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Dynamic Hardy-Littlewood Type Inequalities via Time Scales Involving Nabla Integrals

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Abstract. Hardy-Littlewood inequalities via nabla time scales calculus are studied in this paper. The results are proved by using time scales analogues of chain rule, Hölder's inequality, some algebraic inequalities, and integration by parts. Particular cases include some discrete novel Hardy-Littlewood inequalities by fixing time scales. Moreover, special cases also include some existing integral and discrete Hardy-Littlewood inequalities in literature.

1. Introduction

An English mathematician Hardy proposed classical discrete double series inequality known as the Hardy inequality. Hardy inequality is significant in terms of usability, scope, and range of applications.

In [9], the following discrete inequality is proved by Hardy in 1920

$$\sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{l=1}^{m}F_l\right)^{\varrho} \leq \left(\frac{\varrho}{\varrho-1}\right)^{\varrho}\sum_{m=1}^{\infty}F_m^{\varrho}, \quad \varrho > 1, \tag{1}$$

where F_m is a series of positive terms for $m \geqslant 1$.

Continuous version of (1) is established by Hardy [6] in 1925, by using calculus of variations. He proved that if $F \geq 0$ is integrable over $(0, y)$ and F^{ϱ} is integrable and convergent function over $(0, \infty)$ for $\varrho > 1$, then

$$\int_0^{\infty}\left(\frac{1}{y}\int_0^{y}F(\hbar)d\hbar\right)^{\varrho}dy \leq \left(\frac{\varrho}{\varrho-1}\right)^{\varrho}\int_0^{\infty}F^{\varrho}(y)dy. \tag{2}$$

The constant $\left(\frac{\varrho}{\varrho-1}\right)^{\varrho}$ in (1) and (2) is sharp.

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In [10], Hardy and Littlewood demonstrated that if $\varrho > 1$ and F_m is a series of positive terms, then

$$\sum_{m=1}^{\infty} \frac{1}{m^c} \left(\sum_{l=m}^{\infty} F_l \right)^{\varrho} \leq M \sum_{m=1}^{\infty} \frac{1}{m^{c-\varrho}} F_m^{\varrho}, \quad c < 1, \tag{3}$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^c} \left(\sum_{l=1}^{m} F_l \right)^{\varrho} \leq M \sum_{m=1}^{\infty} \frac{1}{m^{c-\varrho}} F_m^{\varrho}, \quad c > 1, \tag{4}$$

where M is a positive constant (see[21]).

In [8], the continous versions of (3) and (4) were created by Hardy in 1928. He demonstrated that for any integrable function $F(y) > 0$ on $(0, \infty)$, $\varrho > 1$,

$$\int_0^{\infty} \frac{1}{y^n} \left(\int_y^{\infty} F(\hbar) d\hbar \right)^{\varrho} dy \leq \left(\frac{\varrho}{1-n} \right)^{\varrho} \int_0^{\infty} \frac{1}{y^{n-\varrho}} F^{\varrho}(y) dy, \quad n < 1, \tag{5}$$

and

$$\int_0^{\infty} \frac{1}{y^n} \left(\int_0^{y} F(\hbar) d\hbar \right)^{\varrho} dy \leq \left(\frac{\varrho}{n-1} \right)^{\varrho} \int_0^{\infty} \frac{1}{y^{n-\varrho}} F^{\varrho}(y) dy, \quad n > 1. \tag{6}$$

There are number of dynamic Hardy-type inequalities on time scales that have been discovered by several researchers who were inspired by certain useful applications. In recent decades, S. H. Saker *et al*. proved dynamic inequalities of the Hardy and Littlewood types on time scales via delta calculus. They also proved some dynamic inequalities on time scales via delta calculus which contain several integral and discrete inequalities owing to Hardy, Littlewood, Copson, Chow, Levinson, Pachpatte Yang, and Hwang [19, 20]. In [12], El-Deeb proposed some dynamic inequalities of Hardy-Hilbert type on time scales by using delta calculus with the help of the Fenchel-Legendre transform and submultiplicative functions. El-Hamid *et al*.[5] proposed some dynamic Hilbert-type inequalities for two variables on time scales via delta calculus. In [11], Ivan Gaj and Vasil Gochev investigated the behavior of the smallest possible constant in the Hardy inequality for finite sequences.

The objectives of the study are to present some Hardy-Littlewood type inequalities by using nabla calculus. We recover few integral and discrete inequalities that have been documented in the literature as particular cases of the main findings. The following is the order of this article: Section 2 introduces some fundamentals from nabla times scales calculus. Key findings are given in Section 3. The conclusion is presented in Section 4.

2. Preliminaries

Every non-empty closed subset of real numbers $\mathbb{R}$ is referred to a time scale denoted by $\mathbb{T}$. The conventional topology on the real numbers $\mathbb{R}$ is adopted by $\mathbb{T}$. To learn more about time scales calculus, (see[1–3]). Few basic concepts related to time scales theory, are as follow:

Backward jump operator $\rho : \mathbb{T} :\rightarrow \mathbb{T}$ is defined by

$$\rho(\hbar) := \sup\{\ell \in \mathbb{T} : \ell < \hbar\}, \quad \hbar \in \mathbb{T}.$$

Backward graininess function $\nu : \mathbb{T} \to [0, \infty)$, is defined as

$$\nu(\hbar) = \hbar - \rho(\hbar).$$

Definition 2.1. *[2] Let $F : \mathbb{T} \to \mathbb{R}$ be a function and let $\hbar \in \mathbb{T}_k$. we define $F^{\nabla}(\hbar)$ with the characteristic that for any $\epsilon > 0$, there exist a neighborhood N of $\hbar$ (i.e $N = (\hbar - \delta, \hbar + \delta)_{\mathbb{T}}$ for some $\delta > 0$) such that*

$$| \, [F^{\rho}(\hbar) - F(s)] - F^{\nabla}(\hbar) \, [\rho(\hbar) - s] \, | \leq \epsilon \, | \, \rho(\hbar) - s \, | \quad for \ any \ s \in N.$$

We state this $F^{\nabla}(\hbar)$ is the nabla derivative of F at $\hbar$.

Definition 2.2. *[2] A function $F : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous) provided F is continuous at left-dense points and at right-dense points of $\mathbb{T}$, right-hand limits exist and are finite. The set of all such ld-continuous functions is denoted by $C_{ld}(\mathbb{T})$.*

(**Integration by parts**).[1] Let $d, \varrho \in \mathbb{T}$, where $\varrho > d$. If Ω, Ψ are nabla differentiable, then

$$\int_d^{\varrho} \Omega(\hbar) \left[\Psi^{\nabla}(\hbar)\right] \nabla\hbar = [\Omega(\hbar)\Psi(\hbar)]_d^{\varrho} - \int_d^{\varrho} \left[\Omega^{\nabla}(\hbar)\right] \Psi^{\rho}(\hbar) \nabla\hbar. \tag{7}$$

(**Chain rule**).[1] Suppose $\Psi : \mathbb{T} \to \mathbb{R}$ is continuous and nabla differentiable at $\hbar \in \mathbb{T}_k$, and $\omega : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then

$$(\omega \circ \Psi)^{\nabla}(\hbar) = \omega'(\Psi(c))(\Psi^{\nabla}(\hbar)), c \in [\rho(\hbar), \hbar]. \tag{8}$$

(**Hölder inequality**).[3] Let $d, \varrho \in \mathbb{T}$, where $\varrho > d$. If $\omega, \Psi : \mathbb{T} \longrightarrow \mathbb{R}$, then

$$\int_d^{\varrho} |\omega(\hbar)\Psi(\hbar)| \nabla\hbar \leq \left(\int_d^{\varrho} |\omega(\hbar)|^r \nabla\hbar\right)^{\frac{1}{r}} \left(\int_d^{\varrho} |\Psi(\hbar)|^s \nabla\hbar\right)^{\frac{1}{s}}. \tag{9}$$

(**Keller chain rule**).[2] Assume $\Psi \in C_{ld}^{\nabla}(\mathbb{T})$ and let $F : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. Then, $(F \circ \Psi) : \mathbb{T} \to \mathbb{R}$ is nabla differentiable and

$$(F \circ \Psi)^{\nabla}(\hbar) = \left\{\int_0^1 F'\left(\Psi(\rho(\hbar)) - hv(\hbar)\Psi^{\nabla}(\hbar)\right) dh\right\} \Psi^{\nabla}(\hbar). \tag{10}$$

The following inequalities are also used to prove the main results:

$$2^{w-1}\left(\varkappa^{w} + \kappa^{w}\right) \leq (\varkappa + \kappa)^{w} \leq \left(\varkappa^{w} + \kappa^{w}\right), \text{ where } \varkappa, \kappa \geq 0 \text{ and } 0 \leq w \leq 1, \tag{11}$$

and

$$\varkappa^{\varpi} + \kappa^{\varpi} \leq (\varkappa + \kappa)^{\varpi} \leq 2^{\varpi-1}\left(\varkappa^{\varpi} + \kappa^{\varpi}\right), \text{ if } \varkappa, \kappa \geq 0, \quad \varpi \geq 1.$$

3. Main Results

Throughout the results, we assume that the integrals under investigation endure and the functions are nonnegative, ld-continuous, *nabla* differenciable, and locally *nabla* integrable over $[0,\infty)_{\mathbb{T}} := [0,\infty) \cap \mathbb{T}$.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $\varrho \in (0,\infty)_{\mathbb{T}}$ and $r, s > 0$ such that $\frac{r}{s} > 1$ and $\varsigma < 1$. Assume, $\kappa \geq 0$ and the integral $\int_{\varrho}^{\infty} (\rho(\hbar))^{\frac{r}{s}-\varsigma} \kappa^{\frac{r}{s}}(\hbar)\nabla\hbar$ exists. Suppose*

$$\Xi(\hbar) := \int_{\hbar}^{\infty} \kappa(\ell)\nabla\ell, \quad \textit{for any} \quad \hbar \in [\varrho,\infty)_{\mathbb{T}}. \tag{12}$$

Then

$$\int_{\varrho}^{\infty} \frac{(\Xi(\hbar))^{\frac{r}{s}}}{\rho^{\varsigma}(\hbar)} \nabla\hbar \leq \left(\frac{r}{s(1-\varsigma)}\right)^{\frac{r}{s}} \int_{\varrho}^{\infty} \frac{\kappa^{\frac{r}{s}}(\hbar)}{(\rho(\hbar))^{\varsigma-\frac{r}{s}}} \nabla\hbar. \tag{13}$$

Proof. Applying integration by parts formula (7) on left side of (13) with $v^{\nabla}(\hbar) = \frac{1}{\rho^{\varsigma}(\hbar)}$, and $u(\hbar) = (\Xi(\hbar))^{\frac{r}{s}}$, we obtain

$$\int_{\varrho}^{\infty} \frac{(\Xi(\hbar))^{\frac{r}{s}}}{\rho^{\varsigma}(\hbar)} \nabla\hbar = v(\hbar)\Xi^{\frac{r}{s}}(\hbar)\Big|_{\varrho}^{\infty} + \int_{\varrho}^{\infty} (v^{\rho}(\hbar)) \left(-\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} \nabla\hbar, \tag{14}$$

where $v(\hbar) = \int_{\varrho}^{\hbar} \left(\frac{1}{\rho^{\varsigma}(\ell)}\right)\nabla\ell$. Using the chain rule (8) and the reality that $\rho(\ell) \leq \ell$, we have

$$\begin{aligned}
\left(\ell^{1-\varsigma}\right)^{\nabla} &= (1-\varsigma)\int_0^1 [h\rho(\ell) + (1-h)\ell]^{-\varsigma} dh \\
&= (1-\varsigma)\int_0^1 \frac{dh}{[h\rho(\ell) + (1-h)\ell]^{\varsigma}} \\
&\geq (1-\varsigma)\int_0^1 \frac{dh}{[h\rho(\ell) + (1-h)\rho(\ell)]\varsigma} = \frac{(1-\varsigma)}{\rho^{\varsigma}(\ell)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
v^{\rho}(\hbar) &= \int_{\varrho}^{\rho(\hbar)} \frac{1}{\rho^{\varsigma}(\ell)} \nabla\ell \leq \frac{1}{1-\varsigma} \int_{\varrho}^{\rho(\hbar)} \left(\frac{1}{\ell^{\varsigma-1}}\right)^{\nabla} \nabla\hbar \\
&= [\frac{1}{1-\varsigma}\frac{1}{\rho(\hbar)^{\varsigma-1}} - \frac{1}{1-\varsigma}\frac{1}{\varrho^{\varsigma-1}}] \leq \frac{1}{1-\varsigma}(\rho(\hbar))^{1-\varsigma}.
\end{aligned} \tag{15}$$

Combining (14), (15), we get that

$$\int_{\varrho}^{\infty} \frac{(\Xi(\hbar))^{\frac{r}{s}}}{\rho^{\varsigma}(\hbar)} \nabla\hbar \leq \frac{1}{(1-\varsigma)} \int_{\varrho}^{\infty} \frac{1}{(\rho(\hbar))^{\varsigma-1}} \left(-\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla}\right) \nabla\hbar. \tag{16}$$

Using the chain rule $\kappa^{\nabla}(\kappa(\hbar)) = \kappa'(\kappa(c))\kappa^{\nabla}(\hbar)$, where $c \in [\rho(\hbar),\hbar]$, We can observe that there is $c \in [\rho(\hbar),\hbar]$ such that

$$-\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} = -\left(\frac{r}{s}\right)\Xi^{\frac{r}{s}-1}(c)\left(\Xi^{\nabla}(\hbar)\right).$$

Since $\Xi^{\nabla}(\hbar) = -\kappa(\hbar) \leq 0$ *and* $c \leq \hbar$, *we have*

$$-\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} \leq \left(\frac{r}{s}\right)(\Xi(\hbar))^{\frac{r}{s}-1}\kappa(\hbar). \tag{17}$$

Substituting (17) into (16), we have

$$\int_{\varrho}^{\infty} \frac{(\Xi(\hbar))^{\frac{r}{s}}}{\rho^{\varsigma}(\hbar)}\nabla\hbar \leq \frac{r}{s(1-\varsigma)}\int_{\varrho}^{\infty} \frac{(\Xi(\hbar))^{\frac{r}{s}-1}}{(\rho(\hbar))^{\varsigma-1}}\kappa(\hbar)\nabla\hbar.$$

This implies

$$\int_{\varrho}^{\infty} \frac{(\Xi(\hbar))^{\frac{r}{s}}}{\rho^{\varsigma}(\hbar)}\nabla\hbar \leq \frac{r}{s(1-\varsigma)}\int_{\varrho}^{\infty} \frac{(\rho^{\varsigma}(\hbar))^{\frac{r-s}{r}}\kappa(\hbar)}{(\rho(\hbar))^{\varsigma-1}}\frac{(\Xi(\hbar))^{\frac{r}{s}-1}}{(\rho^{\varsigma}(\hbar))^{\frac{(r-s)}{r}}}\nabla\hbar. \tag{18}$$

Applying the Hölder inequality (17) on the term

$$\int_{\varrho}^{\infty}\left[\frac{(\rho^{\varsigma}(\hbar))^{\frac{r-s}{r}}}{(\rho(\hbar))^{\varsigma-1}}\kappa(\hbar)\right]\left[(\rho^{\varsigma}(\hbar))^{-\frac{(r-s)}{r}}(\Xi(\hbar))^{\frac{r}{s}-1}\right]\nabla\hbar,$$

with indices $\frac{r}{s}$ *and* $\frac{r}{r-s}$, *we obtain*

$$\int_{\varrho}^{\infty}\left[\frac{(\rho^{\varsigma}(\hbar))^{\frac{r-s}{r}}}{(\rho(\hbar))^{\varsigma-1}}\kappa(\hbar)\right](\rho^{\varsigma}(\hbar))^{-\frac{(r-s)}{r}}(\Xi(\hbar))^{\frac{r-s}{s}}\nabla\hbar$$
$$\leq \left[\int_{\varrho}^{\infty}\left[\frac{(\rho^{\varsigma}(\hbar))^{\frac{r-s}{r}}}{(\rho(\hbar))^{\varsigma-1}}\kappa(\hbar)\right]^{\frac{r}{s}}\nabla\hbar\right]^{\frac{s}{r}}\left[\int_{\varrho}^{\infty}\frac{(\Xi(\hbar))^{\frac{r}{s}}}{\rho^{\varsigma}(\hbar)}\nabla\hbar\right]^{\frac{r-s}{r}}.$$

Substituting above inequality into (18), we have

$$\int_{\varrho}^{\infty} \frac{(\Xi(\hbar))^{\frac{r}{s}}}{(\rho(\hbar))^{\varsigma}}\nabla\hbar \leq \left(\frac{r}{s(1-\varsigma)}\right)^{\frac{r}{s}}\int_{\varrho}^{\infty}(\rho(\hbar))^{\frac{r}{s}-\varsigma}\kappa^{\frac{r}{s}}(\hbar)\nabla\hbar,$$

which is the desired (13).

Remark 3.2. *When* $\mathbb{T} = \mathbb{R}$ *and* $\frac{r}{s} = \varpi > 1$ *and* $\varsigma < 1$, *we have the following Hardy type inequality which is given in [6].*

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\int_{\hbar}^{\infty}\kappa(\ell)d\ell\right)^{\varpi}d\hbar \leq \left(\frac{\varpi}{1-\varsigma}\right)^{\varpi}\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma-\varpi}}\kappa^{\varpi}(\hbar)d\hbar. \tag{19}$$

Let $\Xi(\hbar) = \int_{\hbar}^{\infty}\kappa(\ell)d\ell$. *Thus, (note that* $\Xi(\infty) = 0$), *we have*

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}(\Xi(\hbar))^{\varpi}d\hbar \leq \left(\frac{\varpi}{1-\varsigma}\right)^{\varpi}\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma-\varpi}}\left(\Xi'(\hbar)\right)^{\varpi}d\hbar.$$

(19) may be thought as an extension of Wirtinger's inequality, (see[17]).

Remark 3.3. *When* $\mathbb{T} = \shortparallel^{\mathbb{N}_{\kappa}} = \{\hbar : \hbar = q^k, k \in \mathbb{N}_{\kappa}, q > 1\}$, $\frac{r}{s} = \varpi > 1, \varrho = 1$ *and* $\varsigma < 1$, *we have*

$$\sum_{k=1}^{\infty} \frac{1}{(q^{k-1})^{\varsigma}} \left(\sum_{m=k+1}^{\infty} q^m (1 - \frac{1}{q}) \kappa(q^m) \right)^{\varpi} \leq \left(\frac{\varpi}{1-\varsigma} \right)^{\varpi} \sum_{k=1}^{\infty} q^k (1 - \frac{1}{q}) (q^{k-1})^{\varpi - \varsigma} \kappa^{\varpi}(q^k).$$

Remark 3.4. *When* $\mathbb{T} = \approx\mathbb{Z}$ *and* $\frac{r}{s} = \varpi > 1$ *and* $\varsigma < 1$, *we have*

$$\sum_{n=\frac{\varrho}{h}+1}^{\infty} \frac{1}{(n-h)^{\varsigma}} \left(\sum_{m=\frac{n}{h}+1}^{\infty} h\kappa(hm) \right)^{\varpi} \leq \left(\frac{\varpi}{1-\varsigma} \right)^{\varpi} \sum_{n=\frac{\varrho}{h}+1}^{\infty} \frac{1}{(n-h)^{\varsigma - \varpi}} h\kappa^{\varpi}(hn).$$

We will suppose there is a constant $K > 0$ for the rest of the results, with

$$\frac{\ell}{\rho(\ell)} \geq \frac{1}{K}, \quad \text{for } \ell \geq \varrho, \quad \text{where} \quad \varrho \in (0, \infty)_{\mathbb{T}}.$$

Theorem 3.5. *Let* $\mathbb{T}$ *be a time scale with* $\varrho \in (0, \infty)_{\mathbb{T}}$ *and* $r, s > 0$ *such that* $\frac{r}{s} > 1$ *and* $\varsigma > 1$. *Assume,* $\kappa \geq 0$ *and the integral* $\int_{\varrho}^{\infty} \hbar^{\frac{r}{s} - \varsigma} \kappa^{\frac{r}{s}}(\hbar) \nabla \hbar$ *exists. Define*

$$\Xi(\hbar) := \int_{\varrho}^{\hbar} \kappa(\ell) \nabla \ell, \quad \textit{for any} \quad \hbar \in [\varrho, \infty)_{\mathbb{T}}. \tag{20}$$

Then

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar) \right)^{\frac{r}{s}} \nabla \hbar \leq \left(\frac{rK^{\varsigma}}{s(\varsigma - 1)} \right)^{\frac{r}{s}} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma - \frac{r}{s}}} \kappa^{\frac{r}{s}}(\hbar) \nabla \hbar. \tag{21}$$

Proof. Applying integration by parts formula (7) on left side of (21) with $u^{\nabla}(\hbar) = \dfrac{1}{\hbar^{\varsigma}}$ *and* $v^{\rho}(\hbar) = (\Xi^{\rho}(\hbar))^{\frac{r}{s}}$, *we obtain*

$$\int_{\varrho}^{\infty} \frac{(\Xi^{\rho}(\hbar))^{\frac{r}{s}}}{\hbar^{\varsigma}} \nabla \hbar = \left[u(\hbar) \Xi^{\frac{r}{s}}(\hbar) \right]_{\varrho}^{\infty} + \int_{\varrho}^{\infty} (-u(\hbar)) \left(\Xi^{\frac{r}{s}}(\hbar) \right)^{\nabla} \nabla \hbar, \tag{22}$$

where

$$u(\hbar) = \int_{\hbar}^{\infty} \left(\frac{-1}{\ell^{\varsigma}} \right) \nabla \ell. \tag{23}$$

Using (8), we have that

$$\begin{aligned} \left(\frac{-1}{\ell^{\varsigma-1}} \right)^{\nabla} &= (\varsigma - 1) \int_0^1 \frac{1}{[h\rho(\ell) + (1-h)\ell]^{\varsigma}} dh \\ &\geq (\varsigma - 1) \int_0^1 \frac{1}{[h\rho(\ell) + (1-h)\rho(\ell)]^{\varsigma}} dh \\ &= \int_0^1 \left(\frac{\varsigma - 1}{\rho^{\varsigma}(\ell)} \right) dh = \frac{(\varsigma - 1)}{\rho^{\varsigma}(\ell)}. \end{aligned} \tag{24}$$

From (23) and (24), we see that

$$\left(\frac{-1}{\ell^{\varsigma-1}}\right)^{\nabla} \geq \frac{(\varsigma-1)}{K^{\varsigma}\ell^{\varsigma}}.$$

Then

$$\int_{\hbar}^{\infty} \frac{-1}{\ell^{\varsigma}} \nabla \ell \geq \frac{-K^{\varsigma}}{(\varsigma-1)} \int_{\hbar}^{\infty} \left(\frac{-1}{\ell^{\varsigma-1}}\right)^{\nabla} \nabla \ell = \frac{-K^{\varsigma}}{(\varsigma-1)} \left(\frac{1}{\hbar^{\varsigma-1}}\right). \tag{25}$$

Hence

$$-u(\hbar) = -\int_{\hbar}^{\infty} \left(\frac{-1}{\ell^{\varsigma}}\right) \nabla \ell \leq \frac{K^{\varsigma}}{\varsigma-1} \left(\frac{1}{\hbar^{\varsigma-1}}\right). \tag{26}$$

From (20), (22), (23) and (26), we have (note that $u(\infty) = 0$ *and* $\Xi(\varrho) = 0$*) that*

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar \leq \frac{K^{\varsigma}}{(\varsigma-1)} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-1}} \left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} \nabla \hbar. \tag{27}$$

Applying (8) on the term $(\Xi^{\frac{r}{s}}(\hbar))^{\nabla}$*, we obtain*

$$\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} = \left(\frac{r}{s}\right) \Xi^{\frac{r}{s}-1}(c) \Xi^{\nabla}(\hbar).$$

Since $\Xi^{\nabla}(\hbar) \geq 0$ *and* $\rho(\hbar) \leq c$*, we have*

$$\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} \leq \left(\frac{r}{s}\right) \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}-1} \kappa(\hbar). \tag{28}$$

Substituting (28) into (27), we have

$$\int_{\varrho}^{\infty} \frac{\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}}}{\hbar^{\varsigma}} \nabla \hbar \leq \frac{rK^{\varsigma}}{s(\varsigma-1)} \int_{\varrho}^{\infty} \frac{\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}-1}}{\hbar^{\varsigma-1}} \kappa(\hbar) \nabla \hbar.$$

It implies that

$$\int_{\varrho}^{\infty} \frac{\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}}}{\hbar^{\varsigma}} \nabla \hbar \leq \frac{rK^{\varsigma}}{s(\varsigma-1)} \int_{\varrho}^{\infty} \left(\left(\hbar^{\varsigma}\right)^{-\frac{(r-s)}{r}} \hbar^{\varsigma-1} \kappa(\hbar)\right) \left[\left(\hbar^{\varsigma}\right)^{-\frac{(r-s)}{r}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}-1}\right] \nabla \hbar. \tag{29}$$

Applying the Hölder inequality (9) on the term

$$\int_{\varrho}^{\infty} \left[\left(\hbar^{\varsigma}\right)^{\frac{(r-s)}{r}} \hbar^{\varsigma-1} \kappa(\hbar)\right] \left(\hbar^{\varsigma}\right)^{-\frac{(r-s)}{r}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r-s}{s}} \nabla \hbar,$$

with indices $\frac{r}{s}$ *and* $\frac{r}{r-s}$*, we see that*

$$\int_{\varrho}^{\infty}\left[\frac{(\hbar^{\varsigma})^{\frac{r-s}{r}}}{\hbar^{\varsigma-1}}\kappa(\hbar)\right](\hbar^{\varsigma})^{-\frac{(r-s)}{r}}\left(\Xi^{\rho}(\hbar)\right)^{\frac{r-s}{s}}\nabla\hbar$$
$$\leq\left[\int_{\varrho}^{\infty}\left[\frac{(\hbar^{\varsigma})^{\frac{r-s}{r}}}{\hbar^{\varsigma-1}}\kappa(\hbar)\right]^{\frac{r}{s}}\nabla\hbar\right]^{\frac{s}{r}}\left[\int_{\varrho}^{\infty}\frac{(\Xi^{\rho}(\hbar))^{\frac{r}{s}}}{\hbar^{\varsigma}}\nabla\hbar\right]^{\frac{r-s}{r}}. \quad (30)$$

Substituting (30) into (29), we have

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}}\nabla\hbar\leq\frac{rK^{\varsigma}}{s(\varsigma-1)}\left[\int_{\varrho}^{\infty}\left[\frac{(\hbar^{\varsigma})^{\frac{r-s}{r}}}{\hbar^{\varsigma-1}}\kappa(\hbar)\right]^{\frac{r}{s}}\nabla\hbar\right]^{\frac{s}{r}}$$
$$\times\left[\int_{\varrho}^{\infty}\frac{(\Xi^{\rho}(\hbar))^{\frac{r}{s}}}{\hbar^{\varsigma}}\nabla\hbar\right]^{\frac{r-s}{r}}. \quad (31)$$

This gives that

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}}\nabla\hbar\leq\left(\frac{rK^{\varsigma}}{s(\varsigma-1)}\right)^{\frac{r}{s}}\int_{\varrho}^{\infty}\hbar^{\frac{r}{s}-\varsigma}\kappa^{\frac{r}{s}}(\hbar)\nabla\hbar,$$

which is the desired (21).

Remark 3.6. *When $\mathbb{T}=\mathbb{R}$ and $\frac{r}{s}=\varpi>1$ and $\varsigma>1$, we have the following Hardy type inequality which is given in [6].*

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\int_{\varrho}^{\hbar}\kappa(\ell)d\ell\right)^{\varpi}d\hbar\leq\left(\frac{\varpi}{\varsigma-1}\right)^{\varpi}\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma-\varpi}}\kappa^{\varpi}(\hbar)d\hbar. \quad (32)$$

Let $\Xi(\hbar)=\int_{\varrho}^{\hbar}\kappa(\ell)d\ell$. Thus, (note that $\Xi(\varrho)=0$), we have

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\Xi(\hbar)\right)^{\varpi}d\hbar\leq\left(\frac{\varpi}{\varsigma-1}\right)^{\varpi}\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma-\varpi}}\left(\Xi'(\hbar)\right)^{\varpi}d\hbar.$$

The inequality (32) may be thought as an extension of Wirtinger's inequality, (see[17]).

When $\varsigma=\varpi>1$ we have traditional Hardy type inequality which is given in [6].

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\int_{\varrho}^{\hbar}\kappa(\ell)d\ell\right)^{\varpi}d\hbar\leq\left(\frac{\varpi}{\varpi-1}\right)^{\varpi}\int_{\varrho}^{\infty}\kappa^{\varpi}(\hbar)d\hbar.$$

Remark 3.7. *When $\mathbb{T}=\shortparallel^{\mathbb{N}_{\nvdash}}=\{\hbar:\hbar=q^{k},k\in\mathbb{N}_{\nvdash},q>1\}$, $\frac{r}{s}=\varpi>1$, $\varrho=1$ and $\varsigma>1$, we have*

$$\sum_{k=1}^{\infty}\frac{1}{(q^{k})^{\varsigma}}\left(\sum_{m=2}^{k-1}q^{m}(1-\frac{1}{q})\kappa(q^{m})\right)^{\varpi}\leq\left(\frac{2^{\varpi}\varpi}{\varsigma-1}\right)^{\varpi}\sum_{k=1}^{\infty}\frac{(1-\frac{1}{q})}{(q^{k})^{\varsigma-\varpi-1}}\kappa^{\varpi}(q^{k}).$$

Remark 3.8. *When $\mathbb{T}=\eqsim\mathbb{Z}$ and $\frac{r}{s}=\varpi>1$ and $\varsigma>1$, we have*

$$\sum_{n=\frac{\varrho}{h}+1}^{\infty}\frac{1}{n^{\varsigma}}\left(\sum_{m=\frac{\varrho}{h}+1}^{n-h}h\kappa(hm)\right)^{\varpi}\leq\left(\frac{2^{\varpi}\varpi}{\varsigma-1}\right)^{\varpi}\sum_{n=\frac{\varrho}{h}+1}^{\infty}\frac{1}{n^{\varsigma-\varpi}}h\kappa^{\varpi}(hn).$$

Theorem 3.9. *Let $\mathbb{T}$ be a time scale with $\varrho \in (0,\infty)_{\mathbb{T}}$ and $r, s > 0$ such that $\frac{r}{s} \geq 2$ and $\varsigma > 1$. Assume, $\kappa \geq 0$ and the integral $\int_{\varrho}^{\infty} \hbar^{\frac{r}{s}-\varsigma}\kappa^{\frac{r}{s}}(\hbar)\nabla\hbar$ exists. $\Xi(\hbar)$ be defined in (20). Then*

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla\hbar \leq \frac{2^{\frac{r}{s}-2} r K^{\varsigma}}{s(\varsigma-1)} \left[\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-\frac{r}{s}}} \kappa^{\frac{r}{s}}(\hbar)\nabla\hbar\right]^{\frac{s}{r}}$$
$$\times\left[\int_{\varrho}^{\infty} \frac{\left(\Xi^{\rho}(\hbar)\right)\Xi^{\frac{r}{s}}}{\hbar^{\varsigma}}\nabla\hbar\right]^{\frac{r-s}{r}} + \frac{2^{\frac{r}{s}-2}K^{\varsigma}}{(\varsigma-1)} \int_{\varrho}^{\infty} \frac{\nu^{\frac{r}{s}-1}(\hbar)}{\hbar^{\varsigma-1}} (\kappa(\hbar))^{\frac{r}{s}}\nabla\hbar.$$

Proof. From inequality (27) of Theorem 3.5, we have

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla\hbar \leq \frac{K^{\varsigma}}{(\varsigma-1)} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-1}} \left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} \nabla\hbar. \tag{33}$$

Utilizing (10) on the term $\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla}$, we have

$$\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} = \frac{r}{s}\int_{0}^{1} \left(\Xi(\hbar) - h\nu(\hbar)\Xi^{\nabla}(\hbar)\right) dh \Xi^{\nabla}(\hbar),$$

where $\Xi^{\nabla}(\hbar) = \kappa(\hbar) \geq 0$, we have

$$\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} = \left(\frac{r}{s}\right)\kappa(\hbar)\int_{0}^{1} [\Xi(\hbar) + \nu h\kappa]^{\frac{r}{s}-1} dh.$$

Utilizing (11) on the term $[\Xi + h\nu\kappa]^{(\frac{r}{s})-1}$, and $\frac{r}{s} \geq 2$, we see

$$\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} \leq \left(\frac{r}{s}\right) 2^{\frac{r}{s}-2}\kappa(\hbar)\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}-1} + 2^{\frac{r}{s}-2}\kappa(\hbar)(\nu\kappa)^{\frac{r}{s}-1}. \tag{34}$$

Substituting (34) into (33), we have

$$\int_{\varrho}^{\infty} \frac{\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}}}{\hbar^{\varsigma}}\nabla\hbar \leq \frac{r\left(2^{\frac{r}{s}-2}\right)K^{\varsigma}}{s(\varsigma-1)} \int_{\varrho}^{\infty} \frac{\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}-1}}{\hbar^{\varsigma-1}} \kappa(\hbar)\nabla\hbar \tag{35}$$
$$+\frac{2^{\frac{r}{s}-2}K^{\varsigma}}{\varsigma-1} \int_{\varrho}^{\infty} \frac{(\nu(\hbar))^{\frac{r}{s}-1}(\kappa(\hbar))^{\frac{r}{s}}}{\hbar^{\varsigma-1}}\nabla\hbar. \tag{36}$$

This implies that

$$\begin{aligned} &\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla\hbar \\ \leq\ & \frac{2^{\frac{r}{s}-2} r K^{\varsigma}}{s(\varsigma-1)} \int_{\varrho}^{\infty} \left[\frac{(\hbar^{\varsigma})^{\frac{r-s}{r}}}{\hbar^{\varsigma-1}}\kappa(\hbar)\right]\left[(\hbar^{\varsigma})^{\frac{-(r-s)}{r}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r-s}{s}}\right]\nabla\hbar \\ +\ & \frac{2^{\frac{r}{s}-2}K^{\varsigma}}{\varsigma-1} \int_{\varrho}^{\infty} \frac{\nu^{\frac{r}{s}-1}(\hbar)}{\hbar^{\varsigma-1}} (f(\hbar))^{\frac{r}{s}}\nabla\hbar. \end{aligned} \tag{37}$$

Uses the Hölder inequality (9) on the term

$$\int_{\varrho}^{\infty}\left[\frac{(\hbar^{\varsigma})^{\frac{r-s}{r}}}{\hbar^{\varsigma-1}}f(\hbar)\right]\left[(\hbar^{\varsigma})^{\frac{-(r-s)}{r}}\left(\Xi^{\rho}(\hbar)\right)^{\frac{r-s}{s}}\right]\nabla\hbar,$$

with indices $\frac{r}{s}$ and $\frac{r}{(r-s)}$, to obtain

$$\begin{aligned}\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}}\nabla\hbar \leq{}& \frac{2^{\frac{r}{s}-2}rK^{\varsigma}}{s(\varsigma-1)}\left[\int_{\varrho}^{\infty}\left[\frac{(\hbar^{\varsigma})^{\frac{r-s}{r}}}{\hbar^{\varsigma-1}}\kappa(\hbar)\right]^{\frac{r}{s}}\nabla\hbar\right]^{\frac{s}{r}}\\ &\times\left[\int_{\varrho}^{\infty}\frac{\left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}}}{\hbar^{\varsigma}}\nabla\hbar\right]^{\frac{r-s}{r}}\\ &+\frac{2^{\frac{r}{s}-2}K^{\varsigma}}{\varsigma-1}\int_{\varrho}^{\infty}\frac{\nu^{\frac{r}{s}-1}(\hbar)}{\hbar^{\varsigma-1}}(\kappa(\hbar))^{\frac{r}{s}}\nabla\hbar,\end{aligned} \tag{38}$$

which is the desired (33).

Remark 3.10. *When $\mathbb{T}=\mathbb{R}$, we have the following Hardy-type inequality which is given in [6]. If $\varpi=\frac{r}{s}\geq 2$ and $\varsigma>1$, then*

$$\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma}}\left(\int_{\varrho}^{\hbar}\kappa(\ell)d\ell\right)^{\varpi}d\hbar \leq \left(\frac{2^{\varpi-2}\varpi}{\varsigma-1}\right)^{\varpi}\int_{\varrho}^{\infty}\frac{1}{\hbar^{\varsigma-\varpi}}\kappa^{\varpi}(\hbar)d\hbar.$$

Remark 3.11. *When $\mathbb{T}={}_{\amalg}\mathbb{N}_{\digamma}=\{\hbar:\hbar=q^{k},k\in\mathbb{N}_{\digamma},q>1\}$, $\frac{r}{s}=\varpi>1$, $\varrho=1$ and $\varsigma>1$, we have*

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(q^{k})^{\varsigma}}\left(\sum_{m=2}^{k-1}q^{m}(1-\frac{1}{q})\kappa(q^{m})\right)^{\varpi} \leq{}& \left(\frac{2^{\varpi}-2}{\varsigma-1}\right)^{\varpi}\sum_{k=1}^{\infty}\frac{(1-\frac{1}{q})}{(q^{k})^{\varsigma-\varpi-1}}\kappa^{\varpi}(q^{k})\\ &\times\left[\sum_{k=1}^{\infty}\frac{1}{(q^{k})^{\varsigma}}\left(\sum_{m=2}^{k-1}q^{m}(1-\frac{1}{q})\kappa(q^{m})\right)^{\varpi}\right]^{\frac{r-s}{r}}\\ &+\left(\frac{2^{\varpi}-2}{\varsigma-1}\right)^{\varpi}\sum_{k=1}^{\infty}\frac{(1-\frac{1}{q}))(q^{k}-q^{k-1})^{\frac{r}{s}-1}}{(q^{k})^{\varsigma-\varpi-1}}\kappa^{\varpi}(q^{k}).\end{aligned}$$

Remark 3.12. *When $\mathbb{T}=\approx\mathbb{Z}$, $\frac{r}{s}=\varpi>1$, and $\varsigma>1$, we have*

$$\begin{aligned}\sum_{n=\frac{\varrho}{h}+1}^{\infty}\frac{1}{(n)^{\varsigma}}\left(\sum_{m=\frac{\varrho}{h}+1}^{n-h}h\kappa(hm)\right)^{\varpi} \leq{}& \left(\frac{2^{\varpi}-2}{\varsigma-1}\right)^{\varpi}\sum_{n=\frac{\varrho}{h}+1}^{\infty}\frac{1}{(n)^{\varsigma-\varpi-1}}h\kappa^{\varpi}(hn)\\ &\times\left[\sum_{n=\frac{\varrho}{h}+1}^{\infty}\frac{1}{(n)^{\varsigma}}\left(\sum_{m=\frac{\varrho}{h}+1}^{n-h}h\kappa(hm)\right)^{\varpi}\right]^{\frac{r-s}{r}}\\ &+\left(\frac{2^{\varpi}-2}{\varsigma-1}\right)^{\varpi}\sum_{n=\frac{\varrho}{h}+1}^{\infty}\frac{h^{\frac{r}{s}-1}}{(n)^{\varsigma-1}}h\kappa^{\varpi}(hn).\end{aligned}$$

Theorem 3.13. *Let a time scale $\mathbb{T}$ with $\varrho\in(0,\infty)_{\mathbb{T}}$ and $r,s>0$ such that $\frac{r}{s}\geq 2$ and $\varsigma>1$. Suppose, $\kappa\geq 0$ and the integral $\int_{\varrho}^{\infty}\hbar^{\frac{r}{s}-\varsigma}\kappa^{\frac{r}{s}}(\hbar)\nabla\hbar$ exists. $\Xi(\hbar)$ be defined in (20). Then*

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar \leq \left(\frac{2^{\frac{r}{s}-1} K^{\varsigma}}{(\varsigma-1)}\right)^{\frac{r}{s}} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-\frac{r}{s}}} \kappa^{\frac{r}{s}}(\hbar) \nabla \hbar. \tag{39}$$

Proof. From inequality (27) of Theorem 3.5, we have

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar \leq \frac{K^{\varsigma}}{(\varsigma-1)} \int_{\varrho}^{\infty} \frac{\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla}}{\ell^{\varsigma-1}} \nabla \hbar. \tag{40}$$

Utilizing (10) and inequality (11), we obtain

$$\begin{aligned} \left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} &\leq 2^{\frac{r}{s}-2} \frac{r}{s} \int_0^1 \left[(h\Xi^{\rho})^{\frac{r}{s}-1} + (1-h)^{\frac{r}{s}-1} \Xi^{\frac{r}{s}-1}\right] dh \Xi^{\nabla}(\hbar) \\ &= 2^{\frac{r}{s}-2} \left[(\Xi^{\rho})^{\frac{r}{s}-1} + \Xi^{\frac{r}{s}-1}\right] \Xi^{\nabla}(\hbar) \\ &\leq 2^{\frac{r}{s}-2} \left[(\Xi^{\rho})^{\frac{r}{s}-1} + (\Xi^{\rho})^{\frac{r}{s}-1}\right] \kappa(\hbar). \end{aligned} \tag{41}$$

This implies that

$$\left(\Xi^{\frac{r}{s}}(\hbar)\right)^{\nabla} \leq 2^{\frac{r}{s}-1} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}-1} \kappa(\hbar).$$

Hence

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar \leq \frac{2^{\frac{r}{s}-1} K^{\varsigma}}{(\varsigma-1)} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-1}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}-1} \kappa(\hbar) \nabla \hbar, \tag{42}$$

and thus

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar \leq \frac{2^{\frac{r}{s}-1} K^{\varsigma}}{(\varsigma-1)} \int_{\varrho}^{\infty} \left[\frac{\hbar^{\varsigma \frac{(r-s)}{r}}}{\hbar^{\varsigma-1}} \kappa(\hbar)\right] \left[(\hbar^{\varsigma})^{-\frac{(r-s)}{r}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r-s}{s}}\right] \nabla \hbar. \tag{43}$$

Utilizing the Hölder inequality (9) on the right hand side with indices $\frac{r}{s}$ and $\frac{r}{r-s}$, we obtain

$$\begin{aligned} &\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar \\ \leq\ & \frac{2^{\frac{r}{s}-1} K^{\varsigma}}{\varsigma-1} \left[\int_{\varrho}^{\infty} \left[(\hbar^{\varsigma})^{\frac{(r-s)}{r}} \hbar^{\varsigma-1} \kappa(\hbar)\right]^{\frac{r}{s}} \nabla \hbar\right]^{\frac{s}{r}} \left[\int_{\varrho}^{\infty} \frac{(\Xi^{\rho})^{\frac{r}{s}})(\hbar)}{\hbar^{\varsigma}} \nabla \hbar\right]^{\frac{r-s}{r}}. \end{aligned} \tag{44}$$

This implies that

$$\left[\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar\right]^{1-\frac{r-s}{r}} \leq \frac{2^{\frac{r}{s}-1} K^{\varsigma}}{\varsigma-1} \left[\int_{\varrho}^{\infty} \left[(\hbar^{\varsigma})^{\frac{(r-s)}{r}} \hbar^{\varsigma-1} \kappa(\hbar)\right]^{\frac{r}{s}} \nabla \hbar\right]^{\frac{s}{r}}.$$

Then

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\Xi^{\rho}(\hbar)\right)^{\frac{r}{s}} \nabla \hbar \leq \left(\frac{2^{\frac{r}{s}-1} K^{\varsigma}}{\varsigma-1}\right)^{\frac{r}{s}} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-\frac{r}{s}}} \kappa^{\frac{r}{s}}(\hbar) \nabla \hbar, \tag{45}$$

which is the desired inequality (39).

Remark 3.14. *when $\mathbb{T} = \mathbb{R}$, we have the following Hardy-type inequality which is given in [6]. If $\varpi = \frac{r}{s} \geq 2$ and $\varsigma > 1$, then*

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} \left(\int_{\varrho}^{\hbar} \kappa(\ell) d\ell \right)^{\varpi} d\hbar \leq \left(\frac{2^{\varpi-1}\varpi}{\varsigma - 1} \right)^{\varpi} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-\varpi}} \kappa^{\varpi}(\hbar) d\hbar.$$

Remark 3.15. *When $\mathbb{T} = {}_{\text{II}}\mathbb{N}_{\digamma} = \{\hbar : \hbar = q^k, k \in \mathbb{N}_{\digamma}, q > 1\}$, $\frac{r}{s} = \varpi > 1$, $\varrho = 1$ and $\varsigma > 1$, we have*

$$\sum_{k=1}^{\infty} \frac{1}{(q^k)^{\varsigma}} \left(\sum_{m=2}^{k-1} q^m (1 - \frac{1}{q}) \kappa(q^m) \right)^{\varpi} \leq \left(\frac{2^{\varpi-1}\varpi}{\varsigma - 1} \right)^{\varpi} \sum_{k=1}^{\infty} \frac{(1 - \frac{1}{q})}{(q^k)^{\varsigma-\varpi-1}} \kappa^{\varpi}(q^k).$$

Remark 3.16. *When $\mathbb{T} = \approx\mathbb{Z}$, $\frac{r}{s} = \varpi > 1$ and $\varsigma > 1$, we have*

$$\sum_{n=\frac{\varrho}{h}+1}^{\infty} \frac{1}{(n)^{\varsigma}} \left(\sum_{m=\frac{\varrho}{h}+1}^{n-h} h\kappa(hm) \right)^{\varpi} \leq \left(\frac{2^{\varpi-1}\varpi}{\varsigma - 1} \right)^{\varpi} \sum_{n=\frac{\varrho}{h}+1}^{\infty} \frac{1}{(n)^{\varsigma-\varpi}} h\kappa^{\varpi}(hn).$$

In the following section, we take into account the situation in which $\frac{r}{s} \leq 2$. We require the inequality to support these findings.

Applying this inequality (11) when $w = \frac{r}{s} - 1 < 1$, we see that

$$\frac{r}{s} \int_0^1 \left[\Lambda + h\nu\Lambda^{\nabla} \right]^{(\frac{r}{s})-1} dh \leq (\frac{r}{s}) \Lambda^{\frac{r}{s}-1} + (\nu\Lambda)^{\frac{r}{s}-1}, \quad \frac{r}{s} \leq 2.$$

From Theorem 3.9, leads to:

Theorem 3.17. *Let $\mathbb{T}$ be a time scale with $\varrho \in (0, \infty)_{\mathbb{T}}$ and $r, s > 0$ such that $\frac{r}{s} \leq 2$ and $\varsigma > 1$. Assume, $\kappa \geq 0$ and the integral $\int_{\varrho}^{\infty} \hbar^{\frac{r}{s}-\varsigma} \kappa^{\frac{r}{s}}(\hbar) \nabla\hbar$ exists. Suppose that $\kappa \geq 0$ is and let $\Xi(\hbar)$ be as defined in (20). Then*

$$\begin{aligned} &\int_{\varrho}^{\infty} \frac{(\Xi^{\rho}(\hbar))^{\frac{r}{s}}}{\hbar^{\varsigma}} \nabla\hbar - K^{\varsigma} \int_{\varrho}^{\infty} \frac{\nu^{\frac{r}{s}-1}(\hbar)}{\hbar^{\varsigma-1}} (\kappa(\hbar))^{\frac{r}{s}} \nabla\hbar \\ \leq\; &\frac{rK^{\varsigma}}{s(\varsigma - 1)} \left[\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-\frac{r}{s}}} \kappa^{\frac{r}{s}}(\hbar) \nabla\hbar \right]^{\frac{s}{r}} \left[\int_{\varrho}^{\infty} \frac{(\Xi^{\rho}(\hbar))^{\frac{r}{s}}}{\hbar^{\varsigma}} \nabla\hbar \right]^{\frac{r-s}{r}}. \end{aligned} \tag{46}$$

Theorem 3.13, leads to:

Theorem 3.18. *Let $\mathbb{T}$ be a time scale with $\varrho \in (0, \infty)_{\mathbb{T}}$ and $r, s > 0$ such that $\frac{r}{s} \leq 2$ and $\varsigma > 1$. Suppose that $\kappa \geq 0$ and let $\Xi(\hbar)$ be as defined in (20). Then*

$$\int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma}} (\Xi^{\rho}(\hbar))^{\frac{r}{s}} \nabla\hbar \leq \left(\frac{2K^{\varsigma}}{\varsigma - 1} \right)^{\frac{r}{s}} \int_{\varrho}^{\infty} \frac{1}{\hbar^{\varsigma-\frac{r}{s}}} \kappa^{\frac{r}{s}}(\hbar) \nabla\hbar.$$

Theorem 3.13, leads to:

Theorem 3.19. *Let $\mathbb{T}$ be a time scale with $\varrho \in (0,\infty)_{\mathbb{T}}$ and $\varsigma < 1$ by utilizing the function $\Xi(\hbar)$ defined in (12). Using the inequality (11) on the term $[h\Xi^{\rho} + (1-h)\Xi)]^{\frac{r}{s}-1}$, when $\frac{r}{s} - 1 \leq 1$, we obtain*

$$\begin{aligned} & \frac{r}{s}\int_0^1 [h\Xi^{\rho} + (1-h)\Xi)\Big]^{\frac{r}{s}-1} dh \\ \leq\ & \frac{r}{s}\int_0^1 \Big[h^{\frac{r}{s}-1}(\Xi^{\rho})^{\frac{r}{s}-1} + (1-h)^{\frac{r}{s}-1}\Xi^{\frac{r}{s}-1}\Big] dh \\ =\ & \Big[(\Xi^{\rho})^{\frac{r}{s}-1} + \Xi^{\frac{r}{s}-1}\Big] \leq 2\Xi^{\frac{r}{s}-1}(\hbar). \end{aligned}$$

This implies that

$$\int_{\varrho}^{\infty} \frac{1}{\rho^{\varsigma}(\hbar)}(\Xi(\hbar))^{\frac{r}{s}}\nabla\hbar \leq \frac{2}{1-\varsigma}\int_{\varrho}^{\infty}\left[\frac{(\rho^{\varsigma}(\hbar))^{\frac{r-s}{r}}}{\rho^{\varsigma-1}(\hbar)}\kappa(\hbar)\right]\left[(\rho^{\varsigma}(\hbar))^{-\frac{(r-s)}{r}}(\Xi(\hbar))^{\frac{r}{s}-1}\right]\nabla\hbar. \quad (47)$$

From Theorem 3.1, we have the following result:

Theorem 3.20. *Let $\mathbb{T}$ be a time scale with $\varrho \in (0,\infty)_{\mathbb{T}}$ and $r, s > 0$ such that $\frac{r}{s} \leq 2$ and $\varsigma < 1$. Suppose, $\kappa \geq 0$ and the integral $\int_{\varrho}^{\infty}(\rho(\hbar))^{\frac{r}{s}-\varsigma}\kappa^{\frac{r}{s}}(\hbar)\nabla\hbar$ exists. Then*

$$\int_{\varrho}^{\infty} \frac{1}{\rho^{\varsigma}(\hbar)}\left(\int_{\hbar}^{\infty}\kappa(\ell)\nabla\ell\right)^{\frac{r}{s}}\nabla\hbar \leq \left(\frac{2}{1-\varsigma}\right)^{\frac{r}{s}}\int_{\varrho}^{\infty}\frac{1}{(\rho(\hbar))^{\varsigma-\frac{r}{s}}}(\kappa(\hbar))^{\frac{r}{s}}\nabla\hbar. \quad (48)$$

4. Conclusion

A number of Hardy-Littlewood inequalities are developed via nabla calculus in the paper. The main results also hold true when the time scale interval $[\varrho,\infty)_{\mathbb{T}}$, is substituted with $[\varrho, d]_{\mathbb{T}}$. The delta analogues of the present results are obtained in [18]. The main results of the present work are also estimated for $\mathbb{T} = \{\eqsim\mathbb{Z}, h > 0\}$ and $\mathbb{T} = \shortparallel^{\mathbb{N}_{\varkappa}} = \{q^k, k \in \mathbb{N}_{\varkappa}, q > 1\}$, which are not discussed before. Moreover, Some special cases coincide with some classical inequalities by choosing $\mathbb{T} = \mathbb{R}$.

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