of functions by K_s^1

Aslan and Akdemir gave the definition of exponential convex function in coordinates in 2022 as follows.

Definition 1.6. *(See [6]) Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$ and $c < d$. The mapping $f : \Delta \longrightarrow R$ is exponential convex on the co-ordinates on Δ, if the following inequality holds,*

$$f(tx + (1-t)z, ty + (1-t)w) \leq t\frac{f(x,y)}{e^{\alpha(x+y)}} + (1-t)\frac{f(z,w)}{e^{\alpha(z+w)}}$$

for all $(x,y),(z,w) \in \Delta, \alpha \in R$,and $t \in [0,1]$.

Definition 1.7. *(See [6]) The mapping $f : \Delta \longrightarrow \mathbb{R}$ is exponential convex on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned} & f(ta + (1-t)b, sc + (1-s)d) \\ \leq \quad & ts\frac{f(a,c)}{e^{\alpha(a+c)}} + t(1-s)\frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t)(1-s)\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

for all $(a,c),(a,d),(b,c),(b,d) \in \Delta, \alpha \in \mathbb{R}$ and $t,s \in [0,1]$.

Theorem 1.8. *(See [6]) Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $f \in L(\Delta)$, $\alpha \in \mathbb{R}$. If f is exponential convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \leq \frac{1}{4}\left[\frac{f(a,c)}{e^{\alpha(a+c)}} + \frac{f(a,d)}{e^{\alpha(a+d)}} + \frac{f(b,c)}{e^{\alpha(b+c)}} + \frac{f(b,d)}{e^{\alpha(b+d)}}\right].$$

Theorem 1.9. *(See[7]) Let $F : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $\partial^2 F/\partial t\partial s \in L(\Delta)$, $\alpha,\beta > 0, \alpha_1 \in \mathbb{R}$. If $\left|\partial^2 F/\partial t\partial s\right|$ is exponential convex function on the co-ordinates on Δ, then the has*

$$\begin{aligned} &\left|\frac{F(a,c) + F(a,d) + F(b,c) + F(b,d)}{4} + A\right| \\ \leq\ & \frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)} \times \left[\frac{\partial^2 F/\partial t\partial s\,(a,c)}{e^{\alpha_1(a+c)}} + \frac{\partial^2 F/\partial t\partial s\,(a,d)}{e^{\alpha_1(a+d)}} + \frac{\partial^2 F/\partial t\partial s\,(b,c)}{e^{\alpha_1(b+c)}} + \frac{\partial^2 F/\partial t\partial s\,(b,d)}{e^{\alpha_1(b+d)}}\right] \end{aligned}$$

Theorem 1.10. *(See[7]) Let $F : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $\partial^2 F/\partial t\partial s \in L(\Delta)$, $\alpha,\beta > 0, \alpha_1 \in \mathbb{R}$. If $\left|\partial^2 F/\partial t\partial s\right|, q > 1$, is exponential convex function on the co-ordinates on Δ, then the has*

$$\begin{aligned} &\left|\frac{F(a,c) + F(a,d) + F(b,c) + F(b,d)}{4} + A\right| \\ \leq\ & \frac{(b-a)(d-c)}{4} \\ &\times \left[\frac{4}{p(\alpha+1)^p(\beta+1)^p} + \frac{\left|\partial^2 F/\partial t\partial s\,(a,c)\right|^q}{qe^{\alpha_1(a+c)}} + \frac{\left|\partial^2 F/\partial t\partial s\,(a,d)\right|^q}{qe^{\alpha_1(a+d)}} + \frac{\left|\partial^2 F/\partial t\partial s\,(b,c)\right|^q}{qe^{\alpha_1(b+c)}} + \frac{\left|\partial^2 F/\partial t\partial s\,(b,d)\right|^q}{qe^{\alpha_1(b+d)}}\right], \end{aligned}$$

where $p^{-1} + q^{-1} = 1$.

For more information on types of exponential convexity in coordinates and s-convexity functions in the first sense, we recommend readers the following articles ([8]-[20]).

2. Second Section

Definition 2.1. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$ and $c < d$. The mapping $f : \Delta \longrightarrow R$ is exponential s-convex in the first sense on the co-ordinates on Δ, if the following inequality holds,*

$$f(tx + (1-t)z, ty + (1-t)w) \leq t^s \frac{f(x,y)}{e^{\alpha(x+y)}} + (1-t^s)\frac{f(z,w)}{e^{\alpha(z+w)}}$$

for all $(x,y),(z,w) \in \Delta, \alpha \in R, s \in (0,1]$ and $t \in [0,1]$.

A definition equivalent to the exponential $s-$ convex function definition in the first sense can be made as follows.

Definition 2.2. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$ and $c < d$. The mapping $f : \Delta \longrightarrow R$ is exponential s_1-convex in the first sense on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned} & f(tx + (1-t)y, sz + (1-s)w) \\ \leq\ & t^{s_1}s^{s_1}\frac{f(x,z)}{e^{\alpha(x+z)}} + t^{s_1}(1-s^{s_1})\frac{f(x,w)}{e^{\alpha(x+w)}} + (1-t^{s_1})s^{s_1}\frac{f(y,z)}{e^{\alpha(y+z)}} + (1-t^{s_1})(1-s^{s_1})\frac{f(y,w)}{e^{\alpha(y+w)}}. \end{aligned}$$

for all $(x,z),(x,w),(y,z),(y,w) \in \Delta, \alpha \in R, s_1 \in (0,1]$ and $t,s \in [0,1]$.

Lemma 2.3. *A function $f : \Delta \longrightarrow R$ will be called exponential s-convex in the first sense on the co-ordinates on Δ, if the partial mappings $f_y : [a,b] \longrightarrow R$, $f_y(u) = e^{\alpha y} f(u,y)$ and $f_x : [c,d] \longrightarrow R$, $f_x(v) = e^{\alpha x} f(x,v)$ are exponential s-convex in the first sense on the co-ordinates on Δ, where defined for all $y \in [c,d]$ and $x \in [a,b]$.*

Proof. From the definition of partial mapping f_x,we can write

$$\begin{aligned} f_x(tv_1 + (1-t)\, v_2) &= e^{\alpha x} f(x, tv_1 + (1-t)\, v_2) \\ &= e^{\alpha x} f(tx + (1-t)x, tv_1 + (1-t)\, v_2) \\ &\leq e^{\alpha x} \left[t^s \frac{f(x,v_1)}{e^{\alpha(x+v_1)}} + (1-t^s) \frac{f(x,v_2)}{e^{\alpha(x+v_2)}} \right] \\ &= t^s \frac{f(x,v_1)}{e^{\alpha v_1}} + (1-t^s) \frac{f(x,v_2)}{e^{\alpha v_2}} \\ &= t^s \frac{f_x(v_1)}{e^{\alpha v_1}} + (1-t^s) \frac{f_x(v_2)}{e^{\alpha v_2}} \end{aligned}$$

Similarly,

$$\begin{aligned} f_y(tu_1 + (1-t)\, u_2) &= e^{\alpha y} f(tu_1 + (1-t)\, u_2, y) \\ &= e^{\alpha y} f(tu_1 + (1-t)\, u_2, ty + (1-t)y) \\ &\leq e^{\alpha y} \left[t^s \frac{f(u_1,y)}{e^{\alpha(u_1+y)}} + (1-t^s) \frac{f(u_2,y)}{e^{\alpha(u_2+y)}} \right] \\ &= t^s \frac{f(u_1,y)}{e^{\alpha u_1}} + (1-t^s) \frac{f(u_2,y)}{e^{\alpha u_2}} \\ &= t^s \frac{f_y(u_1)}{e^{\alpha u_1}} + (1-t^s) \frac{f_y(u_2)}{e^{\alpha u_2}}. \end{aligned}$$

Proof is completed. □

Theorem 2.4. *Let $f : \Delta = [a,b] \times [c,d] \rightarrow R$ be partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $f \in L(\Delta)$, $\alpha \in R$ and $s_1 \in (0,1]$. If f is exponential s_1-convex in the first sense on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dxdy \\ \leq\ & \frac{1}{(s_1+1)^2} \frac{f(a,c)}{e^{\alpha(a+c)}} + \frac{s_1}{(s_1+1)^2} \left(\frac{f(a,d)}{e^{\alpha(a+d)}} + \frac{f(b,c)}{e^{\alpha(b+c)}} \right) + \frac{s_1^2}{(s_1+1)^2} \frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Proof. By the definition of the exponential s_1-convex in the first sense on the co-ordinates on Δ, we can write

$$\begin{aligned} &f\,(ta + (1-t)b, sc + (1-s)d) \\ \leq\ & t^{s_1} s^{s_1} \frac{f(a,c)}{e^{\alpha(a+c)}} + t^{s_1}(1-s^{s_1}) \frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t^{s_1}) s^{s_1} \frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t^{s_1})(1-s^{s_1}) \frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

By integrating both sides of the above inequality with respect to t, s on $[0,1]^2$, we have

$$\begin{aligned} &\int_0^1 \int_0^1 f\,(ta + (1-t)b, sc + (1-s)d)\, dtds \\ \leq\ & \int_0^1 \int_0^1 t^{s_1} s^{s_1} \frac{f(a,c)}{e^{\alpha(a+c)}} dtds + \int_0^1 \int_0^1 t^{s_1}(1-s^{s_1}) \frac{f(a,d)}{e^{\alpha(a+d)}} dtds \\ &+ \int_0^1 \int_0^1 (1-t^{s_1}) s^{s_1} \frac{f(b,c)}{e^{\alpha(b+c)}} dtds + \int_0^1 \int_0^1 (1-t^{s_1})(1-s^{s_1}) \frac{f(b,d)}{e^{\alpha(b+d)}} dtds. \end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Remark 2.5. *If we choose $\alpha = 0$ and $s_1 = 1$ in the above Hadamard-type inequality, the result coincides with the Hadamard-type inequality proved by Dragomir (See [1]).*

Remark 2.6. *If we choose $\alpha = 0$ in the above Hadamard-type inequality, the result coincides with the Hadamard-type inequality proved by Alaromi and Darus (See [19]).*

Remark 2.7. *If we choose $s_1 = 1$ in Theorem (2.4), the result will match Theorem (1.8)*

Theorem 2.8. *Let $f : \Delta = [a,b] \times [c,d] \to R$ be partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ and $f \in L(\Delta)$, $\alpha \in R$ and $s_1 \in (0,1]$. If $|f|$ is exponential s_1-convex in the first sense on the co-ordinates on Δ, $p > 1$ then the following inequality holds;*

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\ \leq\; &\left(\frac{1}{(ps_1+1)^2}\right)^{\frac{1}{p}}\frac{|f(a,c)|}{e^{\alpha(a+c)}} + \left(\frac{ps_1}{(ps_1+1)^2}\right)^{\frac{1}{p}}\left(\frac{|f(a,d)|}{e^{\alpha(a+d)}} + \frac{|f(b,c)|}{e^{\alpha(b+c)}}\right) + \left(\frac{p^2s_1^2}{(ps_1+1)^2}\right)^{\frac{1}{p}}\frac{|f(b,d)|}{e^{\alpha(b+d)}}. \end{aligned}$$

Proof. By the definition of the exponential s_1-convex in the first sense on the co-ordinates on Δ, we can write

$$\begin{aligned} &f(ta+(1-t)b, sc+(1-s)d) \\ \leq\; &t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}} + t^{s_1}(1-s^{s_1})\frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t^{s_1})s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t^{s_1})(1-s^{s_1})\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t, s on $[0,1]^2$, we can write

$$\begin{aligned} &\left|\int_0^1\int_0^1 f(ta+(1-t)b, sc+(1-s)d)\,dtds\right| \\ \leq\; &\int_0^1\int_0^1\left|t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds + \int_0^1\int_0^1\left|t^{s_1}(1-s^{s_1})\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\ &+\int_0^1\int_0^1\left|(1-t^{s_1})s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds + \int_0^1\int_0^1\left|(1-t^{s_1})(1-s^{s_1})\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds \end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\ \leq\; &\left(\int_0^1\int_0^1 t^{ps_1}s^{ps_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}} + \left(\int_0^1\int_0^1 t^{ps_1}(1-s^{s_1})^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}} \\ &+\left(\int_0^1\int_0^1 (1-t^{s_1})^p s^{ps_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\ &+\left(\int_0^1\int_0^1 (1-t^{s_1})^p(1-s^{s_1})^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}} \end{aligned}$$

By using the fact that $\left|1-(1-t)^{\theta}\right|^{\beta} \leq 1-(1-t)^{\theta\beta}$ for $\theta > 0, \beta > 0$ [5] , we can write

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
\leq\;& \left(\int_0^1\int_0^1 t^{ps_1}s^{ps_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1 t^{ps_1}(1-s^{ps_1})dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{ps_1})s^{ps_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{ps_1})(1-s^{ps_1})dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}}
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 2.9. *Let $f:\Delta=[a,b]\times[c,d]\to R$ be partial differentiable mapping on $\Delta=[a,b]\times[c,d]$ and $f\in L(\Delta)$, $\alpha\in R$ and $s_1\in(0,1]$. If $\left|f\right|$ is exponential s_1-convex in the first sense on the co-ordinates on Δ, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$,then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
\leq\;& \left(\frac{1+ps_1+p^2s_1^2}{p(ps_1+1)^2}\right)+\frac{1}{q}\left(\frac{\left|f(a,c)\right|^q}{e^{\alpha q(a+c)}}+\frac{\left|f(a,d)\right|^q}{e^{\alpha q(a+d)}}+\frac{\left|f(b,c)\right|^q}{e^{\alpha q(b+c)}}+\frac{\left|f(b,d)\right|^q}{e^{\alpha q(b+d)}}\right).
\end{aligned}$$

Proof. By the definition of the exponential s_1-convex in the first sense on the co-ordinates on Δ, we can write

$$\begin{aligned}
&f(ta+(1-t)b, sc+(1-s)d) \\
\leq\;& t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s^{s_1})\frac{f(a,d)}{e^{\alpha(a+d)}}+(1-t^{s_1})s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{s_1})(1-s^{s_1})\frac{f(b,d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned}
&\left|\int_0^1\int_0^1 f(ta+(1-t)b, sc+(1-s)d)\,dtds\right| \\
\leq\;& \int_0^1\int_0^1\left|t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t^{s_1}(1-s^{s_1})\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\
&+\int_0^1\int_0^1\left|(1-t^{s_1})s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t^{s_1})(1-s^{s_1})\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds
\end{aligned}$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
\leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^{ps_1}s^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 t^{ps_1}(1-s^{s_1})^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{s_1})^p s^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{s_1})^p(1-s^{s_1})dtds\right)^p+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right).
\end{aligned}$$

By using the fact that $\left|1-(1-t)^{\theta}\right|^{\beta}\leq 1-(1-t)^{\theta\beta}$ for $\theta>0,\beta>0$ [5] , we can write

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
\leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^{ps_1}s^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 t^{ps_1}(1-s^{ps_1})dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{ps_1})s^{ps_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{ps_1})(1-s^{ps_1})dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right).
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Proposition 2.10. *If $f,g:\Delta\to R$ are two exponential s_1-convex in the first sense on the co-ordinates on Δ, then $f+g$ are exponential s_1-convex in the first sense on the co-ordinates on Δ,*

Proof. By the definition of the exponential s_1-convex in the first sense on the co-ordinates on Δ, we can write

$$\begin{aligned}
&f(ta+(1-t)b,sc+(1-s)d)+g(ta+(1-t)b,sc+(1-s)d) \\
\leq\ & t^{s_1}s^{s_1}\left(\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{g(a,c)}{e^{\alpha(a+c)}}\right)+t^{s_1}(1-s^{s_1})\left(\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{g(a,d)}{e^{\alpha(a+d)}}\right) \\
&+(1-t^{s_1})s^{s_1}\left(\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{g(b,c)}{e^{\alpha(b+c)}}\right)+(1-t^{s_1})(1-s^{s_1})\left(\frac{f(b,d)}{e^{\alpha(b+d)}}+\frac{g(b,d)}{e^{\alpha(b+d)}}\right).
\end{aligned}$$

Namely,

$$\begin{aligned}
&(f+g)(ta+(1-t)b,sc+(1-s)d) \\
\leq\ & t^{s_1}s^{s_1}\frac{(f+g)(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s^{s_1})\frac{(f+g)(a,d)}{e^{\alpha(a+d)}}+(1-t^{s_1})s^{s_1}\frac{(f+g)(b,c)}{e^{\alpha(b+c)}}+(1-t^{s_1})(1-s^{s_1})\frac{(f+g)(b,d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

Therefore $(f+g)$ is exponential s_1-convex in the first sense on the co-ordinates on Δ. □

Proposition 2.11. *If $f:\Delta\to R$ is exponential s_1-convex in the first sense on the co-ordinates on Δ and $k\geq 0$ then kf is exponential s_1-convex in the first sense on the co-ordinates on Δ.*

Proof. By the definition of the exponential s_1-convex in the first sense on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+(1-s)d\right) \\ \leq \quad & t^{s_1}s^{s_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s^{s_1})\frac{f(a,d)}{e^{\alpha(a+d)}}+(1-t^{s_1})s^{s_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{s_1})(1-s^{s_1})\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

If both sides are multiplied by k, we have,

$$\begin{aligned} & (kf)\left(ta+(1-t)b, sc+(1-s)d\right) \\ \leq \quad & t^{s_1}s^{s_1}\frac{(kf)(a,c)}{e^{\alpha(a+c)}}+t^{s_1}(1-s^{s_1})\frac{(kf)(a,d)}{e^{\alpha(a+d)}}+(1-t^{s_1})s^{s_1}\frac{(kf)(b,c)}{e^{\alpha(b+c)}}+(1-t^{s_1})(1-s^{s_1})\frac{(kf)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore (kf) is exponential s_1-convex in the first sense on the co-ordinates on Δ. □

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