TURKISH JOURNAL OF SCIENCE
VOLUME 9, ISSUE 2, 101–125
ISSN: 2587–0971

https://www.tjoscience.com

Ruled Surfaces with N_1B_1-Smarandache Base Curve Obtained from the Successor Frame

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Abstract. In this study, ruled surfaces formed by the movement of the Frenet vectors of the successor curve along the Smarandache curve obtained from the principal normal and binormal vectors of the successor curve of a curve were defined. Then, the Gaussian and mean curvatures of each ruled surface were calculated. It has been shown that the ruled surface formed by the tangent vector of the successor curve moving along the Smarandache curve is a developable ruled surface. In addition, it was found that the surface formed by the principal normal vector of the successor curve along the smarandache curve is a minimal developable ruled surface if the principal curve is planar. Conditions are given for other surfaces to be developable or minimal surfaces. Finally, the examples of these surfaces were provided and their shapes were drawn.

1. Introduction

The image of a function with two real variables in three-dimensional space is a surface. Surfaces are used in many fields, such as architecture and engineering [1]. In 1795, Monge defined the ruled surface as the surface formed by the movement of the line along a curve. Any ruled surface is formed as a result of the continuous movement of a line along any curve. These curves are called the base curve and the director curve, respectively. The curvature of surfaces was defined by Gauss in the 19th century, and therefore it was named Gaussian curvature [2]. Gaussian curvatures are related to the dimensions of the surface [3]. Since the average curvature of the surface is a ratio, it is independent of the size of the surface. Thus far, many studies [4–15] on the Gaussian curvatures of surfaces have been conducted.

There are many special curves in differential geometry. One of them is the successor curve. This curve is defined as a new curve; such that the tangent of one curve is the principal normal of the other curve, by Menninger [16] in 2014. Later, Masal [17] investigated the relationships between the position vectors of this curve and defined Successor planes. Thus far, many studies have been conducted on this concept [18, 19]. Another special curve is the Smarandache curve defined in Minkowski space [20–23].

In recent years, many studies have been conducted on ruled surfaces whose base curve is the Smarandache curve. Some of these studies can be accessed from [24–37].

In this paper, we present some special ruled surfaces with N_1B_1-Smarandache curves obtained from their successor frames. We investigate the properties of these ruled surfaces by means of Gaussian and

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Received: 18 July 2024; Accepted: 28 August 2024; Published: 30 September 2024.

Keywords. Smarandache ruled surfaces, Sucessor curve, mean curvature, Gaussian curvature.

2010 *Mathematics Subject Classification*. 53A04, 53A05.

Cited this article as: Uzun, G., Şenyurt, S. & Akdağ Bayram, K. (2024). Ruled Surfaces with N_1B_1-Smarandache Base Curve Obtained from the Successor Frame. Turkish Journal of Science, 9(2), 101–125.

mean curvatures. We then obtain the conditions that which of these surfaces are developable and which of these are minimal. Finally, we visualise the main idea by providing four examples.

2. Preliminaries

This section provides some basic notions needed for the following sections. Throughout this paper, let $\alpha = \alpha(s)$ and $\beta = \beta(s)$ be two differentiable unit speed curves in E^3 and their Frenet apparatus be $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$, respectively. Then,

$$T = \alpha', \quad N = \frac{\alpha''}{\|\alpha''\|}, \quad B = T \wedge N, \quad \kappa = \|\alpha''\|, \quad \tau = \langle N', B \rangle,$$

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N$$

The surface formed by a line moving depending on the parameter of a curve is called a ruled surface, and its parametric expression is $X(s, v) = \alpha(s) + v r(s)$. Here, v is a constant. Besides, α and r are referred to as the base curve and the director curve of X, respectively. The normal vector field N_X, the Gaussian curvature K_X, and the mean curvature H_X of $X(s, v)$ are as follows:

$$N_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|}, \tag{1}$$

$$K_X = \frac{eg - f^2}{EG - F^2}, \quad H_X = \frac{Eg - 2fF + eG}{2(EG - F^2)}, \tag{2}$$

Here,

$$E = \langle X_s, X_s \rangle, \quad F = \langle X_s, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \tag{3}$$

$$e = \langle X_{ss}, N_X \rangle, \quad f = \langle X_{sv}, N_X \rangle, \quad g = \langle X_{vv}, N_X \rangle. \tag{4}$$

Definition 2.1. *[22, 23] If the unit tangent vector of α is the principal normal vector of β, then β is called the Successor curve of α.*

Theorem 2.2. *[22, 23] Let β be the Successor curve of α. The Frenet apparatus of β are as follows:*

$$T_1 = -cos\theta N + sin\theta B, \;\; N_1 = T, \;\; B_1 = sin\theta N + cos\theta B, \;\; \kappa_1 = \kappa cos\theta, \;\; \tau_1 = \kappa sin\theta.$$

Where θ is the angle between binormal vectors B and B_1, $\theta(s) = \theta_0 + \int \tau(s)ds$.

Definition 2.3. *[28] A regular curve in Minkowski space, whose position vector is obtained by Frenet frame vectors on another regular curve, is called a Smarandache Curve.*

Let β be the Successor curve of α. It can be observed that the unit curve γ, inspired in [23], produces Smarandache curves, for all $s \in I \subseteq \mathbb{R}$, such that

$$\gamma(s) = \frac{aT + bN + cB}{\sqrt{a^2 + b^2 + c^2}}, \;\; a, b, c \in \mathbb{R}.$$

Here, if a, b, and c are nonzero, the Smarandache curves produced by $\gamma(s)$ are denoted by TNB-Smarandache Curves. This paper consider NB-Smarandache Curves.

3. Ruled Surfaces with N_1B_1-Smarandache Base Curve Obtained from the Successor Frame

Definition 3.1. *Let the successor curve of the curve α be β. The ruled surface formed by tangent vector T_1 along the N_1B_1-Smarandache curve obtained from the principal normal vector N_1 and binormal vector B_1 of the curve β as follows:*

$$\Phi(s,v)=\frac{1}{\sqrt{2}}(N_1+B_1)+vT_1=\frac{1}{\sqrt{2}}(T+sin\theta N+cos\theta B)+v(-cos\theta N+sin\theta B). \tag{5}$$

Theorem 3.2. *Let the successor curve of the curve α be β. The Gaussian and mean curvature of the ruled surface $\Phi(s,v)$ are as follows:*

$$K_\Phi=\frac{cos^2\theta sin^2\theta}{sin^2\theta+(v\sqrt{2}cos\theta-sin\theta)^2},$$

$$H_\Phi=\frac{\kappa^2\Big(2\sqrt{2}cos^2\theta sin^2\theta+sin\theta(2v^2cos^2\theta-sin^2\theta-1)\Big)+\kappa'\Big(2sin\theta(v\sqrt{2}cos\theta-sin\theta)\Big)-\kappa\tau\Big(sin\theta(cos\theta+v\sqrt{2}sin\theta)\Big)}{\kappa\sqrt{2}(sin^2\theta+(v\sqrt{2}cos\theta-sin\theta)^2)}.$$

Proof. Partial derivatives of Equation (5) are,

$$\Phi_s=\frac{\kappa((v\sqrt{2}cos\theta-sin\theta)T+N)}{\sqrt{2}},\quad \Phi_{sv}=\kappa cos\theta T,\quad \Phi_v=-cos\theta N+sin\theta B,\quad \Phi_{vv}=0,$$

$$\Phi_{ss}=-\frac{\Big(\kappa^2+\kappa'(sin\theta-v\sqrt{2}cos\theta)+\kappa\tau(cos\theta+v\sqrt{2}sin\theta)\Big)T+(\kappa'+\kappa^2(sin\theta+v\sqrt{2}cos\theta))N-\kappa\tau B}{\sqrt{2}}.$$

Thus, from Equation (1) the normal of the surface N_Φ is given as

$$N_\Phi=\frac{sin\theta T-sin\theta(v\sqrt{2}cos\theta-sin\theta)N-cos\theta(v\sqrt{2}cos\theta-sin\theta)B}{\Big(sin^2\theta+(v\sqrt{2}cos\theta-sin\theta)^2\Big)^{\frac{1}{2}}}.$$

Moreover, in Equations (3) and (4) the coefficients of fundamental forms are

$$E_\Phi=\frac{\kappa^2\Big((v\sqrt{2}cos\theta-sin\theta)^2+1\Big)}{2},\quad F_\Phi=-\frac{\kappa cos\theta}{\sqrt{2}},\quad G_\Phi=1,\quad f_\Phi=\frac{\kappa cos\theta sin\theta}{\Big(sin^2\theta+(v\sqrt{2}cos\theta-sin\theta)^2\Big)^{\frac{1}{2}}},\quad g_\Phi=0,$$

$$e_\Phi=\frac{\kappa^2\Big(sin\theta(2v^2cos^2\theta-sin^2\theta-1)\Big)+\kappa'\Big(2sin\theta(v\sqrt{2}cos\theta-sin\theta)\Big)-\kappa\tau\Big(sin\theta(cos\theta+v\sqrt{2}sin\theta)\Big)}{\sqrt{2}\Big(sin^2\theta+(v\sqrt{2}cos\theta-sin\theta)^2\Big)^{\frac{1}{2}}}$$

respectively. Thus, by using Equation (2) the Gaussian and mean curvatures are found. □

Corollary 3.3. *If $\theta=k\pi$ $(k\in\mathbb{N})$, the ruled surface $\Phi(s,v)$ is the minimal developable surface.*

Definition 3.4. *Let the successor curve of the curve α be β. The ruled surface formed by principal normal vector N_1 along the N_1B_1-Smarandache curve obtained from the principal normal vector N_1 and binormal vector B_1 of the curve β as follows:*

$$Q(s,v)=\frac{1}{\sqrt{2}}(N_1+B_1)+vN_1=\frac{1}{\sqrt{2}}(T+sin\theta N+cos\theta B)+vT. \tag{6}$$

Theorem 3.5. *Let the successor curve of the curve α be β. The Gaussian and mean curvature of the ruled surface $Q(s,v)$ are as follows:*

$$K_Q = 0, \quad H_Q = \frac{\tau(1 - v\sqrt{2})}{\kappa(1 + v\sqrt{2})}.$$

Proof. Partial derivatives of Equation (6) are,

$$Q_s = \frac{\kappa(-sin\theta T + (1 + v\sqrt{2})N)}{\sqrt{2}}, \quad Q_v = T, \quad Q_{sv} = \kappa N, \quad Q_{vv} = 0,$$

$$Q_{ss} = \frac{(\kappa^2(v\sqrt{2} - 1) - \kappa' sin\theta - \kappa\tau cos\theta)T + (\kappa'(1 + v\sqrt{2}) - \kappa^2 sin\theta)N + \kappa\tau(1 - v\sqrt{2})B}{\sqrt{2}}.$$

Thus, from Equation (1) the normal of the surface N_Q is given as $N_Q = -B$. Moreover, in equaitons 3 and 4 the coefficients of fundamental forms are

$$E_Q = \frac{\kappa^2\left(sin^2\theta + (1 + v2)^2\right)}{2}, \quad F_Q = -\frac{\kappa sin\theta}{\sqrt{2}}, \quad G_Q = 1, \quad e_Q = \kappa\tau(1 - v\sqrt{2}), \quad f_Q = g_Q = 0$$

respectively. Thus, by using Equation (2) the Gaussian and mean curvatures are found. □

Corollary 3.6. *Let the successor curve of the α curve be β. If α curve is planar, the ruled surface $Q(s,v)$ is the minimal developable surface.*

Definition 3.7. *Let the successor curve of the curve α be β. The ruled surface formed by binormal vector B_1 along the N_1B_1-Smarandache curve obtained from the principal normal vector N_1 and binormal vector B_1 of the curve β as follows:*

$$M(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vB_1 = \frac{1}{\sqrt{2}}(T + sin\theta N + cos\theta B) + v(sin\theta N + cos\theta B). \tag{7}$$

Theorem 3.8. *Let the successor curve of the curve α be β. The Gaussian and mean curvature of the ruled surface $M(s,v)$ are as follows:*

$$K_M = -\frac{2cos^2\theta sin^2\theta}{((1 + v\sqrt{2})^2 sin^2\theta + cos^2\theta)^2},$$

$$H_M = \frac{\kappa^2 cos\theta((1 + v\sqrt{2})^2 sin^2\theta + 1) - \kappa'(1 + v\sqrt{2})sin2\theta - \kappa\tau(1 + v\sqrt{2})}{\sqrt{2}\kappa^2(cos^2\theta + (1 + v\sqrt{2})^2 sin^2\theta)^{\frac{3}{2}}}.$$

Proof. Partial derivatives of Equation (7) are,

$$M_s = \frac{\kappa(-(1 + v\sqrt{2})sin\theta T + N)}{\sqrt{2}}, \quad M_v = sin\theta N + cos\theta B, \quad M_{vv} = 0, \quad M_{sv} = -\kappa sin\theta T,$$

$$M_{ss} = \frac{-(\kappa^2 + (1 + v\sqrt{2})(\kappa' sin\theta + \kappa\tau cos\theta))T + (\kappa' - \kappa^2(1 + v\sqrt{2})sin\theta)N + \kappa\tau B}{\sqrt{2}}.$$

Thus, from Equation (1) the normal of the surface N_M is given as

$$N_M = \frac{cos\theta T - ((1 + v\sqrt{2})sin\theta cos\theta)N - ((1 + v\sqrt{2})sin^2\theta)B}{\left(cos^2\theta + (1 + v\sqrt{2})^2 sin^2\theta\right)^{\frac{1}{2}}}.$$

Moreover, Equations (3) and (4) the coefficients of fundamental forms are

$$E_M = \frac{\kappa^2((1+v\sqrt{2})^2 sin^2\theta + 1)}{2}, \quad F_M = \frac{\kappa sin\theta}{\sqrt{2}}, \quad G_M = 1, \quad g_M = 0,$$

$$e_M = \frac{\kappa^2 cos\theta((1+v\sqrt{2})^2 sin^2\theta - 1) - \kappa'(1+v\sqrt{2})sin2\theta - \kappa\tau(1+v\sqrt{2})}{\sqrt{2}\Big(cos^2\theta + (1+v\sqrt{2})^2 sin^2\theta\Big)^{\frac{1}{2}}}, \quad f_M = \frac{-\kappa sin\theta cos\theta}{\Big(cos^2\theta + (1+v\sqrt{2})^2 sin^2\theta\Big)^{\frac{1}{2}}}$$

respectively. Thus, by using Equation (2) the Gaussian and mean curvatures are found. □

Corollary 3.9. *If the $\theta = \frac{k\pi}{2}$ $(k \in \mathbb{Z})$, the ruled surface $M(s,v)$ is a developable surface.*

Definition 3.10. *Let the successor curve of the curve α be β. The ruled surface formed by the vector T_1N_1 along the N_1B_1-Smarandache curve obtained from the principal normal vector N_1 and binormal vector B_1 of the curve β as follows:*

$$\mu(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1) = \frac{1}{\sqrt{2}}(T + sin\theta N + cos\theta B) + \frac{v}{\sqrt{2}}(T - cos\theta N + sin\theta B). \tag{8}$$

Theorem 3.11. *Let the successor curve of the curve α be β. The Gaussian and mean curvature of the ruled surface $\mu(s,v)$ are as follows:*

$$K_\mu = \frac{sin^2\theta(cos\theta + sin\theta)^2}{\sqrt{2}(1+v)\Big((1+v)sin^2\theta + 2cos\theta(vcos\theta - sin\theta)\Big).\Big(sin^2\theta\Big((vcos\theta - sin\theta)^2 + (1+v)^2\Big) + \Big(cos\theta(vcos\theta - sin\theta) + (1+v)\Big)^2\Big)^{\frac{1}{2}}},$$

$$H_\mu = -\frac{\sqrt{2}\tau\Big(v(v + sin2\theta + (sin^2\theta - cos^2\theta)) + (1-v^2)(cos\theta sin\theta - 1) - \sqrt{2}\kappa\Big((1+v)(cos\theta + sin\theta)^2\Big((1+v)sin^2\theta(v^2cos^2\theta - sin^2\theta)\Big)\Big)}{\sqrt{2}(1+v)\Big((1+v)sin^2\theta + 2cos\theta(vcos\theta - sin\theta)\Big).\Big(sin^2\theta\Big((vcos\theta - sin\theta)^2 + (1+v)^2\Big) + \Big(cos\theta(vcos\theta - sin\theta) + (1+v)\Big)^2\Big)^{\frac{1}{2}}}.$$

Proof. Partial derivatives of Equation (8) are,

$$\mu_s = \frac{\kappa\Big((vcos\theta - sin\theta)T + (1+v)N\Big)}{\sqrt{2}}, \quad \mu_{sv} = \frac{\kappa(cos\theta T + N)}{\sqrt{2}}, \quad \mu_v = \frac{T - cos\theta N + sin\theta B}{\sqrt{2}}, \quad \mu_{vv} = 0,$$

$$\mu_{ss} = \frac{-\Big(\kappa^2(1+v) + \kappa'(sin\theta - vcos\theta) + \kappa\tau(cos\theta + vsin\theta)\Big)T + \Big(\kappa'(1+v) - \kappa^2(sin\theta + vcos\theta\Big)N + \kappa\tau(1-v)B}{\sqrt{2}},$$

Thus, from Equation (1) the normal of the surface N_μ is given as

$$N_\mu = \frac{(1+v)sin\theta T - sin\theta(vcos\theta - sin\theta)N - (cos\theta(vcos\theta - sin\theta) + (1+v))B}{\Big(sin^2\theta\Big((vcos\theta - sin\theta)^2 + (1+v)^2\Big) + \Big(cos\theta(vcos\theta - sin\theta) + (1+v)\Big)^2\Big)^{\frac{1}{2}}}.$$

Moreover, in Equations (3) and (4) the coefficients of fundamental forms are

$$E_\mu = \frac{\kappa^2}{2}((vcos\theta - sin\theta)^2 + (1+v)^2), \quad F_\mu = \frac{\kappa}{2}((vcos\theta - sin\theta) - cos\theta(1+v)), \quad G_\mu = 1,$$

$$e_\mu = \frac{\kappa^2 sin\theta\Big((1+v)^2 - v^2cos^2\theta + sin^2\theta\Big) + \kappa'(1+v)\Big(sin\theta(sin\theta - vcos\theta) + 1\Big) + \kappa\tau(1+v)\Big(sin\theta(cos\theta + vsin\theta) + (1-v)\Big)}{\Big(sin^2\theta\Big((vcos\theta - sin\theta)^2 + (1+v)^2\Big) + \Big(cos\theta(vcos\theta - sin\theta) + (1+v)\Big)^2\Big)^{\frac{1}{2}}},$$

$$f_\mu = \frac{\kappa sin\theta(cos\theta + sin\theta)}{\sqrt{2}\Big(sin^2\theta\Big((vcos\theta - sin\theta)^2 + (1+v)^2\Big) + \Big(cos\theta(vcos\theta - sin\theta) + (1+v)\Big)^2\Big)^{\frac{1}{2}}}, \quad g_\mu = 0.$$

respectively. Thus, by using Equation (2) the Gaussian and mean curvatures are found. □

Corollary 3.12. *If the $\theta = k\pi$ $(k \in \mathbb{Z})$ or $\theta = k\pi + \frac{3}{4}$ $(k \in \mathbb{Z})$, the ruled surface $\mu(s,v)$ is a developable surface.*

Definition 3.13. *Let the successor curve of the curve α be β. The ruled surface formed by the vector T_1B_1 along the N_1B_1-Smarandache curve obtained from the principal normal vector N_1 and binormal vector B_1 of the curve β as follows:*

$$\psi(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1) = \frac{1}{\sqrt{2}}(T + sin\theta N + cos\theta B) + \frac{v}{\sqrt{2}}\Big((sin\theta - cos\theta)N + (sin\theta + cos\theta)B\Big). \tag{9}$$

Theorem 3.14. *Let the successor curve of the curve α be β. The Gaussian and mean curvature of the ruled surface $\psi(s,v)$ are as follows:*

$$K_\psi = -\frac{2(cos^2\theta - sin^2\theta)^2}{\Big(1 + sin2\theta + 2\big(v(cos\theta - sin\theta) - sin\theta\big)^2\Big)^2}, \quad H_\psi = -\frac{(cos^2\theta - sin^2\theta)(sin\theta - cos\theta)}{\sqrt{2}\Big(1 + sin2\theta + 2\big(v(cos\theta - sin\theta) - sin\theta\big)^2\Big)^{\frac{3}{2}}}.$$

Proof. Partial derivatives of Equation (9) are,

$$\psi_s = \frac{\kappa\Big(\big(v(cos\theta - sin\theta) - sin\theta\big)T + N\Big)}{\sqrt{2}}, \quad \psi_{sv} = \frac{\kappa(cos\theta - sin\theta)T}{\sqrt{2}}, \quad \psi_v = \frac{(sin\theta - cos\theta)N + (sin\theta + cos\theta)B}{\sqrt{2}}, \quad \psi_{vv} = 0,$$

$$\psi_{ss} = \frac{\Big(\kappa'\big(v(cos\theta - sin\theta) - sin\theta\big) - \kappa^2 - \kappa\tau\big(cos\theta + v(sin\theta + cos\theta)\big)\Big)T + \Big(\kappa' - \kappa^2\big((sin\theta - v(cos\theta - sin\theta)\big)\Big)N + \kappa\tau B}{\sqrt{2}}.$$

Thus, from Equation (1) the normal of the surface N_ψ is given as

$$N_\psi = \frac{(sin\theta + cos\theta)T - \big(v(cos\theta - sin\theta) - sin\theta\big)(sin\theta + cos\theta)N + \big(v(cos\theta - sin\theta) - sin\theta\big)(sin\theta - cos\theta)B}{\Big((sin\theta + cos\theta)^2 + 2\big(v(cos\theta - sin\theta) - sin\theta\big)^2\Big)^{\frac{1}{2}}}$$

Moreover, in Equations (3) and (4) the coefficients of fundamental forms are

$$E_\psi = \frac{\kappa^2\Big(\big(v(cos\theta - sin\theta) - sin\theta\big)^2 + 1\Big)}{2}, \quad F_\psi = \frac{\kappa(sin\theta - cos\theta)}{2}, \quad G_\psi = 1,$$

$$e_\psi = \frac{-\kappa^2\Big((sin\theta + cos\theta)(v(cos\theta - sin\theta) - sin\theta)^2 + 1\Big) - \kappa\tau\Big(sin2\theta(2v - 1) - 1\Big)}{\sqrt{2}\Big((sin\theta + cos\theta)^2 + 2\big(v(cos\theta - sin\theta) - sin\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$f_\psi = \frac{\kappa(cos^2\theta - sin^2\theta)}{\sqrt{2}\Big((sin\theta + cos\theta)^2 + 2\big(v(cos\theta - sin\theta) - sin\theta\big)^2\Big)^{\frac{1}{2}}}, \quad g_\psi = 0$$

respectively. Thus, by using Equation (2) the Gaussian and mean curvatures are found. □

Corollary 3.15. *If the $\theta = \pi + k\pi$ $(k \in \mathbb{N})$, the ruled surface $\psi(s,v)$ is a developable surface.*

Definition 3.16. *Let the successor curve of the curve α be β. The ruled surface formed by the vector N_1B_1 along the N_1B_1-Smarandache curve obtained from the principal normal vector N_1 and binormal vector B_1 of the curve β as follows:*

$$\eta(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(N_1 + B_1) = \frac{1}{\sqrt{2}}(T + sin\theta N + cos\theta B) + \frac{v}{\sqrt{2}}(T + sin\theta N + cos\theta B). \tag{10}$$

Theorem 3.17. *Let the successor curve of the curve α be β. The Gaussian and mean curvature of the ruled surface $\Gamma(s,v)$ are as follows:*

$$K_\eta = 0, \quad H_\eta = \frac{\tau}{2\kappa(1 + v)(1 + sin^2\theta)}.$$

Proof. Partial derivatives of Equation (10) are,

$$\eta_s = \frac{\kappa(1+v)(-sin\theta T + N)}{\sqrt{2}}, \quad \eta_{sv} = \frac{\kappa(sin\theta T + N)}{\sqrt{2}}, \quad \eta_v = \frac{T + sin\theta N + cos\theta B}{\sqrt{2}},$$

$$\eta_{ss} = \frac{-(1+v)\Big((\kappa' sin\theta + \kappa\tau cos\theta + \kappa^2)T + (\kappa^2 sin\theta - \kappa')N - \kappa\tau B\Big)}{\sqrt{2}}, \quad \eta_{vv} = 0.$$

Thus, from Equation (1) the normal of the surface N_η is given as $N_\eta = -B$. Moreover, in Equations (3) and (4) the coefficients of fundamental forms are

$$E_\eta = \frac{\kappa^2(1+v)^2(sin^2\theta + 1)}{2}, \quad F_\eta = 0, \quad G_\eta = 1, \quad e_\eta = \frac{\kappa\tau(1+v)}{\sqrt{2}}, \quad f_\eta = g_\eta = 0$$

respectively. Thus, by using Equation (2) the Gaussian and mean curvatures are found. □

Corollary 3.18. *If α curve is planar, the ruled surface $\eta(s,v)$ is the minimal developable surface.*

Definition 3.19. *Let the successor curve of the curve α be β. The ruled surface formed by $T_1N_1B_1$ the vector along the N_1B_1-Smarandache curve obtained from the principal normal vector N_1 and binormal vector B_1 of the β curve as follows:*

$$\begin{aligned}\Gamma(s,v) &= \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{3}}(T_1 + N_1 + B_1) \\ &= \frac{1}{\sqrt{2}}(T + sin\theta N + cos\theta B) + \frac{v}{\sqrt{3}}\Big(T + (sin\theta - cos\theta)N + (sin\theta + cos\theta)B\Big).\end{aligned} \tag{11}$$

Theorem 3.20. *Let β be the Successor curve of α. The Gaussian and mean curvature of the $\Gamma(s,v)$ ruled surface are as follows:*

$$K_\Gamma = \frac{6cos^2\theta(sin2\theta - 1)}{\begin{pmatrix}(\sqrt{3}+v\sqrt{2})^2(sin\theta+cos\theta)^2 + \Big((v\sqrt{2}(cos\theta - sin\theta) - \sqrt{3}sin\theta\Big)^2(sin\theta+cos\theta)^2 \\ +(sin\theta - cos\theta)^2\Big(v\sqrt{2}(cos\theta - sin\theta) - \sqrt{3}sin\theta - 1\Big)^2\end{pmatrix} \\ .\Big(\Big(v\sqrt{2}(cos\theta - sin\theta) - \sqrt{3}sin\theta\Big)^2 + (\sqrt{3}+v\sqrt{2})^2 - cos^2\theta\Big)},$$

$$H_\Gamma = \frac{\begin{matrix}\kappa^2(sin\theta+cos\theta)\Big(\Big(2v^2(1-sin2\theta) - 3sin^2\theta\Big) - (\sqrt{3}+v\sqrt{2})^2 + 2\sqrt{6}\kappa^2cos^2\theta\Big) \\ +\kappa'\Big(2\sqrt{3}(\sqrt{3}+v\sqrt{2})(sin\theta+cos\theta)sin\theta\Big) + \kappa\tau(\sqrt{3}+v\sqrt{2})(-2v\sqrt{2} - \sqrt{3} - sin\theta + cos\theta)\end{matrix}}{2\kappa^2\begin{pmatrix}(\sqrt{3}+v\sqrt{2})^2(sin\theta+cos\theta)^2 + \Big((v\sqrt{2}(cos\theta - sin\theta) - \sqrt{3}sin\theta\Big)^2(sin\theta+cos\theta)^2 \\ +(sin\theta - cos\theta)^2\Big(v\sqrt{2}(cos\theta - sin\theta) - \sqrt{3}sin\theta - 1\Big)^2\end{pmatrix}^{\frac{1}{2}} \\ .\Big(\Big(v\sqrt{2}(cos\theta - sin\theta) - \sqrt{3}sin\theta\Big)^2 + (\sqrt{3}+v\sqrt{2})^2 - cos^2\theta\Big)}.$$

Proof. Partial derivatives of Equation (11) are,

$$\Gamma_s = \frac{\kappa\Big(\Big(v\sqrt{2}(cos\theta - sin\theta) - \sqrt{3}sin\theta\Big)T + (\sqrt{3}+v\sqrt{2})N\Big)}{\sqrt{6}}, \quad \Gamma_{sv} = \frac{\kappa\Big((cos\theta - sin\theta)T + N\Big)}{\sqrt{3}},$$

$$\Gamma_v = \frac{T + (sin\theta - cos\theta)N + (sin\theta+cos\theta)B}{\sqrt{3}}, \quad \Gamma_{vv} = 0,$$

$$\Gamma_{ss} = \frac{\begin{matrix}-\Big(\kappa^2(\sqrt{3}+v\sqrt{2}) + \kappa'(\sqrt{3}sin\theta + v\sqrt{2}(cos\theta - sin\theta)) + \kappa\tau(\sqrt{3}cos\theta + v\sqrt{2}(sin\theta+cos\theta))\Big)T \\ +\Big(\kappa'(\sqrt{3}+v\sqrt{2}) - \kappa^2(\sqrt{3}sin\theta + v\sqrt{2}(cos\theta - sin\theta)))\Big)N + \kappa\tau(\sqrt{3}+v\sqrt{2})B\end{matrix}}{\sqrt{6}}.$$

Thus, from Equation (1) the normal of the surface N_Γ is given as

$$N_\Gamma = \frac{\begin{array}{c}\left((\sqrt{3}+v\sqrt{2})(sin\theta+cos\theta)\right)T-\left((v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta\right)(sin\theta+cos\theta)N\\ +(sin\theta-cos\theta)\left(v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta-1\right)B\end{array}}{\left(\begin{array}{c}(\sqrt{3}+v\sqrt{2})^2(sin\theta+cos\theta)^2+\left((v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta\right)^2(sin\theta+cos\theta)^2\\ +(sin\theta-cos\theta)^2\left(v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta-1\right)^2\end{array}\right)^{\frac{1}{2}}}.$$

Moreover, in Equations (3) and (4) the coefficients of fundamental forms are

$$E_\Gamma = \frac{\kappa^2\left(\left(v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta\right)^2+(\sqrt{3}+v\sqrt{2})^2\right)}{6},\quad F_\Gamma=\frac{-\kappa cos\theta}{\sqrt{6}},\quad G_\Gamma=1,$$

$$e_\Gamma = \frac{\begin{array}{l}\kappa^2(sin\theta+cos\theta)\left(\left(2v^2(1-sin2\theta)-3sin^2\theta\right)-(\sqrt{3}+v\sqrt{2})^2\right)+\kappa'\left(2\sqrt{3}(\sqrt{3}+v\sqrt{2})(sin\theta+cos\theta)sin\theta\right)\\ +\kappa\tau(\sqrt{3}+v\sqrt{2})(-2v\sqrt{2}-\sqrt{3}-sin\theta+cos\theta)\end{array}}{\sqrt{6}\left(\begin{array}{c}(\sqrt{3}+v\sqrt{2})^2(sin\theta+cos\theta)^2+\left((v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta\right)^2(sin\theta+cos\theta)^2\\ +(sin\theta-cos\theta)^2\left(v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta-1\right)^2\end{array}\right)^{\frac{1}{2}}},$$

$$f_\Gamma = \frac{\kappa cos\theta(cos\theta+sin\theta)}{\left(\begin{array}{c}(\sqrt{3}+v\sqrt{2})^2(sin\theta+cos\theta)^2+\left((v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta\right)^2(sin\theta+cos\theta)^2\\ +(sin\theta-cos\theta)^2\left(v\sqrt{2}(cos\theta-sin\theta)-\sqrt{3}sin\theta-1\right)^2\end{array}\right)^{\frac{1}{2}}},\quad g_\Gamma=0$$

respectively. Thus, by using Equation (2) the Gaussian and mean curvatures are found. □

Corollary 3.21. *If the $\theta=\frac{\pi}{2}+k\pi$ $(k\in\mathbb{N})$, the ruled surface $\Gamma(s,v)$ is a developable surface.*

Example 3.22. *Let β Salkowski curve [31] be the Successor curve of α. The equation of this curve for $m=\frac{1}{3}$ is as follows:*

$$\beta(s)=\frac{3}{\sqrt{10}}\left(\begin{array}{c}-\frac{\sqrt{10}-1}{4\sqrt{10}+8}\left(sin(\frac{\sqrt{10}+2}{\sqrt{10}})s\right)-\frac{\sqrt{10}-1}{4\sqrt{10}-8}\left(sin(\frac{\sqrt{10}-2}{\sqrt{10}})s\right)-\frac{1}{2}sins,\\ -\frac{\sqrt{10}-1}{4\sqrt{10}+8}\left(cos(\frac{\sqrt{10}+2}{\sqrt{10}})s\right)+\frac{\sqrt{10}-1}{4\sqrt{10}-8}\left(cos(\frac{\sqrt{10}-2}{\sqrt{10}})s\right)+\frac{1}{2}coss,\ \frac{3}{4}cos(\frac{2s}{\sqrt{10}})\end{array}\right)$$

The Successor frames of β curve $\{T_1,N_1,B_1\}$ are as follows:

$$T_1(s)=\left(-cosscos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}sinssin\frac{s}{\sqrt{10}},\ -sinscos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}cosssin\frac{s}{\sqrt{10}},\ \frac{3}{\sqrt{10}}sin\frac{s}{\sqrt{10}}\right),$$

$$N_1(s)=\left(\frac{3}{\sqrt{10}}sins,\ -\frac{3}{\sqrt{10}}coss,\ -\frac{1}{\sqrt{10}}\right),$$

$$B_1(s)=\left(-cosssin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}sinscos\frac{s}{\sqrt{10}},\ -sinscos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}cosscos\frac{s}{\sqrt{10}},\ \frac{3}{\sqrt{10}}cos\frac{s}{\sqrt{10}}\right)$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures 1- 7;

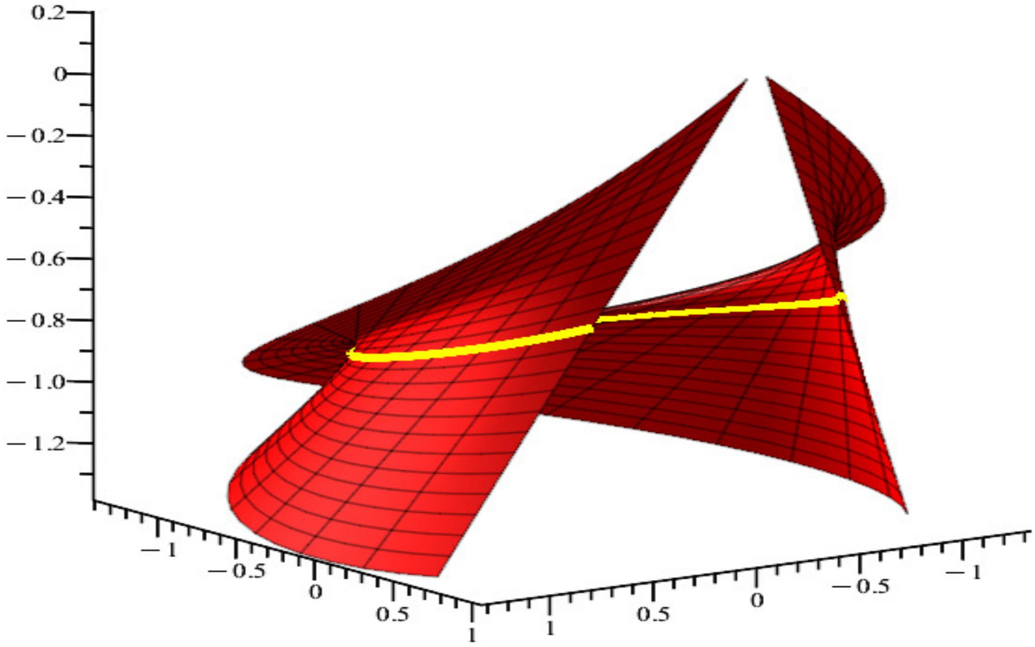

Figure 1: The ruled surface $\Phi(s, v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vT_1$

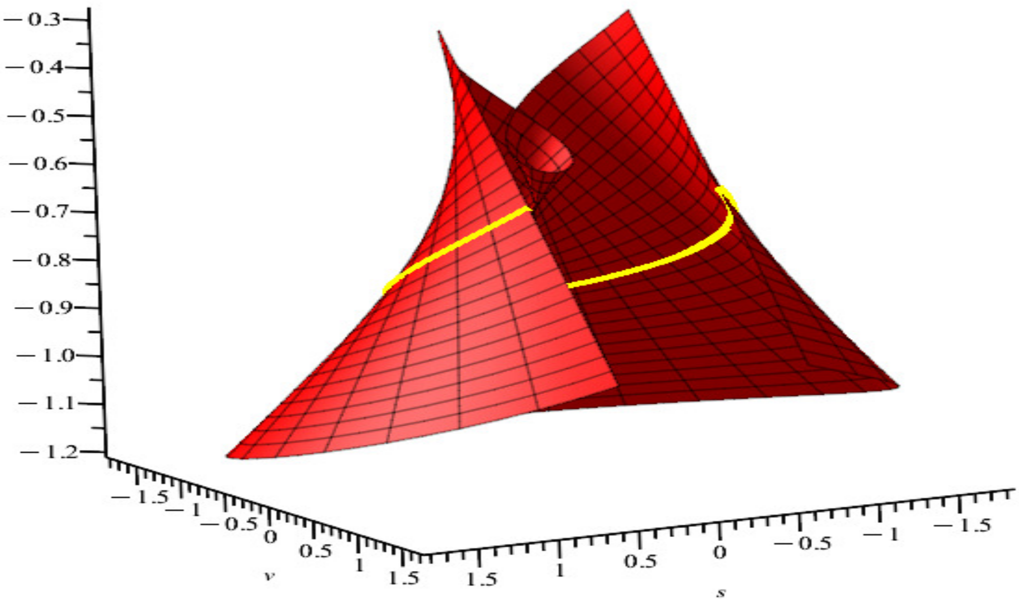

Figure 2: The ruled surface $Q(s, v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vN_1$

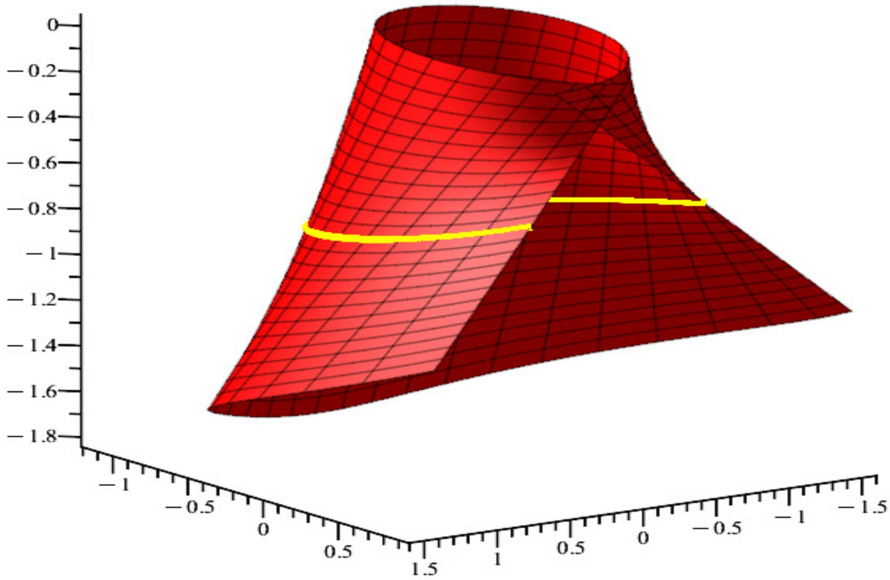

Figure 3: The ruled surface $M(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vB_1$

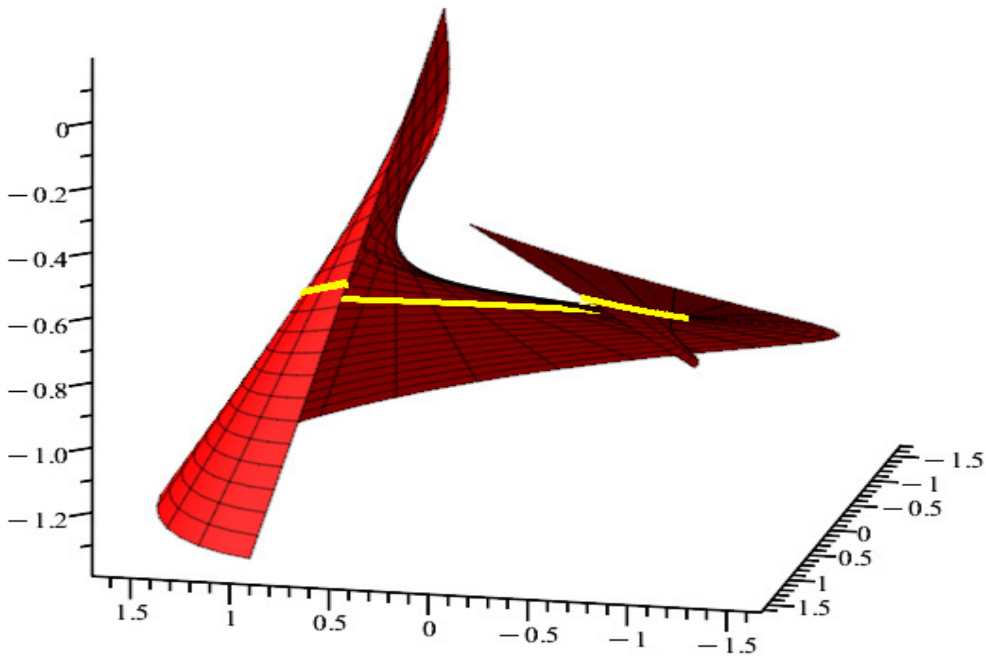

Figure 4: The ruled surface $\mu(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

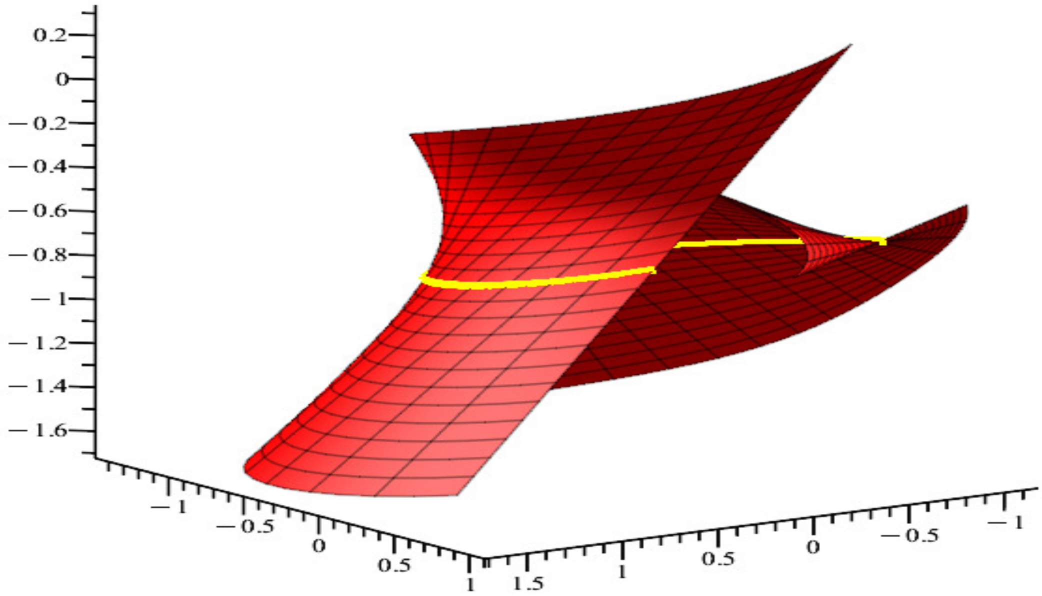

Figure 5: The ruled surface $\psi(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

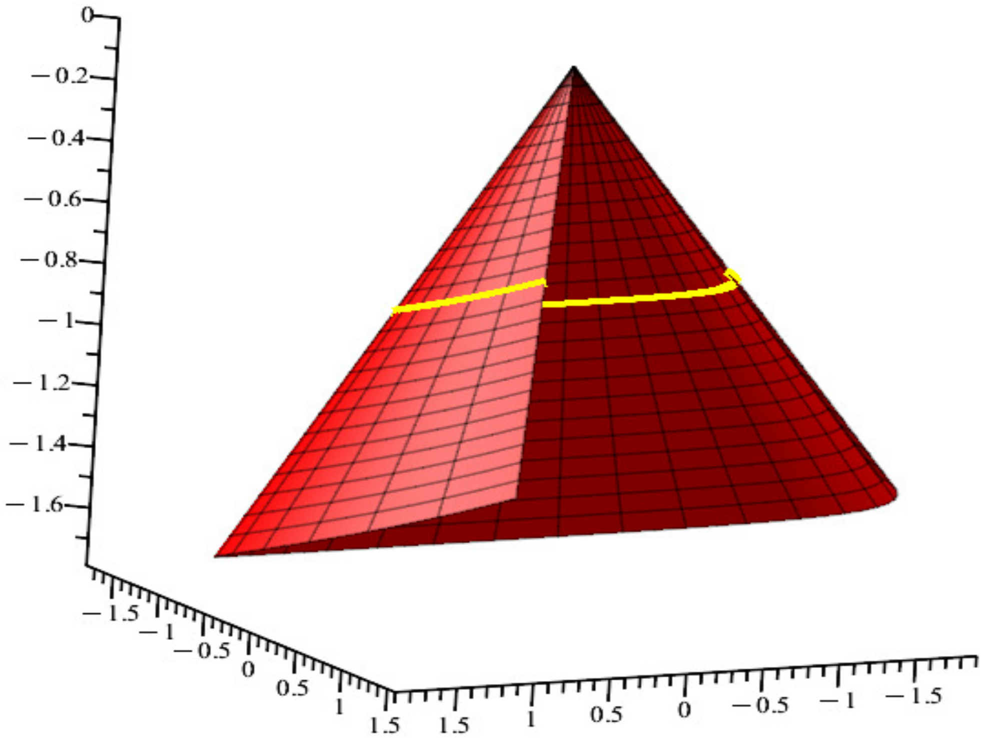

Figure 6: The ruled surface $\eta(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(N_1 + B_1)$

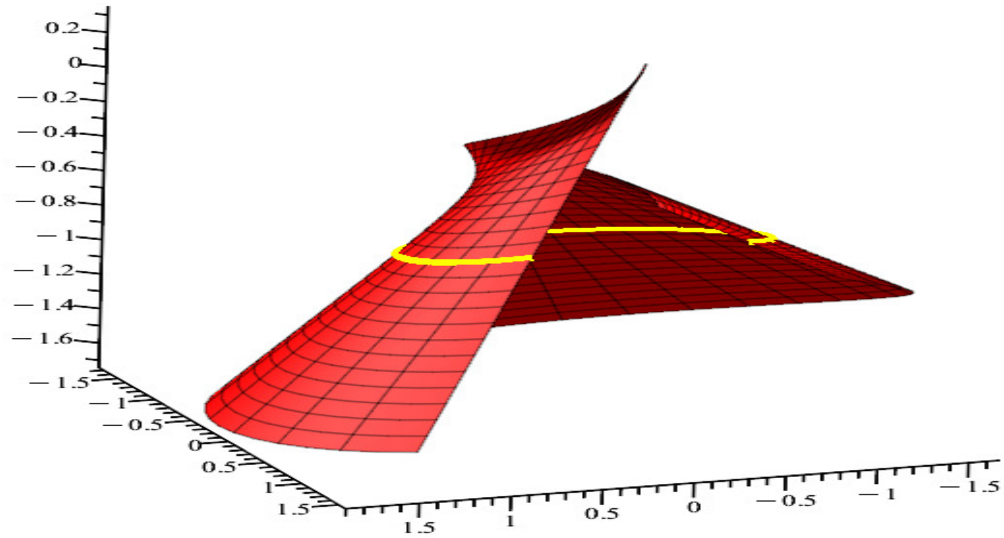

Figure 7: The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{2}}(N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

Example 3.23. *Let the Salkowski curve in Example 3.22 be the main curve. From [31] and Theorem 2.2 the Successor frames are as follows:*

$$T_1(s)=\left(\begin{array}{c} -\cos\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\frac{3}{\sqrt{10}}\sin s\Big)+\sin\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(-\cos s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\Big),\\ \cos\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\frac{3}{\sqrt{10}}\cos s\Big)-\sin\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(-\sin s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\Big),\\ \cos\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\frac{1}{\sqrt{10}}+\sin\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\Big) \end{array}\right),$$

$$N_1(s)=\Big(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\Big),$$

$$B_1(s)=\left(\begin{array}{c} \sin\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\frac{3}{\sqrt{10}}\sin s\Big)-\cos\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\Big),\\ -\sin\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\frac{3}{\sqrt{10}}\cos s\Big)-\cos\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\Big),\\ -\sin\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\frac{1}{\sqrt{10}}+\cos\Big(\int \tan\frac{s}{\sqrt{10}}ds\Big)\Big(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\Big) \end{array}\right).$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures 8-14;

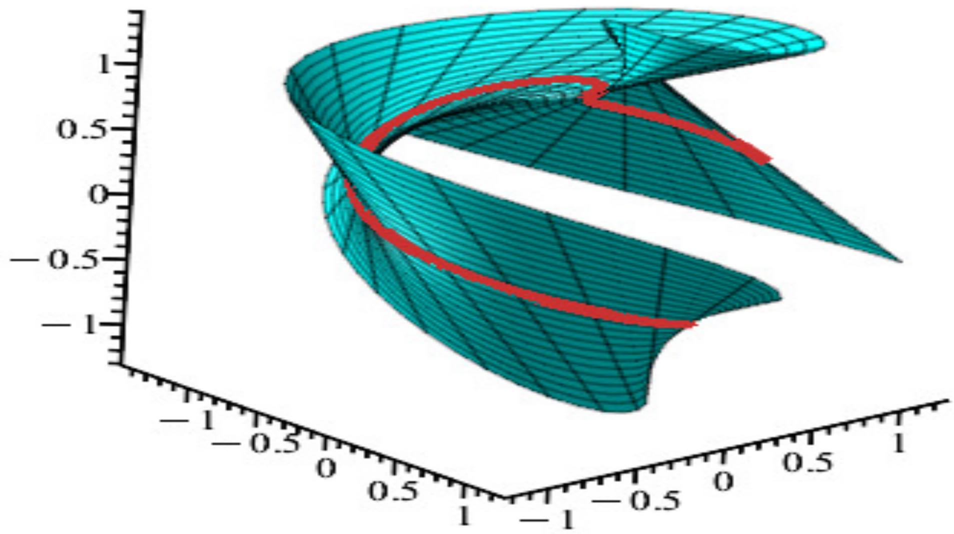

Figure 8: The ruled surface $\Phi(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vT_1$

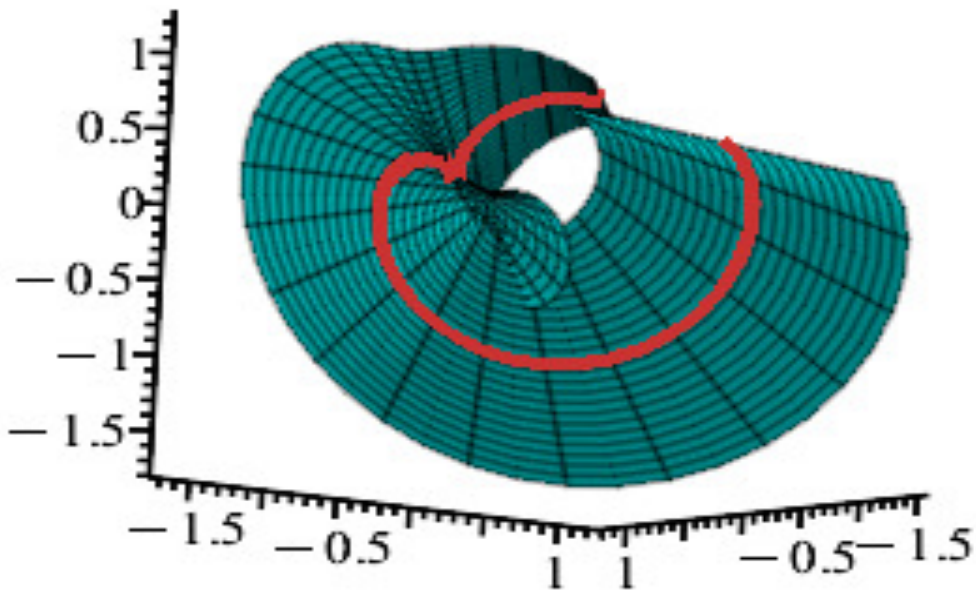

Figure 9: The ruled surface $Q(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vN_1$

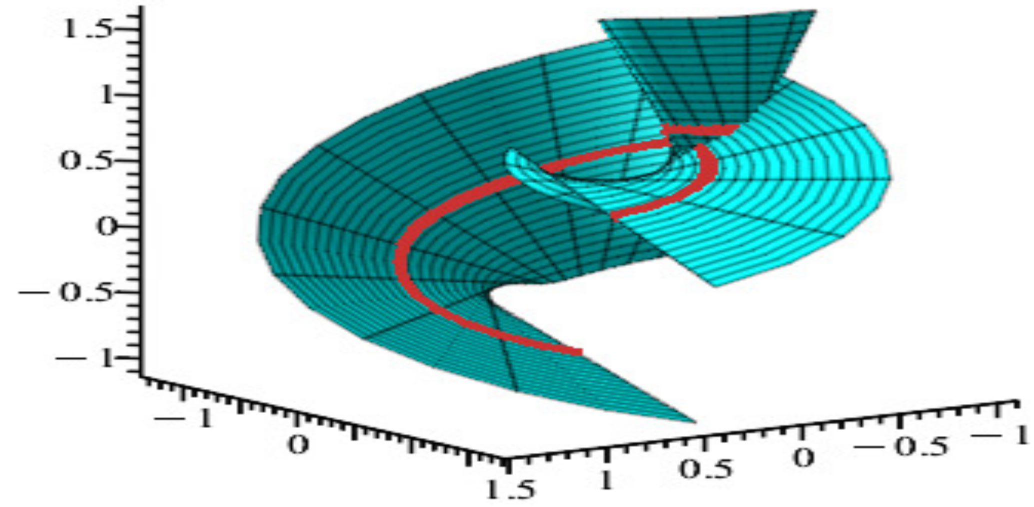

Figure 10: The ruled surface $M(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vB_1$

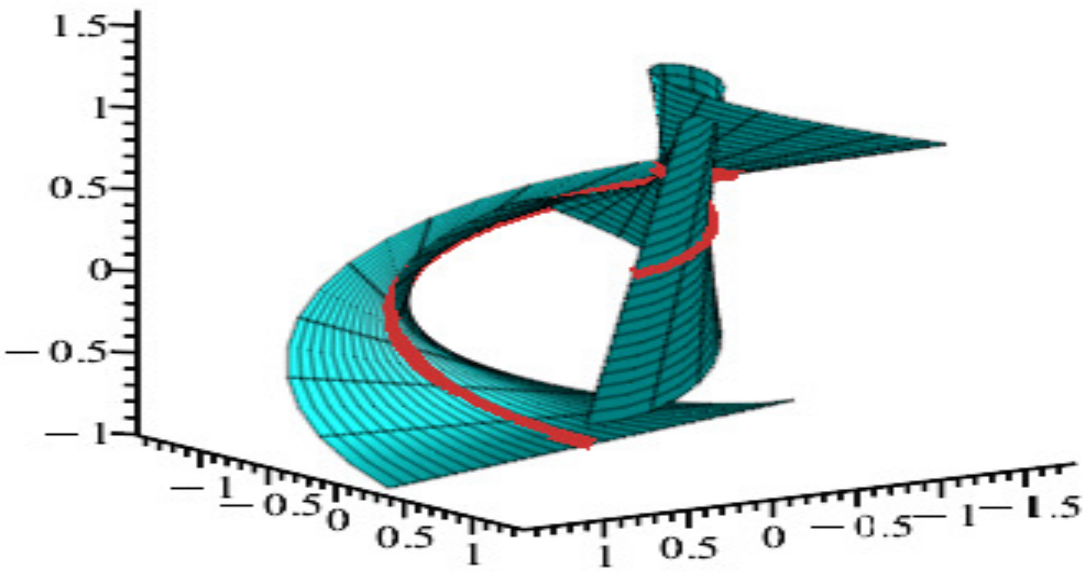

Figure 11: The ruled surface $\mu(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

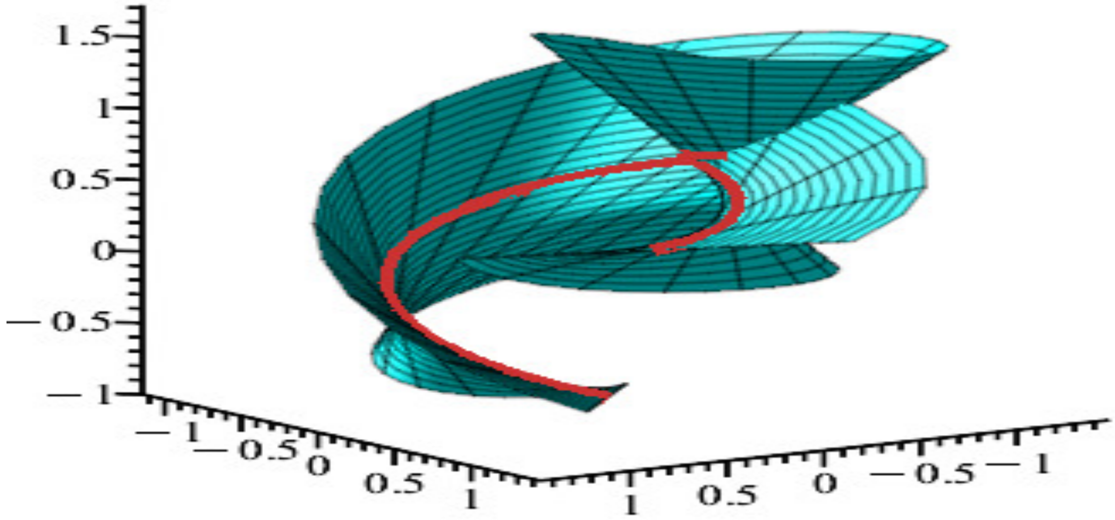

Figure 12: The ruled surface $\psi(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

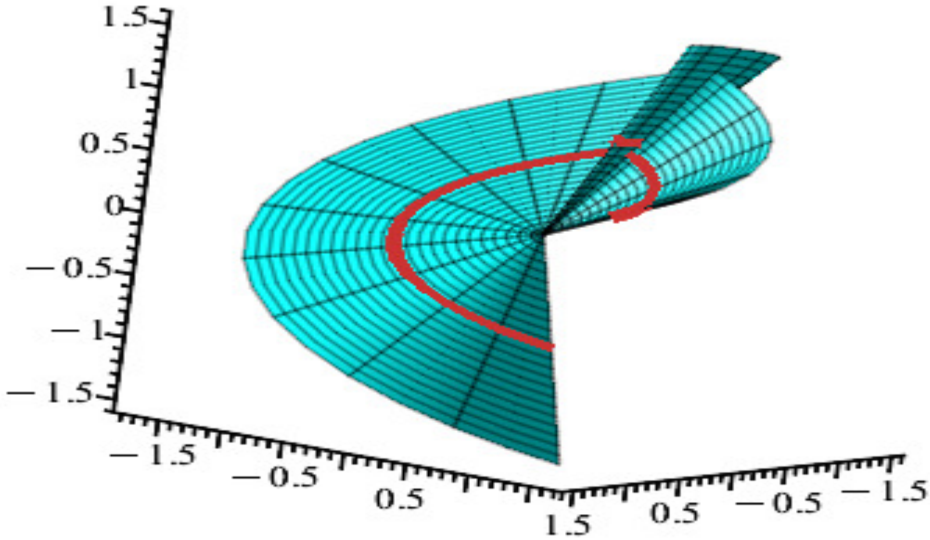

Figure 13: The ruled surface $\eta(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(N_1 + B_1)$

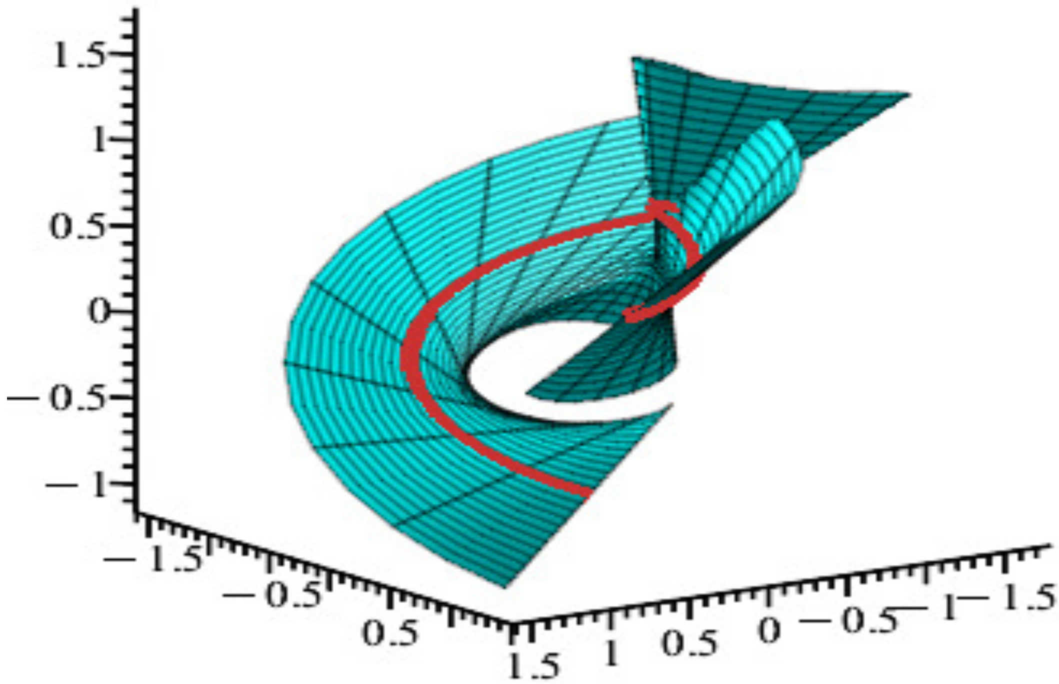

Figure 14: The ruled surface $\Gamma(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{3}}(T_1 + N_1 + B_1)$

Example 3.24. *Let β^* anti Salkowski curve [31] be the Successor curve of α. The equation of this curve for $m = \frac{1}{3}$ is as follows:*

$$\beta^*(s) = \frac{\sqrt{10}}{40}\begin{pmatrix} -\frac{5}{2\sqrt{10}}\Big(\frac{3}{\sqrt{10}}cos(\frac{1}{5} + cos(\frac{2}{\sqrt{10}})s)\Big) + \frac{6}{5}sinssin\frac{2}{\sqrt{10}}s, \\ -\frac{5}{2\sqrt{10}}\Big(\frac{3}{\sqrt{10}}sin(\frac{1}{5} + cos(\frac{2}{\sqrt{10}})s)\Big) + \frac{6}{5}cosssin\frac{2}{\sqrt{10}}s, \quad -\frac{9\sqrt{10}}{40}\Big(\frac{2}{\sqrt{10}}s + sin(\frac{2}{\sqrt{10}})s\Big) \end{pmatrix}.$$

The Successor frames of β^* curve $\{T_1^*, N_1^*, B_1^*\}$ are as follows:

$$T_1^*(s) = \Big(-cosssin\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}sinscos\frac{s}{\sqrt{10}}, \ -sinssin\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}cosscos\frac{s}{\sqrt{10}}, \ -\frac{3}{\sqrt{10}}cos\frac{s}{\sqrt{10}} \Big),$$

$$N_1^*(s) = \Big(\frac{3}{\sqrt{10}}sins, \ -\frac{3}{\sqrt{10}}coss, \ \frac{1}{\sqrt{10}} \Big),$$

$$B_1^*(s) = \Big(-cosscos\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}sinssin\frac{s}{\sqrt{10}}, \ -sinscos\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}cosssin\frac{s}{\sqrt{10}}, \ \frac{3}{\sqrt{10}}sin\frac{s}{\sqrt{10}} \Big).$$

The graphs of the ruled surfaces obtained from these frames for $s \in [-\pi, \pi]$ and $v \in [-1, 1]$ are shown figure {15-21};

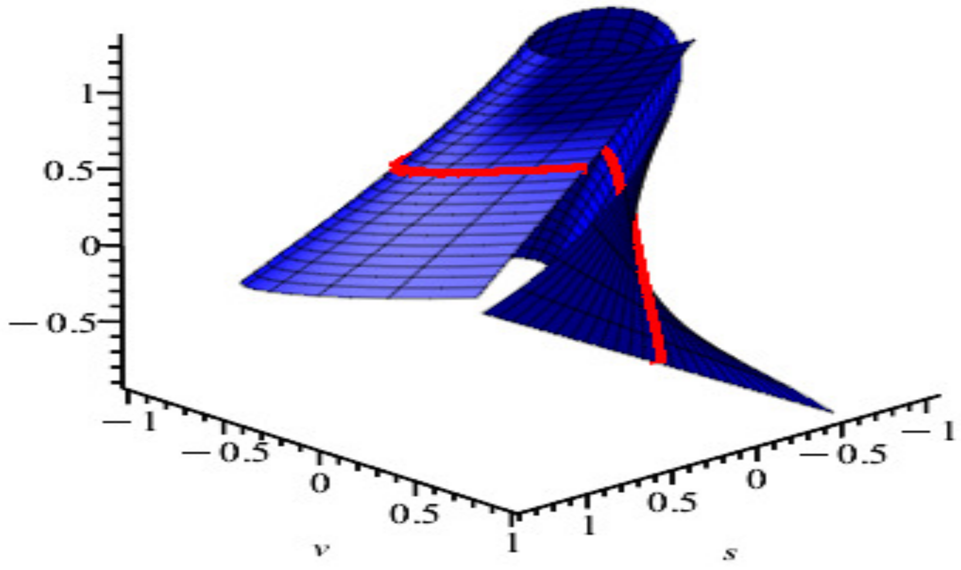

Figure 15: The ruled surface $\Phi(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vT_1$

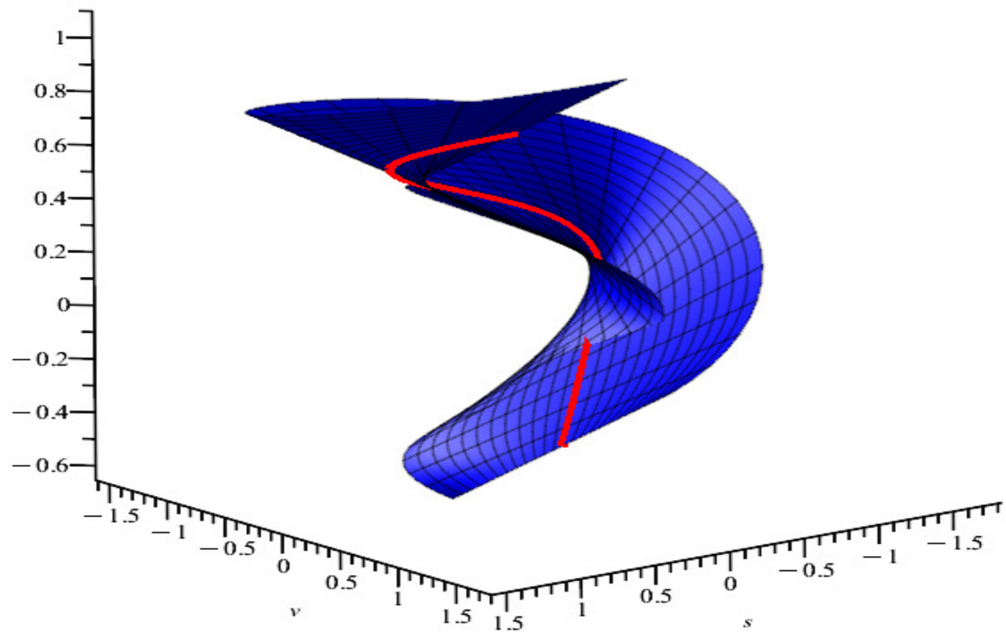

Figure 16: The ruled surface $Q(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vN_1$

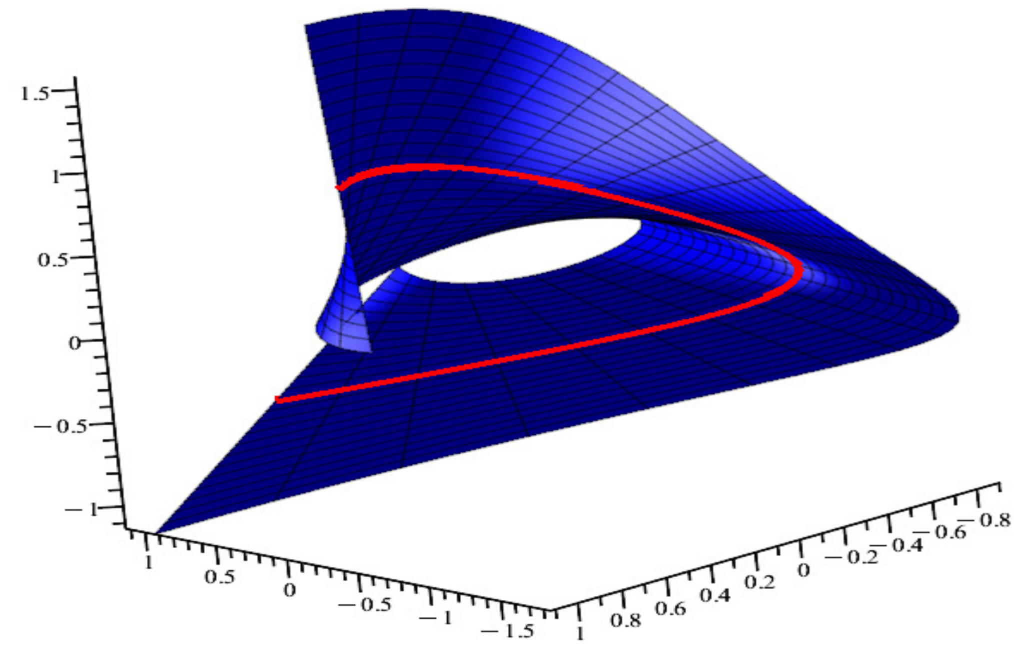

Figure 17: The ruled surface $M(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vB_1$

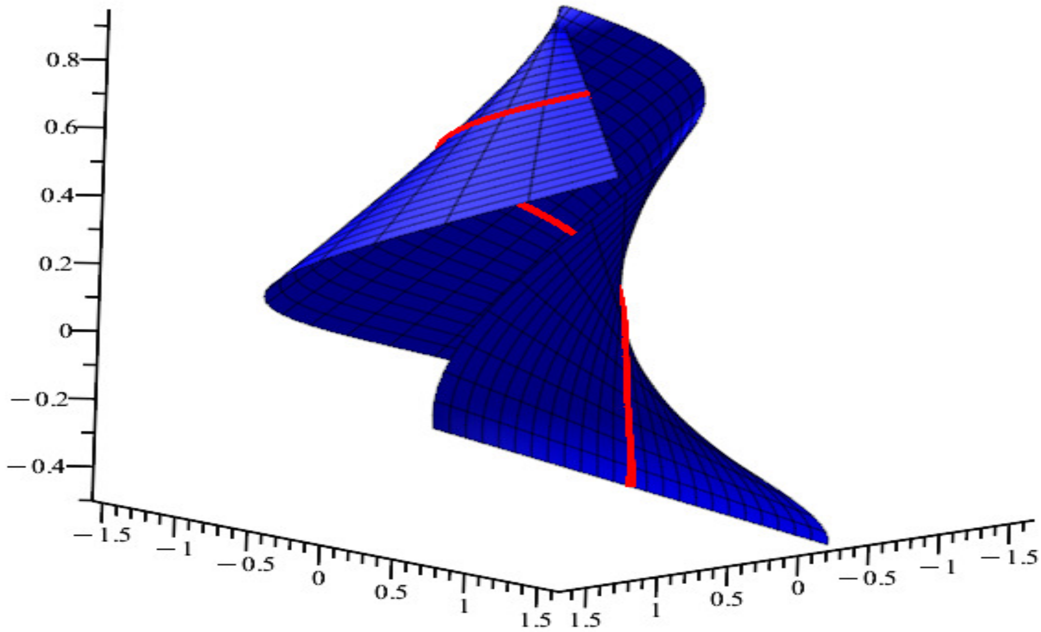

Figure 18: The ruled surface $\mu(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

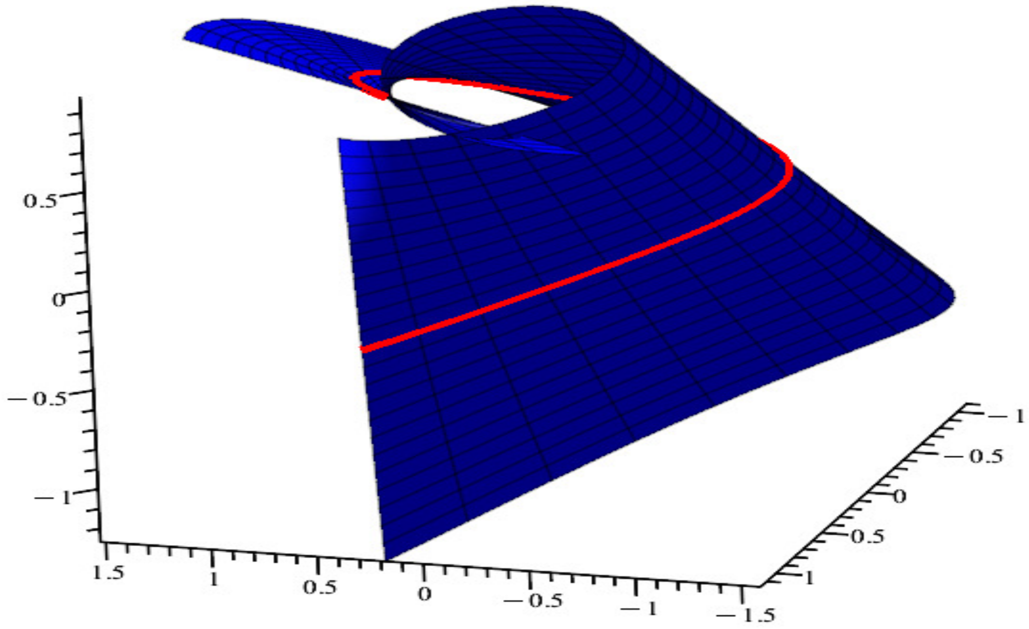

Figure 19: The ruled surface $\psi(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

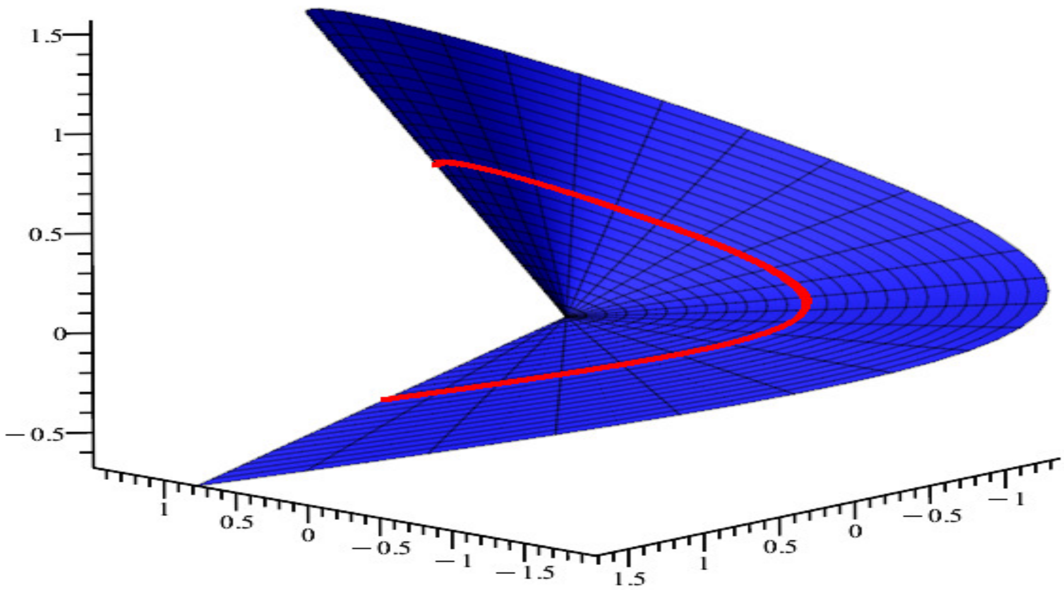

Figure 20: The ruled surface $\eta(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(N_1 + B_1)$

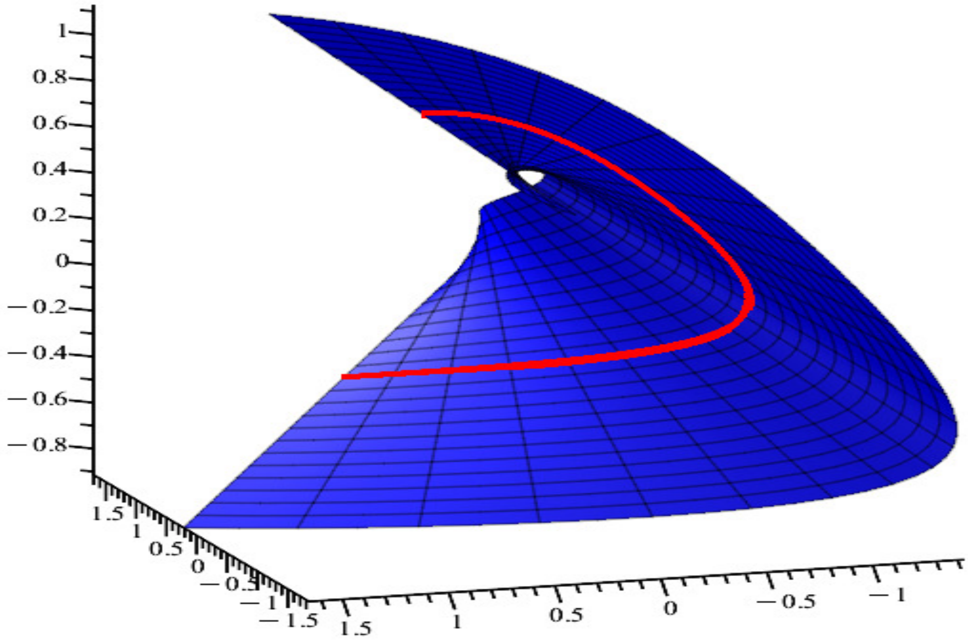

Figure 21: The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{2}}(N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

Example 3.25. *Let the Salkowski curve in Example 3.24 be the main curve. From [31] and Theorem 2.2 the Successor frames are as follows:*

$$T_1^*(s)=\begin{pmatrix} -cos(s+c)\Big(\frac{3}{\sqrt{10}}sins\Big)+sin(s+c)\Big(-cosscos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}sinssin\frac{s}{\sqrt{10}}\Big), \\ cos(s+c)(\frac{3}{\sqrt{10}}coss)+sin(s+c)\Big(-sinscos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}cosssin\frac{s}{\sqrt{10}}\Big), \\ -cos(s+c)\frac{1}{\sqrt{10}}+sin(s+c)\Big(\frac{3}{\sqrt{10}}sin\frac{s}{\sqrt{10}}\Big) \end{pmatrix},$$

$$N_1^*(s)=\Big(-cosssin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}sinscos\frac{s}{\sqrt{10}},\ -sinssin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}cosscos\frac{3}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}cos\frac{s}{\sqrt{10}}\Big),$$

$$B_1^*(s)=\begin{pmatrix} sin(s+c)\Big(\frac{3}{\sqrt{10}}sins\Big)+cos(s+c)\Big(-cosscos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}sinssin\frac{s}{\sqrt{10}}\Big), \\ sin(s+c)(\frac{3}{\sqrt{10}}coss)+cos(s+c)\Big(-cosscos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}sinssin\frac{s}{\sqrt{10}}\Big), \\ sin(s+c)\frac{1}{\sqrt{10}}+cos(s+c)\Big(\frac{3}{\sqrt{10}}sin\frac{s}{\sqrt{10}}\Big) \end{pmatrix}.$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures {22-28};

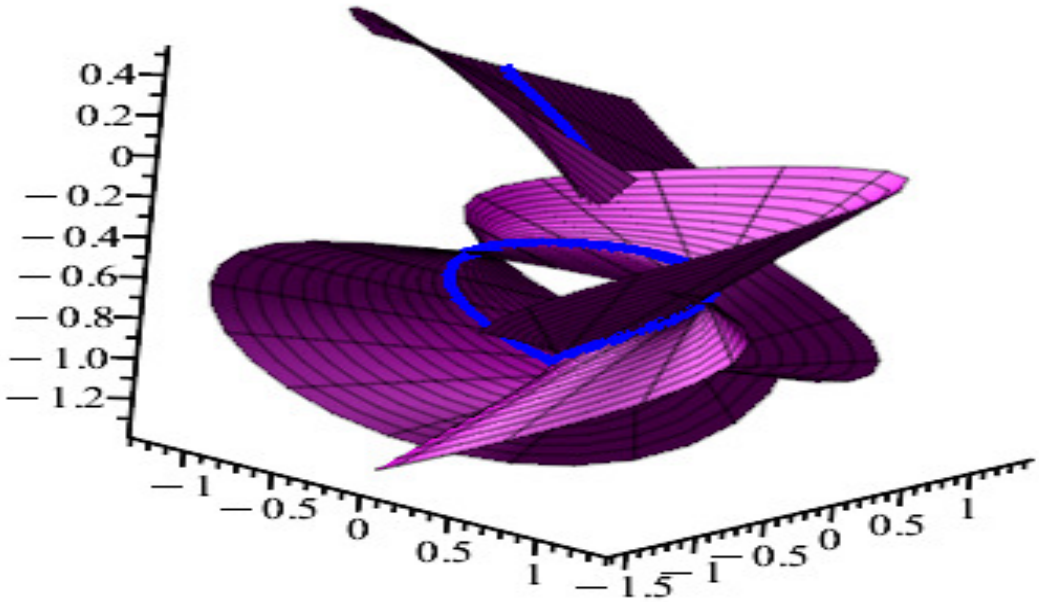

Figure 22: The ruled surface $\Phi(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vT_1$

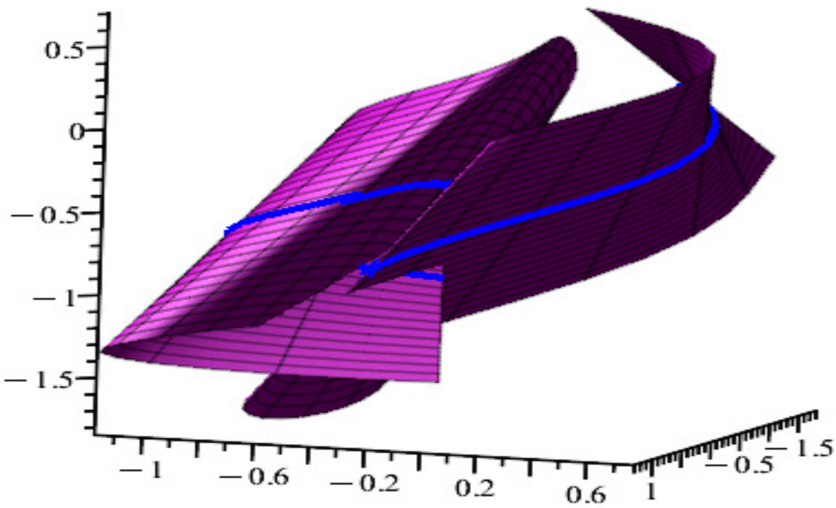

Figure 23: The ruled surface $Q(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vN_1$

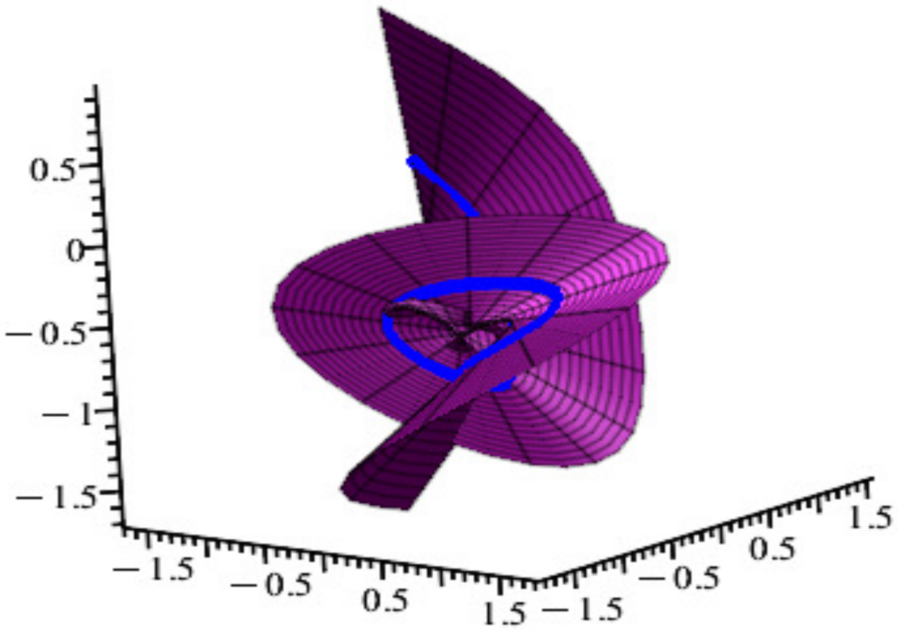

Figure 24: The ruled surface $M(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + vB_1$

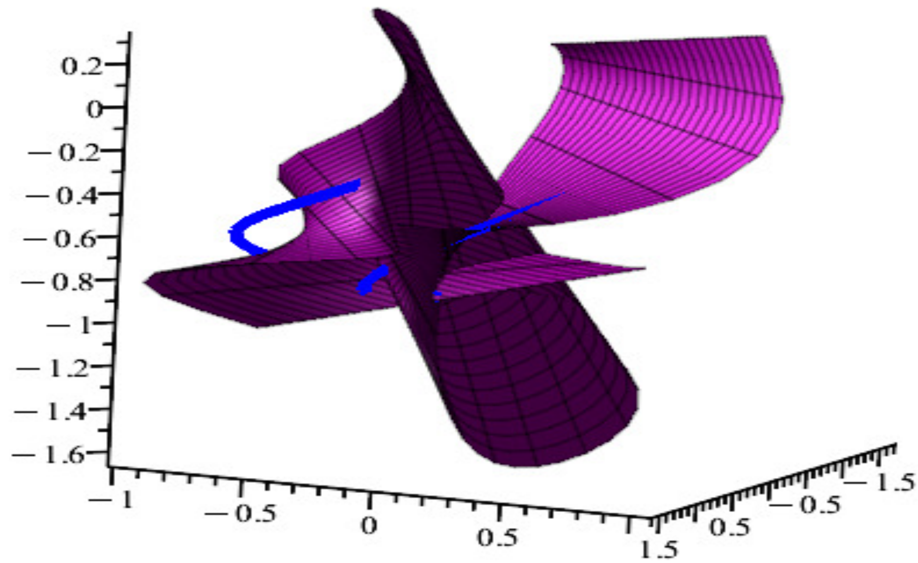

Figure 25: The ruled surface $\mu(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

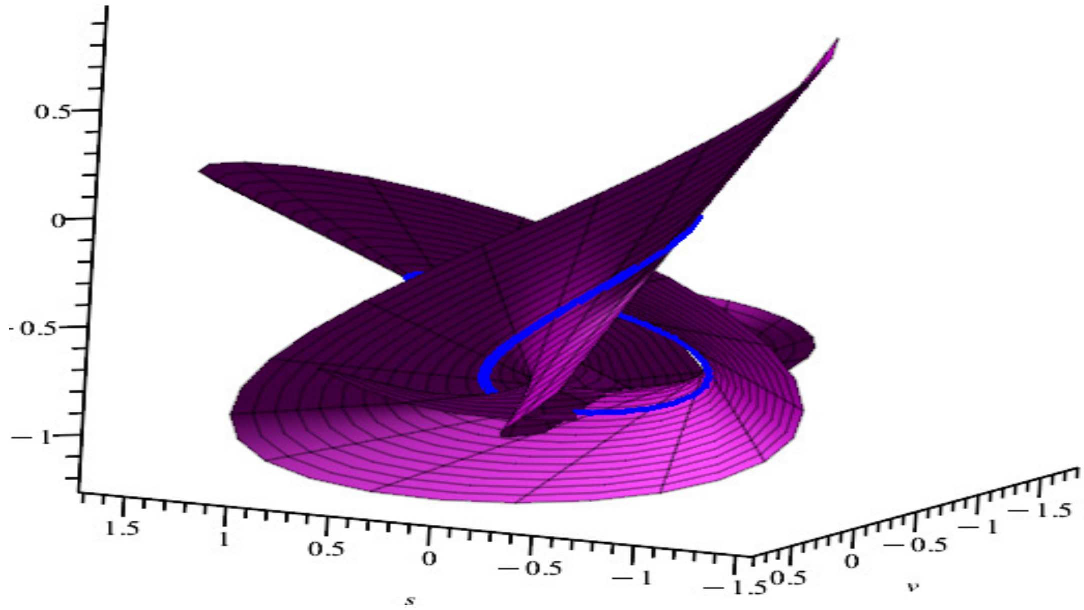

Figure 26: The ruled surface $\psi(s,v)=\frac{1}{\sqrt{2}}(N_1+B_1)+\frac{v}{\sqrt{2}}(T_1+B_1)$

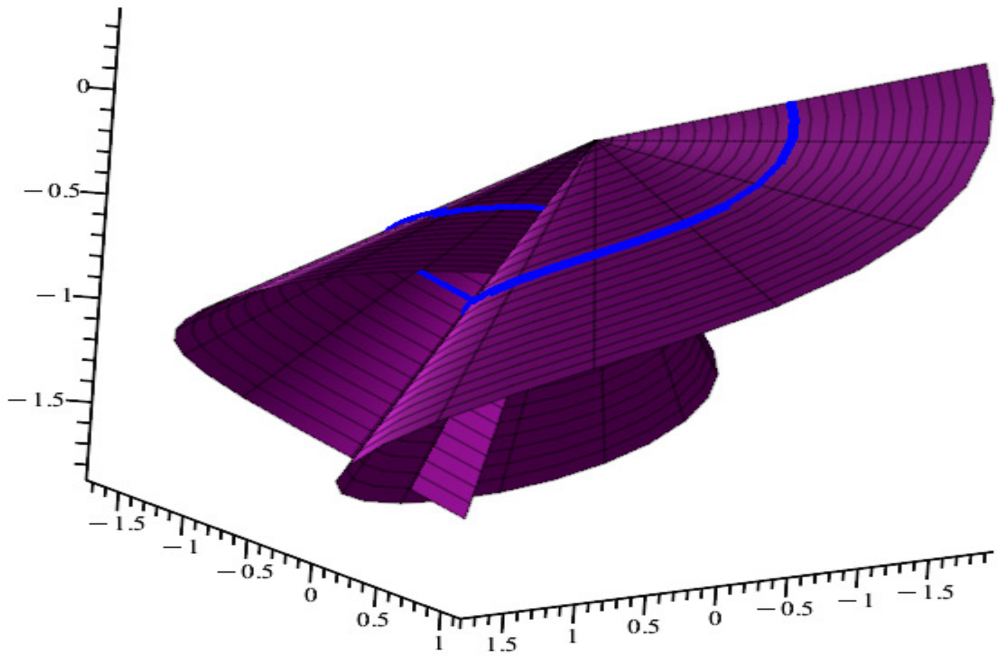

Figure 27: The ruled surface $\eta(s,v)=\frac{1}{\sqrt{2}}(N_1+B_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

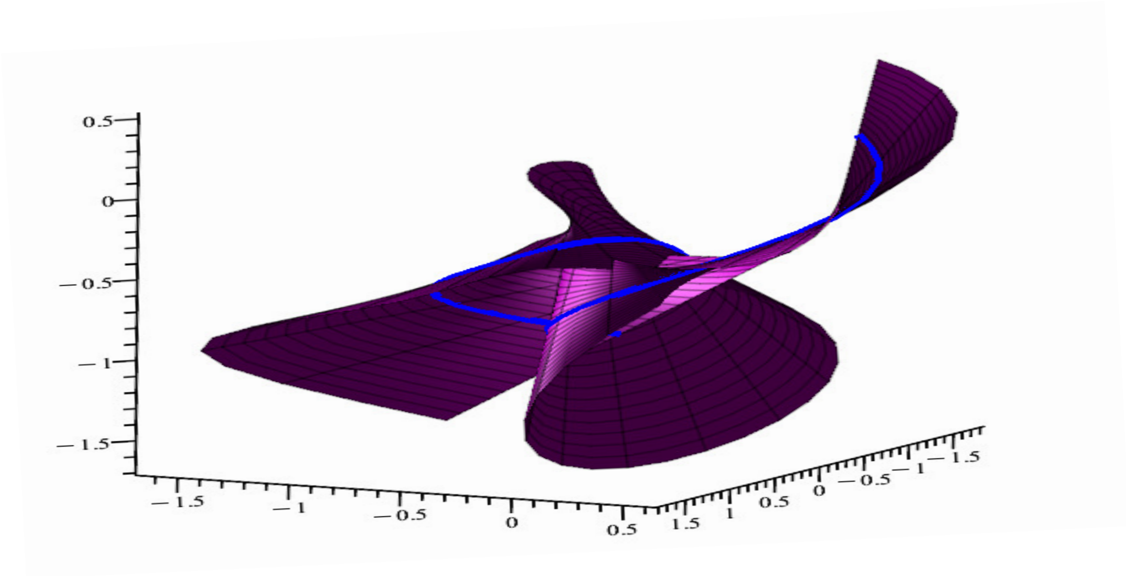

Figure 28: The ruled surface $\Gamma(s,v) = \frac{1}{\sqrt{2}}(N_1 + B_1) + \frac{v}{\sqrt{3}}(T_1 + N_1 + B_1)$

4. Conclusion

This study defined ruled surfaces which one their base curve are N_1B_1-Smarandache curve. There base curves targent vector, normal vector and binormal vector is successor curves Frenet aparatus. The Gaussian and mean curvatures of the surfaces were obatined using the coefficients of the first and the second fundamental forms. The conditions for the surfaces to be developable and minimal were given. These surfaces were drawn. This paper can be studied in Euclidean, Lorentz and dual space. New ruled surfaces can be defined and similar work can be done, by changing the base curve. Also, the singularity of surfaces can be examined.

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