TURKISH JOURNAL OF SCIENCE
VOLUME 9, ISSUE 1, 80–90
ISSN: 2587–0971

https://www.tjoscience.com

On q- Integer Representation with a Special Sequence

Serpil HALICI[a], Ömür DEVECİ[b], Şule ÇÜRÜK[c]

[a]*Department of Mathematics, Faculty of Sciences and Arts University of Pamukkale, Türkiye*
[b]*Department of Mathematics, Kafkas University, Kars Türkiye*
[c]*Department of Mathematics, Faculty of Sciences and Arts University of Pamukkale, Türkiye*

Abstract. In this study, we considered a different sequence with bicomplex coefficients. We investigated some important properties of this new number sequence that we created by giving the $q-$ form and finding the exponential generating function. Moreover, we obtained the fundamental combinatorial identities related with these sequences.

1. Introduction

Bicomplex numbers form a commutative algebra, which is a generalization of complex numbers. Unlike algebra of complex numbers, bicomplex algebra is isomorphic to the direct sum of two algebras on complex numbers[3]. In this one, apart from the obvious idempotent elements, two idempotent elements, the zero divisors and the orthogonal element e_1 and e_2 can be created. With the help of these idempotent elements, every bicomplex number z can be written in a unique way. These idempotent elements $\{e_1, e_2\}$ form a base for the bicomplex numbers on the scalar field complex numbers. Most of the properties of complex numbers are still provided in bicomplex algebra[13]. Bicomplex numbers are used in many fields. Some of its uses are describing fractals, studying integer sequences, and computer graphics applications[8, 14]. In [5], the author introduced bicomplex numbers whose coefficients were selected from the sequence of Jacobsthal-Lucas numbers and studied some known identities. In [2], the author has studied bicomplex third-order Jacobsthal numbers and gave the Binet formula, generating function and some properties of this sequence. In [8], the authors have investigated new families of Fibonacci and Lucas octonions with q integer components in detail. In [4], the author has defined and also examined the Horadam bicomplex numbers. Any bicomplex number z is written by complex coefficients[12].

$$z = z_1\mathbf{1} + z_2\mathbf{j} = (z_1 - \mathbf{i}\, z_2)\, e_1 + (z_1 + \mathbf{i}\, z_2)\, e_2 \tag{1}$$

where $z_1 = b_0 + \mathbf{i}\, b_1$, $z_2 = b_2 + \mathbf{i}\, b_3$, $\mathbf{i}^2 = \mathbf{j}^2 = -1$. Bicomplex numbers whose are coefficients from Jacobsthal and Jacobsthal-Lucas numbers are defined as follows[5]:

$$\mathbb{BC}J_n = J_n + J_{n+1}\mathbf{i} + J_{n+2}\mathbf{j} + J_{n+3}\mathbf{ij}, \ \mathbb{BC}j_n = j_n + j_{n+1}\mathbf{i} + j_{n+2}\mathbf{j} + j_{n+3}\mathbf{ij}. \tag{2}$$

Corresponding author: ŞÇ mail address: sule9220@gmail.com ORCID:0000-0002-4514-6156, SH ORCID:0000-0002-8071-0437, ÖD ORCID:0000-0001-5870-5298.
Received: 5 March 2024; Accepted: 12 April 2024; Published: 30 April 2024
Keywords. Bicomplex numbers, Quantum integers, Recurrences.
2010 *Mathematics Subject Classification*. 30G35, 05A30, 11B37.
Cited this article as: Halıcı, S., Deveci, Ö., & Çürük Ş. (2024). On q- Integer Representation with a Special Sequence Turkish Journal of Science, 9(1), 80-90.

J_n and j_n are *nth* Jacobsthal and Jacobsthal-Lucas numbers:

$$J_n = \frac{1}{3}\left(2^n - (-1)^n\right),\ j_n = \left(2^n - (-1)^n\right). \tag{3}$$

Since quantum calculus plays an important role in physics, number theory, and other areas of mathematics, we consider two different types of bicomplex sequences with components involving quantum integers. Therefore, we have briefly summarized some properties of quantum integers. For any integer number n, the equations $[n]_q$ and $[-n]_q$ are satistified.

$$[n]_q = \sum_{k=0}^{n-1} q^k,\ \ [-n]_q = -\sum_{k=1}^{n} q^{-k}. \tag{4}$$

Where $[1]_q = 1$ and $[-1]_q = -\frac{1}{q}$. The number $[-n]_q$ is also called a $q-$ integer [6]. Also, $-q^n[-n]_q = [n]_q$, $n < 0$, is satisfied. Considering the ring of integers, in case $q = 1$, the quantum integers turn into known integers. The following algebraic operations are frequently used in quantum calculations, with m, n being natural numbers.

$$[m+n]_q = [m]_q + q^m [n]_q,\ \ [mn]_q = [m]_q [n]_{q^m}. \tag{5}$$

Elements of second-order integer sequences and $q-$integers can be converted to each other. Therefore, studying $q-$integers has great advantages in terms of computational ease and usability. An example is the work of Pashaev and Nalci's Golden quantum oscillator, and Binet-Fibonacci calculus[11]. We can write down some of the work done using these numbers. In [7], the authors have derived families of multilinear and multilateral generating functions for some polynomials based on q integers. Akkus and Kızılaslan have examined the quantum approach to Fibonacci quaternions in a study they conducted in 2019[1]. Kome et al., on the other hand, have made a quantum calculus approximation to the dual bicomplex Fibonacci numbers[9]. In [8], the authors have studied q octonions and gave Binet formulas, exponential generating functions. In the studies performed in the references [10, 12], the authors examined in detail the arithmetic operations in bicomplex space and the structures of bicomplex functions.
In this study, we created a different number sequence using $q-$Jacobsthal bicomplex numbers. In addition, we made it easier to calculate some identities, which give the basic structures of integer sequences and occupy an important place in the literature, by using q calculus.

2. Bicomplex sequence with coefficients from $q-$Jacobsthal numbers and its properties

In the studies performed in the references [1, 9], the authors examined the $q-$Fibonacci bicomplex numbers and duals in detail.
In this section, we give bicomplex numbers using the nth q-Jacobsthal and Jacobsthal-Lucas numbers. These numbers are denoted by $J_n(\alpha, q)$ and $j_n(\alpha, q)$ and are as follows.

$$J_n(\alpha, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - (q\alpha)^n}{\alpha - q\alpha} = \frac{1}{2}\alpha^n [n]_q \tag{6}$$

and

$$j_n(\alpha, q) = \alpha^n + \beta^n = \alpha^n\left(1 + q^n\right), \tag{7}$$

respectively. In here, $\alpha = 2$ and $q = \frac{-1}{\alpha}$.
Now, using the $q-$Jacobsthal and Jacobsthal-Lucas numbers, respectively, let's define the *nth* terms of two different bicomplex number sequences as follows.

$$\mathbb{BC}J_n(\alpha, q) = \frac{1}{2}\alpha^n \left\{[n]_q + \mathbf{i}\alpha [n+1]_q + \mathbf{j}\alpha^2 [n+2]_q + \mathbf{ij}\,\alpha^3 [n+3]_q\right\} \tag{8}$$

and

$$\mathbb{BC}j_n(\alpha, q) = \alpha^n \left\{(1+q^n) + \mathbf{i}\,\alpha\left(1+q^{n+1}\right) + \mathbf{j}\,\alpha^2\left(1+q^{n+2}\right) + \mathbf{ij}\,\alpha^3\left(1+q^{n+3}\right)\right\}. \tag{9}$$

We can also write the following equation as a separate notation for this sequence.

$$\mathbb{BC}j_n(\alpha,q)=\alpha^n\frac{[2n]_q}{[n]_q}+\boldsymbol{i}\alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}+\boldsymbol{j}\alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}+\boldsymbol{ij}\alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}. \tag{10}$$

We have listed below some of the equations provided by the general terms of the sequences we have just defined.

$$\mathbb{BC}J_n(\alpha,q)\pm\mathbb{BC}J_m(\alpha,q)=\frac{1}{1-q}\left\{\underline{\alpha}\left(\alpha^{n-1}\pm\alpha^{m-1}\right)-\underline{\gamma}\left(\alpha^{n-1}q^n\pm\alpha^{m-1}q^m\right)\right\}. \tag{11}$$

$$\mathbb{BC}j_n(\alpha,q)\pm\mathbb{BC}j_m(\alpha,q)=\underline{\alpha}\left(\alpha^n\pm\alpha^m\right)-\underline{\gamma}\left(\alpha^nq^n\pm\alpha^mq^m\right). \tag{12}$$

$$\mathbb{BC}J_n(\alpha,q)\,\mathbb{BC}J_m(\alpha,q)=\frac{\alpha^{m+n-2}}{(1-q)^2}\left\{(\underline{\alpha}^2-q^m\underline{\alpha}\,\underline{\gamma}-q^n\underline{\alpha}\,\underline{\gamma}+q^{m+n}\underline{\gamma}^2\right\}. \tag{13}$$

$$\mathbb{BC}j_n(\alpha,q)\,\mathbb{BC}j_m(\alpha,q)=\alpha^{m+n}\left\{(\underline{\alpha}^2+q^m\underline{\alpha}\,\underline{\gamma}+q^n\underline{\alpha}\,\underline{\gamma}+q^{m+n}\underline{\gamma}^2\right\}. \tag{14}$$

In the following Corollary, we have given some basic equations that the general terms of the sequences we have examined provide according to the standard basis $\{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{ij}\}$.

Corollary 2.1. *The following equalities related to numbers $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$ are satisfied.*

$$\mathbb{BC}J_n(\alpha,q)+\overline{\mathbb{BC}}\,J_n(\alpha,q)=2J_n(\alpha,q). \tag{15}$$

$$\mathbb{BC}J_n(\alpha,q)+\mathbb{BC}^{\mathbf{i}}\,J_n(\alpha,q)=2\left\{J_n(\alpha,q)+\mathbf{j}\,J_{n+2}(\alpha,q)\right\}. \tag{16}$$

$$\mathbb{BC}J_n(\alpha,q)+\mathbb{BC}^{\mathbf{j}}\,J_n(\alpha,q)=2\left\{J_n(\alpha,q)+\mathbf{i}\,J_{n+1}(\alpha,q)\right\}. \tag{17}$$

$$\mathbb{BC}J_n(\alpha,q)+\mathbb{BC}^{\mathbf{ij}}\,J_n(\alpha,q)=2\left\{J_n(\alpha,q)+\mathbf{ij}\,J_{n+3}(\alpha,q)\right\}. \tag{18}$$

$$\mathbb{BC}j_n(\alpha,q)+\overline{\mathbb{BC}}\,j_n(\alpha,q)=2j_n(\alpha,q). \tag{19}$$

$$\mathbb{BC}j_n(\alpha,q)+\mathbb{BC}^{\mathbf{i}}\,j_n(\alpha,q)=2\left\{j_n(\alpha,q)+\mathbf{j}\,j_{n+2}(\alpha,q)\right\}. \tag{20}$$

$$\mathbb{BC}j_n(\alpha,q)+\mathbb{BC}^{\mathbf{j}}\,j_n(\alpha,q)=2\left\{j_n(\alpha,q)+\mathbf{i}\,j_{n+1}(\alpha,q)\right\}. \tag{21}$$

$$\mathbb{BC}j_n(\alpha,q)+\mathbb{BC}^{\mathbf{ij}}\,j_n(\alpha,q)=2\left\{j_n(\alpha,q)+\mathbf{ij}\,j_{n+3}(\alpha,q)\right\}. \tag{22}$$

As it is known, one of the most commonly used formulas in the analysis of integer sequences is the Binet formula. This formula is widely used in practice as it characterizes the elements of the sequence.

Theorem 2.2. *For the bicomplex $q-$ Jacobsthal sequences, Binet formulas are*

$$\textbf{\textit{i)}}\ \mathbb{BC}J_n(\alpha,q)=\alpha^{n-1}\left(\frac{\underline{\alpha}-q^n\underline{\gamma}}{1-q}\right).$$

$$\textbf{\textit{ii)}}\ \mathbb{BC}j_n(\alpha,q)=\alpha^n\left(\underline{\alpha}+q^n\underline{\gamma}\right).$$

Where

$$\underline{\alpha}=1+\mathbf{i}\alpha+\mathbf{j}\,\alpha^2+\mathbf{ij}\,\alpha^3,\ \underline{\gamma}=1+\mathbf{i}\alpha q+\mathbf{j}\,(\alpha q)^2+\mathbf{ij}\,(\alpha q)^3. \tag{23}$$

Proof. **i)** From the equality (8), we write

$$\mathbb{BC}J_n(\alpha,q)=\frac{1}{2}\alpha^n\left\{[n]_q+\mathbf{i}\,\alpha\,[n+1]_q+\mathbf{j}\,\alpha^2\,[n+2]_q+\mathbf{ij}\,\alpha^3\,[n+3]_q\right\},$$

$$\mathbb{BC}J_n(\alpha,q)=\frac{\alpha^n}{2(1-q)}\left\{(1-q^n)+\mathbf{i}\,\alpha\left(1-q^{n+1}\right)+\mathbf{j}\,\alpha^2\left(1-q^{n+2}\right)+\mathbf{ij}\,\alpha^3\left(1-q^{n+3}\right)\right\},$$

$$\mathbb{BC}J_n(\alpha,q)=\frac{\alpha^n}{2(1-q)}\left(1+\mathbf{i}\,\alpha+\mathbf{j}\,\alpha^2+\mathbf{ij}\,\alpha^3\right)+\frac{(\alpha q)^n}{2(1-q)}\left(1+\mathbf{i}\,\alpha q+\mathbf{j}\,(\alpha q)^2+\mathbf{ij}\,(\alpha q)^3\right).$$

If we substitute the values of the alpha line and beta line in the last equation then we get

$$\mathbb{BC}J_n(\alpha,q)=\alpha^{n-1}\left(\frac{\underline{\alpha}-q^n\underline{\gamma}}{1-q}\right)$$

which completes the proof.
ii) From the equality (9), we can write

$$\mathbb{BC}j_n(\alpha,q)=\alpha^n\left\{(1+q^n)+\mathbf{i}\,\alpha\left(1+q^{n+1}\right)+\mathbf{j}\,\alpha^2\left(1+q^{n+2}\right)+\mathbf{ij}\,\alpha^3\left(1+q^{n+3}\right)\right\},$$

$$\mathbb{BC}j_n(\alpha,q)=2\alpha^{n-1}\left(1+\mathbf{i}\,\alpha+\mathbf{j}\,\alpha^2+\mathbf{ij}\,\alpha^3\right)+2(\alpha q)^{n-1}\left(q+\mathbf{i}\,\alpha q^2+\mathbf{j}\,\alpha^2q^3+\mathbf{ij}\,\alpha^3q^4\right).$$

If necessary operations are performed on the last equation, the following equation is obtained, which completes the proof. So,

$$\mathbb{BC}j_n(\alpha,q)=\alpha^n\left(\underline{\alpha}+q^n\underline{\gamma}\right).$$

We would like to point out right away that if the values α, q are substituted in the above last formula, then the Theorem 2.4 in [5] is obtained. □

The algebraic operations and relations between the elements of these sequences that we examined are given in the Corollary below. We noted that q calculus is very useful in reviewing all the properties involving sequence elements.

Corollary 2.3. *For the numbers* $\mathbb{BC}J_n(\alpha,q)$*, the following equalities are satisfied.*

$$\mathbb{BC}J_{n+1}(\alpha,q)+\mathbb{BC}J_n(\alpha,q)=\alpha^n\underline{\alpha}. \tag{24}$$

$$\mathbb{BC}J_{n+1}(\alpha,q)+\mathbb{BC}J_{n-1}(\alpha,q)=\frac{\alpha^{n-2}}{1-q}\left\{5\underline{\alpha}-2q^{n-1}\underline{\gamma}\right\}. \tag{25}$$

$$\mathbb{BC}J_{n+1}(\alpha,q)-\mathbb{BC}J_n(\alpha,q)=\frac{\alpha^{n-1}}{1-q}\left(\underline{\alpha}+2q^n\underline{\gamma}\right). \tag{26}$$

$$\mathbb{BC}J_{n+1}(\alpha,q)-\mathbb{BC}J_{n-1}(\alpha,q)=\frac{\alpha^{n-2}\underline{\alpha}}{1-q}. \tag{27}$$

Proof. It is sufficient to see that only one of these equalities is true. For this purpose,

$$\mathbb{BC}J_{n+1}(\alpha,q)+\mathbb{BC}J_{n-1}(\alpha,q)=\frac{1}{1-q}\left\{\alpha^{n-2}\underline{\alpha}\left(\alpha^2+1\right)-\alpha^{n-2}q^{n-1}\underline{\gamma}\left(1+\alpha^2q^2\right)\right\},$$

$$\mathbb{BC}J_{n+1}(\alpha,q)+\mathbb{BC}J_{n-1}(\alpha,q)=\frac{\alpha^{n-2}}{1-q}\left\{5\underline{\alpha}-2q^{n-1}\underline{\gamma}\right\}$$

which is desired. □

Corollary 2.4. *For* $\mathbb{BC}j_n(\alpha,q)$*, the following equalities are satisfied.*

$$\mathbb{BC}j_{n+1}(\alpha,q)+\mathbb{BC}j_n(\alpha,q)=3\alpha^n\underline{\alpha}. \tag{28}$$

$$\mathbb{BC}j_{n+1}(\alpha,q)+\mathbb{BC}j_{n-1}(\alpha,q)=\alpha^{n-1}(5\underline{\alpha}-2q^{n-1}\underline{\gamma}). \tag{29}$$

$$\mathbb{BC}j_{n+1}(\alpha,q)-\mathbb{BC}j_n(\alpha,q)=2\alpha^n\left(\underline{\alpha}-q^n\underline{\gamma}\right). \tag{30}$$

$$\mathbb{BC}j_{n+1}(\alpha,q)-\mathbb{BC}j_{n-1}(\alpha,q)=3\alpha^{n-1}\underline{\alpha}. \tag{31}$$

Proof. It is sufficient to see that only one of these equalities is true. For this, we write

$$\mathbb{BC}j_{n+1}(\alpha,q)+\mathbb{BC}j_n(\alpha,q)=\alpha^{n+1}\underline{\alpha}+(\alpha q)^{n+1}\underline{\gamma}+\alpha^n\underline{\alpha}+(\alpha q)^n\underline{\gamma},$$

$$\mathbb{BC}j_{n+1}(\alpha,q)+\mathbb{BC}j_n(\alpha,q)=\alpha^n\underline{\alpha}(\alpha+1)+(\alpha q)^n\underline{\gamma}\ (\alpha q+1)=3\alpha^n\underline{\alpha}.$$

□

The relations between the terms $\mathbb{BC}J_n$ and $\mathbb{BC}j_n$ are given in the following Theorem.

Theorem 2.5. *For $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$ the following equalities are satisfied.*

$$\textbf{\textit{i)}}\ \mathbb{BC}J_{n+1}(\alpha,q)+2\mathbb{BC}J_{n-1}(\alpha,q)=\mathbb{BC}j_n(\alpha,q).$$

$$\textbf{\textit{ii)}}\ \mathbb{BC}j_{n+1}(\alpha,q)+2\mathbb{BC}j_{n-1}(\alpha,q)=9\mathbb{BC}J_n(\alpha,q).$$

Proof. **i)** One can get

$$\mathbb{BC}J_{n+1}(\alpha,q)+2\mathbb{BC}J_{n-1}(\alpha,q)=\frac{\alpha^{n-1}\underline{\alpha}\ (\alpha+1)-(\alpha q)^{n-1}\underline{\gamma}\left(1+\alpha q^2\right)}{1-q},$$

$$\mathbb{BC}J_{n+1}(\alpha,q)+2\mathbb{BC}J_{n-1}(\alpha,q)=\frac{2\left\{3\alpha^{n-1}\underline{\alpha}\ -\ \frac{3}{2}(\alpha q)^{n-1}\underline{\gamma}\right\}}{3},$$

$$\mathbb{BC}J_{n+1}(\alpha,q)+2\mathbb{BC}J_{n-1}(\alpha,q)=2\left\{\alpha^{n-1}\underline{\alpha}\ -\ \alpha^{-1}(\alpha q)^{n-1}\underline{\gamma}\right\},$$

$$\mathbb{BC}J_{n+1}(\alpha,q)+2\mathbb{BC}J_{n-1}(\alpha,q)=\alpha^n\underline{\alpha}+\ (\alpha q)^n\underline{\gamma}$$

and

$$\mathbb{BC}J_{n+1}(\alpha,q)+2\mathbb{BC}J_{n-1}(\alpha,q)=\mathbb{BC}j_n(\alpha,q).$$

ii) For the $\mathbb{BC}j_n(\alpha,q)$, we write

$$\mathbb{BC}j_{n+1}(\alpha,q)+2\mathbb{BC}j_{n-1}(\alpha,q)=\alpha^{n+1}\underline{\alpha}+\ (\alpha q)^{n+1}\underline{\gamma}+2\alpha^{n-1}\underline{\alpha}+\ (\alpha q)^{n-1}\underline{\gamma},$$

$$\mathbb{BC}j_{n+1}(\alpha,q)+2\mathbb{BC}j_{n-1}(\alpha,q)=\alpha^n\underline{\alpha}\ \ (\alpha+1)+\alpha^n q^{n-1}\ \ \underline{\gamma}\left(1+\alpha q^2\right),$$

$$\mathbb{BC}j_{n+1}(\alpha,q)+2\mathbb{BC}j_{n-1}(\alpha,q)=6\alpha^{n-1}\left(\underline{\alpha}\ -\ q^n\ \underline{\gamma}\ \right),$$

$$\mathbb{BC}j_{n+1}(\alpha,q)+2\mathbb{BC}j_{n-1}(\alpha,q)=9\mathbb{BC}J_n(\alpha,q).$$

So, the proof is completed. □

The exponential generating function for the $q-$ Jacobsthal and $q-$ Jacobsthal-Lucas bicomplex numbers are given in the following theorem.

Theorem 2.6. *For the bicomplex numbers $\mathbb{BC}J_n(\alpha,q)$, $\mathbb{BC}j_n(\alpha,q)$, we have*

$$\textbf{\textit{i)}}\sum_{n=0}^{\infty}\mathbb{BC}J_n(\alpha,q)\frac{x^n}{n!}=\frac{1}{2(1-q)}\left(\underline{\alpha}\,e^{\alpha x}-\underline{\gamma}\,e^{\alpha qx}\right).$$

$$\textbf{\textit{ii)}}\sum_{n=0}^{\infty}\mathbb{BC}j_n(\alpha,q)\frac{x^n}{n!}=\left(\underline{\alpha}\,e^{\alpha x}+\underline{\gamma}\,e^{\alpha qx}\right).$$

Proof. ***i)*** In the generating function definition if we use the Binet formula of the $\mathbb{BC}J_n(\alpha,q)$, then we get the following equalities.

$$\sum_{n=0}^{\infty}\mathbb{BC}J_n(\alpha,q)\frac{x^n}{n!}=\sum_{n=0}^{\infty}\alpha^{n-1}\frac{\underline{\alpha}-q^n\underline{\gamma}}{1-q}\frac{x^n}{n!},$$

$$\sum_{n=0}^{\infty}\mathbb{BC}J_n(\alpha,q)\frac{x^n}{n!}=\frac{\underline{\alpha}}{1-q}\sum_{n=0}^{\infty}\alpha^{n-1}\frac{x^n}{n!}-\frac{\underline{\gamma}}{1-q}\sum_{n=0}^{\infty}\alpha^{n-1}q^n\frac{x^n}{n!},$$

$$\sum_{n=0}^{\infty}\mathbb{BC}J_n(\alpha,q)\frac{x^n}{n!}=\frac{1}{2(1-q)}\left(\underline{\alpha}e^{\alpha x}-\underline{\gamma}e^{\alpha qx}\right).$$

ii) We calculate the exponential generating function for the numbers $\mathbb{BC}j_n(\alpha,q)$ as

$$\sum_{n=0}^{\infty}\mathbb{BC}j_n(\alpha,q)\frac{x^n}{n!}=\sum_{n=0}^{\infty}\left\{\alpha^n\underline{\alpha}+(\alpha q)^n\underline{\gamma}\right\}\frac{x^n}{n!},$$

$$\sum_{n=0}^{\infty}\mathbb{BC}j_n(\alpha,q)\frac{x^n}{n!}=\underline{\alpha}\sum_{n=0}^{\infty}\alpha^n\frac{x^n}{n!}+\underline{\gamma}\sum_{n=0}^{\infty}(\alpha q)^n\frac{x^n}{n!},$$

$$\sum_{n=0}^{\infty}\mathbb{BC}j_n(\alpha,q)\frac{x^n}{n!}=\left(\underline{\alpha}\,e^{\alpha x}+\underline{\gamma}\,e^{\alpha qx}\right).$$

Thus, the proof is completed. □

In the next section, we give some fundamental identities involving the terms $\mathbb{BC}J_n(\alpha,q)$, $\mathbb{BC}j_n(\alpha,q)$ and are related to each other in this study.

3. Some fundamental identities of involving the terms $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$

First, using q calculus we give the Cassini's identity which has an important place in the literature for Fibonacci-like number sequences.

Theorem 3.1. *For $n\in\mathbb{Z}^+$, we have*

$$\textit{i)}\ \mathbb{BC}J_{n+1}(\alpha,q)\,\mathbb{BC}J_{n-1}(\alpha,q)-\mathbb{BC}J_n^{\,2}(\alpha,q)=\frac{\alpha^{2(n-1)}\underline{\alpha}\,\underline{\gamma}\,q^n\left(2-q-q^{-1}\right)}{(1-q)^2}.$$

$$\textit{ii)}\ \mathbb{BC}j_{n+1}(\alpha,q)\,\mathbb{BC}j_{n-1}(\alpha,q)-\mathbb{BC}j_n^{\,2}(\alpha,q)=\alpha^{2n}\underline{\alpha}\,\underline{\gamma}\,q^n\left(q+q^{-1}-2\right).$$

Proof. **i)** From the Binet formula, we write

$$LHS=\frac{\alpha^{2n-2}}{(1-q)^2}\left\{\underline{\alpha}^2-\underline{\alpha}\,\underline{\gamma}q^{n-1}-\underline{\alpha}\,\underline{\gamma}\,q^{n+1}+q^{2n}\underline{\gamma}^2-\underline{\alpha}^2+2\underline{\alpha}\,\underline{\gamma}q^n-q^{2n}\underline{\gamma}^2\right\},$$

$$\mathbb{BC}J_{n+1}(\alpha,q)\,\mathbb{BC}J_{n-1}(\alpha,q)-\mathbb{BC}J_n^2(\alpha,q)=\frac{\alpha^{2(n-1)}\,\underline{\alpha}\,\underline{\gamma}q^n\left(2-q-q^{-1}\right)}{(1-q)^2}.$$

ii) The proof of this equality can be done in a similar way. □

In the following theorem, we give the Catalan's identity for the $q-$Jacobsthal and $q-$Jacobsthal-Lucas bicomplex numbers. The Catalan's identity, which is a generalization of Cassini's identity, was given by E.C. Catalan(1814-1894).

Theorem 3.2. *For $n \in \mathbb{Z}^+$ and $n \geq r$, we have*

$$\textit{i)}\ \mathbb{BC}J_{n+r}(\alpha,q)\,\mathbb{BC}J_{n-r}(\alpha,q)-\mathbb{BC}J_n^2(\alpha,q)=\frac{\alpha^{2(n-1)}\underline{\alpha}\,\underline{\gamma}\,q^n\ (2-q^r+q^{-r})}{(1-q)^2}.$$

$$\textit{ii)}\ \mathbb{BC}j_{n+r}(\alpha,q)\,\mathbb{BC}j_{n-r}(\alpha,q)-\mathbb{BC}j_n^2(\alpha,q)=\alpha^{2n}\underline{\alpha}\,\underline{\gamma}q^n\,(q^r+q^{-r}-2).$$

Proof. **i)**

$$LHS=\frac{\alpha^{2(n-1)}}{(1-q)^2}\left\{\underline{\alpha}^2-\underline{\alpha}\,\underline{\gamma}q^{n-r}-\underline{\alpha}\,\underline{\gamma}q^{n+r}+\ q^{2n}\underline{\gamma}^2-\underline{\alpha}^2+2\underline{\alpha}\,\underline{\gamma}q^n-q^{2n}\underline{\gamma}^2\right\}.$$

So, we obtain

$$\mathbb{BC}J_{n+r}(\alpha,q)\,\mathbb{BC}J_{n-r}(\alpha,q)-\mathbb{BC}J_n^2(\alpha,q)=\frac{\alpha^{2(n-1)}\underline{\alpha}\,\underline{\gamma}q^n\,(2-q^r+q^{-r})}{(1-q)^2}.$$

ii) Using a similar method, we get

$$LHS=\left(\alpha^{n+r}\underline{\alpha}+(\alpha q)^{n+r}\underline{\gamma}\right)\left(\alpha^{n-r}\underline{\alpha}+(\alpha q)^{n-r}\underline{\gamma}\right)-\left(\alpha^n\underline{\alpha}+(\alpha q)^n\underline{\gamma}\right)^2,$$

$$LHS=\alpha^{2n}\underline{\alpha}^2+\alpha^{2n}q^{n-r}\underline{\alpha}\,\underline{\gamma}+\alpha^{2n}q^{n+r}\underline{\alpha}\,\underline{\gamma}+(\alpha q)^{2n}\underline{\gamma}^2-\alpha^{2n}\underline{\alpha}^2-2\alpha^{2n}q^n\underline{\alpha}\,\underline{\gamma}-(\alpha q)^{2n}\underline{\gamma}^2,$$

$$\mathbb{BC}j_{n+r}(\alpha,q)\,\mathbb{BC}j_{n-r}(\alpha,q)-\mathbb{BC}{j_n}^2(\alpha,q)=\alpha^{2n}q^n\underline{\alpha}\,\underline{\gamma}\,(q^r+q^{-r}-2).$$

Note that the last equality gives the Catalan's identity for the $q-$ Jacobsthal-Lucas bicomplex numbers. Thus, the proof is completed. □

Specifically, in Theorem 3.2 if we write $r=1$, then we get the Cassini identities for the terms $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$.
In the following Theorem, we give the d'Ocagne identity involving the terms $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$.

Theorem 3.3. *For $m,n \in \mathbb{Z}^+$, we have*

$$\textit{i)}\ \mathbb{BC}J_m(\alpha,q)\mathbb{BC}J_{n+1}(\alpha,q)-\mathbb{BC}J_n(\alpha,q)\mathbb{BC}J_{m+1}(\alpha,q)=\frac{\alpha^{m+n-1}\underline{\alpha}\,\underline{\gamma}}{(1-q)^2}(q-1)(q^m-q^n).$$

$$\textit{ii)}\ \mathbb{BC}j_m(\alpha,q)\mathbb{BC}j_{n+1}(\alpha,q)-\mathbb{BC}j_n(\alpha,q)\mathbb{BC}j_{m+1}(\alpha,q)=\alpha^{m+n+1}\underline{\alpha}\,\underline{\gamma}\,(1-q)(q^m-q^n).$$

Proof. **i)** Using the Binet formula in the *LHS*, we obtained

$$\alpha^{m-1}\left(\frac{\underline{\alpha}-q^m\underline{\gamma}}{1-q}\right)\alpha^n\left(\frac{\underline{\alpha}-q^{n+1}\underline{\gamma}}{1-q}\right)-\alpha^{n-1}\left(\frac{\underline{\alpha}-q^n\underline{\gamma}}{1-q}\right)\alpha^m\left(\frac{\underline{\alpha}-q^{m+1}\underline{\gamma}}{1-q}\right).$$

From this fact

$$\mathbb{BC}J_m(\alpha,q)\ \mathbb{BC}J_{n+1}(\alpha,q)\ -\mathbb{BC}J_n(\alpha,q)\ \mathbb{BC}J_{m+1}(\alpha,q)=\frac{\alpha^{m+n-1}\underline{\alpha}\,\underline{\gamma}}{(1-q)^2}(q-1)(q^m-q^n)$$

can be written. Thus, the proof is completed.
ii) The value $\mathbb{BC}j_m(\alpha,q)\ \mathbb{BC}j_{n+1}(\alpha,q)-\mathbb{BC}j_n(\alpha,q)\ \mathbb{BC}j_{m+1}(\alpha,q)$ is equal to this

$$\left(\alpha^m\underline{\alpha}+(\alpha q)^m\underline{\gamma}\right)\left(\alpha^{n+1}\underline{\alpha}+(\alpha q)^{n+1}\underline{\gamma}\right)-\left(\alpha^n\underline{\alpha}+(\alpha q)^n\underline{\gamma}\right)\left(\alpha^{m+1}\underline{\alpha}+(\alpha q)^{m+1}\underline{\gamma}\right).$$

$$\mathbb{BC}j_m(\alpha,q)\ \mathbb{BC}j_{n+1}(\alpha,q)-\mathbb{BC}j_n(\alpha,q)\ \mathbb{BC}j_{m+1}(\alpha,q)=\alpha^{m+n+1}\underline{\alpha}\,\underline{\gamma}\,(1-q)(q^m-q^n).$$

Thus, the proof is completed. □

Theorem 3.4. *For the terms $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$, we have*

$$\textbf{\textit{i)}}\ \mathbb{BC}J_{n-1}(\alpha,q)\,\mathbb{BC}J_m(\alpha,q)+\mathbb{BC}J_n(\alpha,q)\,\mathbb{BC}J_{m+1}(\alpha,q)=\frac{\alpha^{m+n-3}}{(1-q)^2}\left(A+\alpha^2B\right).$$

$$\textbf{\textit{ii)}}\ \mathbb{BC}j_{n-1}(\alpha,q)\,\mathbb{BC}j_m(\alpha,q)+\mathbb{BC}j_n(\alpha,q)\,\mathbb{BC}j_{m+1}(\alpha,q)=\alpha^{m+n+1}\left(C+\alpha^2D\right).$$

Proof. **i)** For $m,n\in\mathbb{Z}^+$, LHS of equality is as follows.

$$LHS=\alpha^{n-2}\left(\frac{\underline{\alpha}-q^{n-1}\underline{\gamma}}{1-q}\right)\alpha^{m-1}\left(\frac{\underline{\alpha}-q^{m}\underline{\gamma}}{1-q}\right)+\alpha^{n-1}\left(\frac{\underline{\alpha}-q^{n}\underline{\gamma}}{1-q}\right)\alpha^{m}\left(\frac{\underline{\alpha}-q^{m+1}\underline{\gamma}}{1-q}\right),$$

$$LHS=\frac{\alpha^{m+n-3}}{(1-q)^2}\left\{\left(\underline{\alpha}^2-(q^m+q^{n-1})\underline{\alpha}\,\underline{\gamma}+q^{m+n-1}\underline{\gamma}^2\right)+\alpha^2\left(\underline{\alpha}^2-(q^{m+1}+q^{n})\underline{\alpha}\,\underline{\gamma}+q^{m+n+1}\underline{\gamma}^2\right)\right\}.$$

We get the following equality by doing the necessary calculations.

$$\mathbb{BC}J_{n-1}(\alpha,q)\,\mathbb{BC}J_m(\alpha,q)+\mathbb{BC}J_n(\alpha,q)\,\mathbb{BC}J_{m+1}(\alpha,q)=\frac{\alpha^{m+n-3}}{(1-q)^2}\left(A+\alpha^2B\right),$$

where,

$$A=\underline{\alpha}^2-(q^m+q^{n-1})\underline{\alpha}\,\underline{\gamma}+q^{m+n-1}\underline{\gamma}^2,$$
$$B=\underline{\alpha}^2-(q^{m+1}+q^{n})\underline{\alpha}\,\underline{\gamma}+q^{m+n+1}\underline{\gamma}^2.$$

ii) For LHS, we calculate

$$LHS=\left(\alpha^{n-1}\underline{\alpha}+(\alpha q)^{n-1}\underline{\gamma}\right)\left(\alpha^{m}\underline{\alpha}+(\alpha q)^{m}\underline{\gamma}\right)+\left(\alpha^{n}\underline{\alpha}+(\alpha q)^{n}\underline{\gamma}\right)\left(\alpha^{m+1}\underline{\alpha}+(\alpha q)^{m+1}\underline{\gamma}\right),$$

$$LHS=\alpha^{m+n-1}\left\{\left(\underline{\alpha}^2+(q^m+q^{n-1})\underline{\alpha}\,\underline{\gamma}+q^{m+n-1}\underline{\gamma}^2\right)+\alpha^2\left(\underline{\alpha}^2+(q^{m+1}+q^{n})\underline{\alpha}\,\underline{\gamma}+q^{m+n+1}\underline{\gamma}^2\right)\right\}.$$

Then, we get

$$\mathbb{BC}j_{n-1}(\alpha,q)\,\mathbb{BC}j_m(\alpha,q)+\mathbb{BC}j_n(\alpha,q)\,\mathbb{BC}j_{m+1}(\alpha,q)=\alpha^{m+n+1}\left(C+\alpha^2D\right),$$

where

$$C=\underline{\alpha}^2+(q^m+q^{n-1})\underline{\alpha}\,\underline{\gamma}+q^{m+n-1}\underline{\gamma}^2,\ D=\underline{\alpha}^2+(q^{m+1}+q^{n})\underline{\alpha}\,\underline{\gamma}+q^{m+n+1}\underline{\gamma}^2.$$

Thus, the proof is completed. □

In the next Theorem, we give Vajda's identity, which is a generalization of the Catalan's identity and has an important place in integer sequences.

Theorem 3.5. *For $n\in\mathbb{Z}^+$, we have*

$$\textbf{\textit{i)}}\ \mathbb{BC}J_{n+i}(\alpha,q)\,\mathbb{BC}J_{n+j}(\alpha,q)-\mathbb{BC}J_n(\alpha,q)\,\mathbb{BC}J_{n+i+j}(\alpha,q)=\frac{\alpha^{2(n-1)+i+j}q^n\underline{\alpha}\,\underline{\gamma}}{(1-q)^2}\left(q^j-1)(q^i-1\right).$$

$$\textbf{\textit{ii)}}\ \mathbb{BC}j_{n+i}(\alpha,q)\,\mathbb{BC}j_{n+j}(\alpha,q)-\mathbb{BC}j_n(\alpha,q)\,\mathbb{BC}j_{n+i+j}(\alpha,q)=\alpha^{2n+i+j}q^n\underline{\alpha}\,\underline{\gamma}\left(1-q^i)(q^j-1\right).$$

Proof. **i)** For LHS, we can write

$$LHS=\alpha^{2(n-1)+i+j}\left\{\left(\frac{\underline{\alpha}-q^{n+i}\underline{\gamma}}{1-q}\right)\left(\frac{\underline{\alpha}-q^{n+j}\underline{\gamma}}{1-q}\right)-\left(\frac{\underline{\alpha}-q^{n}\underline{\gamma}}{1-q}\right)\left(\frac{\underline{\alpha}-q^{n+i+j}\underline{\gamma}}{1-q}\right)\right\},$$

$$LHS = \frac{\alpha^{2(n-1)+i+j} q^n \underline{\alpha}\, \underline{\gamma}(q^{i+j} + 1 - q^j - q^i)}{(1-q)^2},$$

$$LHS = \frac{\alpha^{2(n-1)+i+j} q^n \underline{\alpha}\, \underline{\gamma}}{(1-q)^2} \left(q^j - 1)(q^i - 1\right) = RHS.$$

ii) $\mathbb{BC}j_{n+i}(\alpha,q)\,\mathbb{BC}j_{n+j}(\alpha,q) - \mathbb{BC}j_n(\alpha,q)\,\mathbb{BC}j_{n+i+j}(\alpha,q)$ is equal to this.

$$\left(\alpha^{n+i}\underline{\alpha} + (\alpha q)^{n+i}\underline{\gamma}\right)\left(\alpha^{n+j}\underline{\alpha} + (\alpha q)^{n+j}\underline{\gamma}\right) - \left(\alpha^{n}\underline{\alpha} + (\alpha q)^{n}\underline{\gamma}\right)\left(\alpha^{n+i+j}\underline{\alpha} + (\alpha q)^{n+i+j}\underline{\gamma}\right).$$

If we make the necessary calculations and arrangements, then we obtain

$$\mathbb{BC}j_{n+i}(\alpha,q)\,\mathbb{BC}j_{n+j}(\alpha,q) - \mathbb{BC}j_n(\alpha,q)\,\mathbb{BC}j_{n+i+j}(\alpha,q) = \alpha^{2n+i+j} q^n \underline{\alpha}\,\underline{\gamma}\left(q^j + q^i - q^{i+j} - 1\right),$$

$$\mathbb{BC}j_{n+i}(\alpha,q)\,\mathbb{BC}j_{n+j}(\alpha,q) - \mathbb{BC}j_n(\alpha,q)\,\mathbb{BC}j_{n+i+j}(\alpha,q) = \alpha^{2n+i+j} q^n \underline{\alpha}\,\underline{\gamma}\left(1 - q^i)(q^j - 1\right).$$

Thus, the proof is completed. □

Another generalization of the Catalan identity is Gelin-Cesaro identity. In the following Theorem, we give the Gelin-Cesaro identity involving the terms $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$.

Theorem 3.6. *Gelin-Cesaro identities involving the terms $\mathbb{BC}J_n(\alpha,q)$ and $\mathbb{BC}j_n(\alpha,q)$ are follows.*

$$\textit{i)}\ \mathbb{BC}J_{n-2}(\alpha,q)\,\mathbb{BC}J_{n-1}(\alpha,q)\,\mathbb{BC}J_{n+1}(\alpha,q)\,\mathbb{BC}J_{n+2}(\alpha,q) - \mathbb{BC}{J_n}^4(\alpha,q) = \left(\frac{\alpha^{n-1}}{1-q}\right)^4 (K_1 + K_2).$$

$$\textit{ii)}\ \mathbb{BC}j_{n-2}(\alpha,q)\mathbb{BC}j_{n-1}(\alpha,q)\mathbb{BC}j_{n+1}(\alpha,q)\mathbb{BC}j_{n+2}(\alpha,q) - \mathbb{BC}{j_n}^4(\alpha,q) = -\alpha^{4n}(K_1 - K_2).$$

Where

$$K_1 = \underline{\alpha}\,\underline{\gamma} q^{n-2}(\underline{\alpha}^2 + \underline{\gamma}^2 q^{2n})\left(-q^4 - q^3 + 4q^2 - q - 1\right), \tag{32}$$

$$K_2 = \underline{\alpha}^2\underline{\gamma}^2 q^{2n-3}\left(q^6 + q^4 - 4q^3 + q^2 + 1\right). \tag{33}$$

Proof. **i)** $\mathbb{BC}J_{n-2}(\alpha,q)\mathbb{BC}J_{n-1}(\alpha,q)\mathbb{BC}J_{n+1}(\alpha,q)\mathbb{BC}J_{n+2}(\alpha,q) - \mathbb{BC}{J_n}^4(\alpha,q)$

$$= \left(\frac{\alpha^{n-1}}{1-q}\right)^4 \left\{\left(\underline{\alpha} - q^{n-2}\underline{\gamma}\right)\left(\underline{\alpha} - q^{n-1}\underline{\gamma}\right)\left(\underline{\alpha} - q^{n+1}\underline{\gamma}\right)\left(\underline{\alpha} - q^{n+2}\underline{\gamma}\right) - \left(\underline{\alpha} - q^{n}\underline{\gamma}\right)^4\right\},$$

$$= \left(\frac{\alpha^{n-1}}{1-q}\right)^4 \left\{-\underline{\alpha}^3\underline{\gamma} q^{n+2} - \underline{\alpha}^3\underline{\gamma} q^{n+1} + \underline{\alpha}^2\underline{\gamma}^2 q^{2n+3} - \underline{\alpha}^3\underline{\gamma} q^{n-1} + \underline{\alpha}^2\underline{\gamma}^2 q^{2n+1} - \underline{\alpha}\underline{\gamma}^3 q^{3n+2} - \underline{\alpha}^3\underline{\gamma} q^{n-2}\right.$$
$$\left. + \underline{\alpha}^2\underline{\gamma}^2 q^{2n-1} - \underline{\alpha}\underline{\gamma}^3 q^{3n+1} + \underline{\alpha}^2\underline{\gamma}^2 q^{2n-3} - \underline{\alpha}\underline{\gamma}^3 q^{3n-1} - \underline{\alpha}\underline{\gamma}^3 q^{3n-2} + 4\underline{\alpha}^3\underline{\gamma} q^{n} - 4\underline{\alpha}^2\underline{\gamma}^2 q^{2n} + 4\underline{\alpha}\underline{\gamma}^3 q^{3n}\right\}.$$

If we do the necessary arrangements, then we get

$$= \left(\frac{\alpha^{n-1}}{1-q}\right)^4 \left\{\underline{\alpha}\underline{\gamma} q^{n-2}\left(\underline{\alpha}^2 + \underline{\gamma}^2 q^{2n}\right)\left(-q^4 - q^3 + 4q^2 - q - 1\right) + \underline{\alpha}^2\underline{\gamma}^2 q^{2n-3}\left(q^6 + q^4 - 4q^3 + q^2 + 1\right)\right\}.$$

After sum and simplification, the following equation is obtained.

$$\mathbb{BC}J_{n-2}(\alpha,q)\mathbb{BC}J_{n-1}(\alpha,q)\mathbb{BC}J_{n+1}(\alpha,q)\mathbb{BC}J_{n+2}(\alpha,q) - \mathbb{BC}{J_n}^4(\alpha,q) = \left(\frac{\alpha^{n-1}}{1-q}\right)^4 (K_1 + K_2).$$

ii) Let us now show the truth of the second claim of the Theorem.

$$LHS = \left(\alpha^{n-2}\underline{\alpha} + (\alpha q)^{n-2}\underline{\gamma}\right)\left(\alpha^{n-1}\underline{\alpha} + (\alpha q)^{n-1}\underline{\gamma}\right)\left(\alpha^{n+1}\underline{\alpha} + (\alpha q)^{n+1}\underline{\gamma}\right)\left(\alpha^{n+2}\underline{\alpha} + (\alpha q)^{n+2}\underline{\gamma}\right) - \left(\alpha^{n}\underline{\alpha} + (\alpha q)^{n}\underline{\gamma}\right)^4,$$

$$LHS = \alpha^{4n}q^{n-2}\underline{\alpha}^3\underline{\gamma}\left(q^4 + q^3 - 4q^2 + q + 1\right) + \alpha^{4n}q^{2n-3}\underline{\alpha}^2\underline{\gamma}^2\left(q^6 + q^4 - 4q^3 + q^2 + 1\right)$$
$$+\alpha^{4n}q^{3n-2}\underline{\alpha}\underline{\gamma}^3\left(q^4 + q^3 - 4q^2 + q + 1\right),$$

$$LHS = \alpha^{4n}\left\{\underline{\alpha}\,\underline{\gamma}q^{n-2}(\underline{\alpha}^2 + \underline{\gamma}^2q^{2n})\left(q^4 + q^3 - 4q^2 + q + 1\right) + q^{2n-3}\underline{\alpha}^2\underline{\gamma}^2\left(q^6 + q^4 - 4q^3 + q^2 + 1\right)\right\}.$$

From the last equality, we get

$$\mathbb{BC}j_{n-2}(\alpha,q)\mathbb{BC}j_{n-1}(\alpha,q)\mathbb{BC}j_{n+1}(\alpha,q)\mathbb{BC}j_{n+2}(\alpha,q) - \mathbb{BC}{j_n}^4(\alpha,q) = -\alpha^{4n}\left(K_1 - K_2\right).$$

Thus, the proof is completed. □

With the help of the classical binomial definition, $q-$Jacobsthal and Jacobsthal-Lucas bicomplex numbers are given in the following Theorem.

Theorem 3.7. *For nonnegative integer n, we have*

$$\textbf{\textit{i)}}\ \sum_{k=0}^{n}\binom{n}{k}\left(\alpha[2]_q\right)^k\left(-\alpha^2q\right)^{n-k}\mathbb{BC}J_k(\alpha,q) = \mathbb{BC}J_{2n}(\alpha,q).$$

$$\textbf{\textit{ii)}}\ \sum_{k=0}^{n}\binom{n}{k}(\alpha[2]_q)^k(-\alpha^2q)^{n-k}\mathbb{BC}j_k(\alpha,q) = \mathbb{BC}j_{2n}(\alpha,q).$$

Proof. **i)** From the Binet formula, we write

$$LHS = \sum_{k=0}^{n}\binom{n}{k}\{\alpha(1+q)\}^k\left(-\alpha^2q\right)^{n-k}\alpha^{k-1}\left(\frac{\underline{\alpha} - q^k\underline{\gamma}}{1-q}\right),$$

$$LHS = \frac{\underline{\alpha}}{2(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left\{\alpha^2(1+q)\right\}^k\left(-\alpha^2q\right)^{n-k} - \frac{\underline{\gamma}}{2(1-q)}\sum_{k=0}^{n}\binom{n}{k}\left\{\alpha^2(1+q)\right\}^k\left(-\alpha^2q\right)^{n-k}.$$

Thus, we get

$$\sum_{k=0}^{n}\binom{n}{k}\left(\alpha[2]_q\right)^k\left(-\alpha^2q\right)^{n-k}\mathbb{BC}J_k(\alpha,q) = \alpha^{2n-1}\left(\frac{\underline{\alpha} - q^{2n}\underline{\gamma}}{1-q}\right) = \mathbb{BC}J_{2n}(\alpha,q).$$

ii) For the second equality we can use the same method.

$$LHS = \sum_{k=0}^{n}\binom{n}{k}\{\alpha(1+q)\}^k\left(-\alpha^2q\right)^{n-k}\left(\alpha^k\underline{\alpha} + (\alpha q)^k\underline{\gamma}\right),$$

$$LHS = \underline{\alpha}\sum_{k=0}^{n}\binom{n}{k}\left\{\alpha^2(1+q)\right\}^k\left(-\alpha^2q\right)^{n-k} + \underline{\gamma}\sum_{k=0}^{n}\binom{n}{k}\left\{\alpha^2(1+q)\right\}^k\left(-\alpha^2q\right)^{n-k}.$$

Thus, we get

$$\sum_{k=0}^{n}\binom{n}{k}\left(\alpha[2]_q\right)^k\left(-\alpha^2q\right)^{n-k}\mathbb{BC}j_k(\alpha,q) = \alpha^{2n}\underline{\alpha} + (\alpha q)^{2n}\underline{\gamma} = \mathbb{BC}j_{2n}(\alpha,q).$$

Thus, the proof is completed. □

4. Conclusion

In this study, we investigated bicomplex numbers whose coefficients are $q-$Jacobsthal and $q-$Jacobsthal-Lucas numbers. We obtained some fundamental and important properties of newly defined numbers. And then, we calculated the Binet formula for these numbers. In addition to this, we derived some basic identities for these numbers, such as Cassini and Catalan identities, which have an important place in the literature.

References

[1] Akkus, I., Kizilaslan, G. (2019). Quaternions: Quantum calculus approach with applications. Kuwait Journal of Science, 46, 1–13.

[2] Cerda-Morales, G. (2023). On bicomplex third-order Jacobsthal numbers. Complex Variables and Elliptic Equations, 68, 44–56.

[3] Davenport, C. M. A. (1996). Commutative Hypercomplex Algebra with Associated Function Theory. USA: Birkhauser Boston Inc., 213–227.

[4] Halici, S. (2019). On Bicomplex Fibonacci Numbers and their generalization. In Model and Theories in Social System, Springer, Cham, 3, 509–524.

[5] Halici, S. (2020). On Bicomplex Jacobsthal-Lucas Numbers. Journal of Mathematical Sciences and Modelling, 3, 139–143.

[6] Kac, V. G., Cheung, P. (2020). Quantum calculus. New York: Springer.

[7] Kızılateş, C., Cekim, B. (2018). New families of generating functions for q-Fibonacci and the related polynomials. Ars Combinatoria, 136, 397–404.

[8] Kızılateş, C., Polatlı, E. (2021). New families of Fibonacci and Lucas octonions with q-integer components. Indian Journal of Pure and Applied Mathematics, 52, 231–240.

[9] Kome, C., Kome, S., Catarino, P. (2021). Quantum Calculus Approach to the Dual Bicomplex Fibonacci and Lucas Numbers. Journal of Mathematıcal Extensıon, 16, 1–17.

[10] Luna-Elizarraras, M. E., Shapiro, M., Struppa, D. C., Vajiac, A. (2012). Bicomplex numbers and their elementary functions. Cubo (Temuco), 14, 61–80.

[11] Pashaev, O. K., Nalci, S. (2011). Golden quantum oscillator and Binet–Fibonacci calculus. Journal of Physics A: Mathematical and Theoretical, 45, 015303(23pp).

[12] Price, G. B. (1991). An introduction to multicomplex spaces and functions. New York: CRC Press.

[13] Segre, C. (1892). Le rappresentazioni reali delle forme complesse a gli enti iperalgebrici. Mathematische Annalen, 40, 413–467.

[14] Wang, X. Y., Song, W. (2013). The generalized M-J sets for bicomplex numbers. Nonlinear Dynamics, 72, 17–26.