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On Repdigits as Product of Pell and Narayana Numbers

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Abstract. In this paper, we examine the solution of a Diophantine equation involving two integer sequences. More specifically, we find all repdigits (i.e., numbers with only one repeating digit in the decimal expansion) that can be written as a product of Pell number and a Narayana number. Our approach to solving this problem is to combine Baker theory with the theory of continued fractions.

1. Introduction

Diophantine equations involving recurrence sequences have been studied for a long time. One of the most interesting of these equations is the equations involving repdigits.

A repdigit (short for "repeated digit") T is a natural number composed of repeated instances of the same digit in its decimal expansion. That is, T is of the form

$$x \cdot \left(\frac{10^t - 1}{9}\right)$$

for some positive integers x, t with $t \geq 1$ and $1 \leq x \leq 9$.

Some of the most recent papers related to this concepts are [3–6]. In this note, we use Pell and Narayana sequences in our main result.

Pell sequence, one of the most familiar binary recurrence sequence, is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$. Some of the terms of the Pell sequence are given by $0, 1, 2, 5, 12, 29, 70, \ldots$. Its characteristic polynomial is of the form $x^2 - 2x - 1 = 0$ whose roots are $\nu = 1 + \sqrt{2}$ (the silver ratio) and $\eta = 1 - \sqrt{2}$. Binet's formula enables us to rewrite the Pell sequence by using the roots α and β as

$$P_n = \frac{\nu^n - \eta^n}{2\sqrt{2}}. \tag{1}$$

Also, it is known that

$$\nu^{n-2} \leq P_n \leq \nu^{n-1} \tag{2}$$

and

$$P_n = \frac{\nu^n}{2\sqrt{2}} + \lambda \tag{3}$$

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where $|\lambda| \leq 1/\left(2\sqrt{2}\right)$.

The characteristic polynomial of Narayana sequence $\{\mathcal{N}_n\}_{(n\geq 0)}$ is:

$$\varphi(x) = x^3 - x^2 - 1.$$

and the characteristic roots are:

$$\alpha := 1/3\left(\sqrt[3]{1/2\left(29-3\sqrt{93}\right)} + \sqrt[3]{1/2\left(3\sqrt{93}+29\right)} + 1\right), \tag{4}$$

$$\beta := 1/3 - 1/6\left(1-i\sqrt{3}\right)\sqrt[3]{1/2\left(29-3\sqrt{93}\right)} - 1/6\left(1+i\sqrt{3}\right)\sqrt[3]{1/2\left(3\sqrt{93}+29\right)}, \tag{5}$$

$$\gamma := 1/3 - 1/6\left(1+i\sqrt{3}\right)\sqrt[3]{1/2\left(29-3\sqrt{93}\right)} - 1/6\left(1-i\sqrt{3}\right)\sqrt[3]{1/2\left(3\sqrt{93}+29\right)}. \tag{6}$$

Then, Binet-like formula for Narayana numbers is

$$\mathcal{N}_n := a\alpha^n + b\beta^n + c\gamma^n \tag{7}$$

It can be obtain that

$$a := \frac{\alpha^2}{\alpha^3+2},\ b := \frac{\beta^2}{\beta^3+2},\ and\ c := \frac{\gamma^2}{\gamma^3+2} \tag{8}$$

and the minimal polynomial of a over integers is $31x^3 - 3x - 1$.
Also,

$$\mathcal{N}_n = a\alpha^n + \theta \tag{9}$$

where $\theta < 1/\alpha^{n+2}$ for all $n > 1$. We also have the following property.

Theorem 1.1. *Let $\{\mathcal{N}_n\}_{n\geq 0}$ be the Narayana sequence. Then,*

$$\alpha^{n-2} \leq \mathcal{N}_n \leq \alpha^{n-1} \tag{10}$$

for $n \geq 1$.

It is easy to see that $\alpha \in \{1.46, 1.47\}$, $\left|\beta\right| = \left|\gamma\right| \in \{0.82, 0.83\}$, $a \in \{0.61, 0.62\}$ and $|b| = |c| \in \{0.57, 0.58\}$
In this study, our main result is the following:

Theorem 1.2. *The only positive integer triples (n, t, x) with $1 \leq x \leq 9$ satisfying the Diophantine equation*

$$\mathcal{N}_n P_n = x \cdot \left(\frac{10^t - 1}{9}\right) \tag{11}$$

as follows:

$$(n, t, x) \in \{(1, 1, 1), (2, 1, 2), (3, 1, 5)\}.$$

2. Preliminaries

Before proceeding with the proof of our main result, let us give some necessary information for proof. We give the definition of the logarithmic height of an algebraic number and its some properties.

Definition 2.1. *Let z be an algebraic number of degree d with minimal polynomial*

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \cdot \prod_{i=1}^{d}(x - z_i)$$

where a_i's are relatively prime integers with $a_0 > 0$ and z_i's are conjugates of z. Then

$$h(z) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|z_i|, 1\}\right)\right)$$

is called the logarithmic height of z. *The following proposition gives some properties of logarithmic height that can be found in [7].*

Proposition 2.2. *Let $z, z_1, z_2, \ldots, z_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

1. $h(z_1 \cdots z_t) \le \sum_{i=1}^{t} h(z_i)$
2. $h(z_1 + \cdots + z_t) \le \log t + \sum_{i=1}^{t} h(z_i)$
3. $h(z^m) = |m|\, h(z)$.

We will use the following theorem (see [1] or Theorem 9.4 in [8]) and lemma (see [9] which is a variation of the result due to [2]) for proving our results.

Theorem 2.3. *Let $z_1, z_2, \ldots, z_s$ be nonzero elements of a real algebraic number field $\mathbb{F}$ of degree D, $b_1, b_2, \ldots, b_s$ rational integers. Set*

$$B := \max\{|b_1|, \ldots, |b_s|\}$$

and

$$\Lambda := z_1^{b_1} \ldots z_s^{b_s} - 1.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{s+4} \cdot (s+1)^{5.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log(sB)) \cdot A_1 \cdots A_s$$

where

$$A_i \ge \max\{D \cdot h(z_i), \left|\log z_i\right|, 0.16\}$$

for all $1 \le i \le s$. If $\mathbb{F} = \mathbb{R}$, then

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

Lemma 2.4. *Let A, B, μ be some real numbers with $A > 0$ and $B > 1$ and let γ be an irrational number and M be a positive integer. Take p/q as a convergent of the continued fraction of γ such that $q > 6M$. Set $\varepsilon := \left\|\mu q\right\| - M\left\|\gamma q\right\| > 0$ where $\|\cdot\|$ denotes the distance from the nearest integer. Then, there is no solution to the inequality*

$$0 < \left|u\gamma - v + \mu\right| < AB^{-w}$$

in positive integers u, v and w with

$$u \le M \text{ and } w \ge \frac{\log \frac{Aq}{\varepsilon}}{\log B}.$$

3. The Proof of Theorem 1.2

Let us write Equations (3) and (9) in Equation (11). We get

$$(a\alpha^n + \theta)\left(\frac{\nu^n}{2\sqrt{2}} + \lambda\right) = x \cdot \left(\frac{10^t - 1}{9}\right).$$

After some manipulations using $|\theta| < 1/2$ and $|\lambda| \le 1/\left(2\sqrt{2}\right)$, we have

$$\left|\frac{a(\alpha\nu)^n}{2\sqrt{2}} - \frac{x \cdot 10^t}{9}\right| < 1.35 \cdot \alpha^n.$$

To convert this inequality into form in Theorem 2.3, let us divide both sides by $\frac{a(\alpha\nu)^n}{2\sqrt{2}}$. So, we have

$$\left|1 - 10^t \cdot (\alpha\nu)^{-n} \cdot \left(\left(x \cdot 2\sqrt{2}\right) / (9a)\right)\right| < 6.26 \cdot \nu^{-n}. \tag{12}$$

Set

$$\Gamma := 10^t \cdot (\alpha\nu)^{-n} \cdot \left(\left(x \cdot 2\sqrt{2}\right) / (9a)\right) - 1.$$

It can be easily obtain that $\Gamma \neq 0$.

Now, we are in the position to apply Theorem 2.3 to the inequality (12). Set

$$(m_1, m_2, m_3) = \left(10, \alpha\nu, \left(x \cdot 2\sqrt{2}\right) / (9a)\right) \text{ and } (c_1, c_2, c_3) = (t, -n, 1).$$

Since $\mathbb{Q}(m_1, m_2, m_3) = \mathbb{Q}(\alpha, \nu)$, we know that $D \leq 6$. So, we can take

$$\begin{aligned} 14 = A_1 &\geq 6 \cdot h(10) = 6 \cdot \log(10) \sim 13.82 \\ 3.5 = A_2 &\geq 6 \cdot h(\alpha\nu) < 6 \cdot (\log(\alpha)/3 + \log(\nu)/2) \sim 3.4 \\ 44 = A_3 &\geq 6 \cdot h\left(x \cdot 2\sqrt{2}/(9a)\right) \leq 6\left(h(x) + h\left(2\sqrt{2}\right) + h(9) + h(a)\right) < 43.92. \end{aligned}$$

Now, let us try to find the value of B. From the inequalities (2) and (10), we can write

$$\alpha^{n-1} \cdot \nu^{n-1} \geq \mathcal{N}_n P_n = x \cdot \left(10^t - 1\right)/9 > 10^{t-1}$$

and this inequality implies that

$$1.81t - 0.82 < n. \tag{13}$$

Since $t < 1.81t - 0.82$ for $t > 1$, we can write $t < n$ from the inequality (13). Thus, we can take

$$B := n.$$

So, due to the Theorem 2.3 we have

$$|\Gamma| > \exp(-C \cdot (1 + \log n) \cdot 14 \cdot 3.5 \cdot 44)$$

where $C := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2 \cdot (1 + \log 6)$. From the inequality (12), we get

$$\frac{6.26}{\nu^n} > \exp(-C \cdot (1 + \log n) \cdot 14 \cdot 3.5 \cdot 44).$$

Taking logarithm of both sides of the above inequality and considering $C < 1.44 \cdot 10^{13}$ and $1 + \log n < 2\log n$ for $n \geq 3$, we get

$$n < 2.53 \cdot 10^{17}. \tag{14}$$

We get

$$t < 1.4 \cdot 10^{17}. \tag{15}$$

by the inequality (13).

Now, let us improve the bounds (14) and (15). Set

$$\Psi := t\log 10 - n\log(\alpha\nu) + \log\left(\left(x \cdot 2\sqrt{2}\right) / (9a)\right).$$

So, we can rewrite the inequality (12) as

$$\left|1 - e^{\Psi}\right| < \frac{6.26}{\nu^n}.$$

If $\Psi > 0$, then

$$\Psi < e^{\Psi} - 1 < 6.26 \cdot \nu^{-n}.$$

Otherwise, i.e., $\Psi < 0$, then we get

$$|\Psi| < 6.26 \cdot \nu^{-n+1}. \tag{16}$$

Now, suppose $\Psi > 0$ (for the case $\Psi < 0$ operations are similar). From the inequality (16), we obtain

$$\begin{aligned} 0 &< t\log 10 - n\log(\alpha\nu) + \log\left(\left(x \cdot 2\sqrt{2}\right) / (9a)\right) \\ &< 6.26 \cdot \nu^{-(n-1)}. \end{aligned}$$

Dividing both sides of the above inequality by $\log(\alpha\nu)$, we obtain

$$0 < t \cdot \frac{\log 10}{\log(\alpha\nu)} - n + \frac{\log\left(\left(x \cdot 2\sqrt{2}\right) / (9a)\right)}{\log(\alpha\nu)} < 19.45 \cdot \varphi^{-(n-1)}.$$

In here, $\gamma := \log 10/\log(\alpha\nu)$ is an irrational number. Hence, we can apply the Lemma 2.4 to the above inequality with the parameters

$$\mu := \frac{\log\left(\left(x \cdot 2\sqrt{2}\right) / (9a)\right)}{\log(\alpha\nu)}, \ A := 19.45, \ B := \nu \text{ and } w := n - 1.$$

We can choose $M := 1.4 \cdot 10^{17}$ from the bound (15). So, 43th convergence of γ is satisfies the condition $q > 6M$. From this convergent, we get the smallest ε as 0.113535. Thus, we have

$$\frac{\log(19.45 \cdot 1314312833617044573/0.113535)}{\log\varphi} \sim 53.17 \leq n - 1$$

and so, we get $n < 55$. Considering this bound on n, we obtain $t < 31$ from the inequality (13). Thus, in Mathematica, the solutions of the equation (11) as follows:

$$\{(1,1,1), (2,1,2), (3,1,5)\}.$$

This completes the proof.

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