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Bivariate Pell and Bivariate Pell-Lucas Hybrinomials

Yasemin TAŞYURDU[a]

[a]*Erzincan Binali Yıldırım University, Faculty of Arts and Sciences, Department of Mathematics, Erzincan, Turkey*

Abstract. In this paper, we define bivariate Pell and bivariate Pell-Lucas hybrinomials and derive their recurrence relations. We obtain sequences of both the bivariate Pell hybrinomials and the bivariate Pell-Lucas hybrinomials and provide Binet formulas that allow us to calculate the nth terms of these sequences. Furthermore, we derive generating functions and matrix representations of these hybrinomials and establish their various properties.

1. Introduction

Numbers and polynomials defined by recurrence relations have been extensively studied due to their wide range of applications in modern science. Well-known examples include Fibonacci, Lucas, Pell and Pell-Lucas numbers and their polynomial generalizations, which have been studied for their importance in various fields such as mathematics, number theory, computer science and statistics. Many recurrence sequences have been introduced as generalizations of the Fibonacci numbers, considering the initial terms, the distances between terms in the recurrence relation, and the coefficients of the added terms. For instance, Fibonacci numbers are defined by recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = 0$, $F_1 = 1$ and then, Pell numbers, $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$ with $P_0 = 0$, $P_1 = 1$ are defined by adding only the different coefficient to the recurrence relation and Pell-Lucas numbers, $Q_n = 2Q_{n-1} + Q_{n-2}$, $n \geq 2$ with $Q_0 = 2$, $Q_1 = 2$ are defined by considering both the coefficients in the recurrence relation and the initial terms differently [1, 2]. In [3–6], the basic structures and properties of the Pell and Pell-Lucas numbers and their corresponding problem approaches and generalizations, are presented. In [7], a new generalization of the Fibonacci numbers, which includes all the above generalizations and extends the definition of both distance and coefficient with respect to each term in the recurrence relation, is defined as five-parameter generalized Fibonacci numbers.

Various polynomials related to numbers defined by recurrence relations have been introduced as generalizations of both numbers and polynomials [8–11]. Pell polynomials, which are the generalization of Pell numbers by adding a variable x to the recurrence relation, are defined by the recurrence relation

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \quad P_n(x) = 0, \; P_1(x) = 1$$

and Pell-Lucas polynomials, which are the generalization of Pell-Lucas numbers by adding a variable x to both the recurrence relation and the initial terms differently, are defined by the recurrence relation

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Corresponding author: YT mail address: ytasyurdu@erzincan.edu.tr ORCID: 0000-0002-9011-8269
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$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), \quad Q_n(x) = 2,\ Q_1(x) = 2x$$

for $n \geq 2$ [12]. New generalizations of both the Pell and Pell-Lucas numbers and polynomials, called bivariate Pell polynomials and bivariate Pell-Lucas polynomials, are introduced by using a new parameter variable y in the recurrence relations of the Pell and Pell-Lucas polynomials, respectively. These polynomials are defined by the recurrence relations

$$P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y), \quad P_n(x,y) = 0,\ P_1(x,y) = 1 \tag{1}$$

and

$$Q_n(x,y) = 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y), \quad Q_n(x,y) = 2,\ Q_1(x,y) = 2xy \tag{2}$$

for $n \geq 2$ where $x, y \neq 0$, $y^2x^2 + y \neq 0$, respectively. Obviously, for $y = 1$ we obtain the Pell and Pell-Lucas polynomials and $P_n(x,1) = P_n(x)$, $Q_n(x,1) = Q_n(x)$ where $P_n(x)$ is the nth Pell polynomial and $Q_n(x)$ is the nth Pell-Lucas polynomial, respectively. Also, for $x = y = 1$ we obtain the Pell and Pell-Lucas numbers and $P_n(1,1) = P_n$, $Q_n(1,1) = Q_n$ where P_n is the nth Pell number and Q_n is the nth Pell-Lucas number, respectively.

Binet formulas for the nth bivariate Pell polynomial and the nth bivariate Pell-Lucas polynomial are given by

$$P_n(x,y) = \frac{\sigma^n(x,y) - \rho^n(x,y)}{\sigma(x,y) - \rho(x,y)} \tag{3}$$

and

$$Q_n(x,y) = \sigma^n(x,y) + \rho^n(x,y) \tag{4}$$

where $\sigma(x,y) = xy + \sqrt{x^2y^2 + y}$ and $\rho(x,y) = xy - \sqrt{x^2y^2 + y}$ are the roots of the quadratic equation $t^2 - 2xyt - y = 0$ of equations (1) and (2), respectively. Also, the roots $\sigma(x,y)$ and $\rho(x,y)$ hold

$$\begin{aligned} \sigma(x,y) + \rho(x,y) &= 2xy \\ \sigma(x,y)\rho(x,y) &= -y \\ \sigma(x,y) - \rho(x,y) &= 2\sqrt{x^2y^2 + y}. \end{aligned}$$

In recent years, studies on the structures and properties of the bivariate Pell and bivariate Pell-Lucas polynomials have been presented. In [13], these polynomials were examined and several identities and sum formulas were derived using different matrices. In [14], a symmetric function was introduced to derive a new generating function for the bivariate Pell-Lucas polynomials, and new symmetric functions were developed to present interesting properties of these polynomials. In [15], some sums and connection formulas with identities involving these polynomials were obtained.

In mathematical theories, in addition to numbers and polynomials defined by recurrence relations, there are also classes of numbers and polynomials that arise from the combination of different algebraic structures. One such class is hybrid numbers, a new non-commutative number system, represent a generalization of complex, dual and hyperbolic numbers with applications in various fields of mathematics. In [16], the set of hybrid numbers is introduced and denoted by $\mathbb{K}$, which contains complex, dual and hyperbolic numbers. The set of hybrid numbers, $\mathbb{K}$, is defined as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} : a,b,c,d \in \mathbb{R},\ \mathbf{i}^2 = -1,\ \boldsymbol{\varepsilon}^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i} \right\}.$$

For $z_1 = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$ and $z_2 = x + y\mathbf{i} + z\boldsymbol{\varepsilon} + t\mathbf{h}$ are defined as

Equality: $z_1 = z_2$ only if $a = x$, $b = y$, $c = z$, $d = t$

Addition: $z_1 + z_2 = (a + x) + (b + y)\mathbf{i} + (c + z)\boldsymbol{\varepsilon} + (d + t)\mathbf{h}$

Subtraction: $z_1 - z_2 = (a - x) + (b - y)\mathbf{i} + (c - z)\boldsymbol{\varepsilon} + (d - t)\mathbf{h}$

Multiplication: $z_1z_2 = ax - by + dt + bz + cy + (ay + bx + bt - dy)\mathbf{i}$
$+ (az + cx + bt - dy + dz - ct)\boldsymbol{\varepsilon} + (at + dx + cy - bz)\mathbf{h}.$

The multiplication of a hybrid number $z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ by the real scalar k is defined as

$$kz = ka + kb\mathbf{i} + kc\varepsilon + kd\mathbf{h}.$$

In hybrid numbers, multiplication is associative but not commutative. Addition, on the other hand, is both associative and commutative. The additive identity is represented by $0 = 0 + 0\mathbf{i} + 0\varepsilon + 0\mathbf{h}$. The additive inverse of a hybrid number $z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is $-z = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. Therefore, the set $(\mathbb{K}; +)$ forms an Abelian group. For more comprehensive understanding of hybrid numbers, readers can refer to [16]. The multiplication table for the basis of hybrid numbers is as follows:

Table 1: The multiplication of hybrid units of $\mathbb{K}$

.	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$1+\mathbf{h}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

In the following years, new generalizations of sequences of numbers and polynomials defined by recurrence relations using hybrid numbers have been introduced and called hybrid sequences. Specifically, in [17–19], hybrid sequences of the Fibonacci, Lucas, Pell, Pell-Lucas and Horadam numbers, which were defined by second-order recurrence relations, were presented. In [20], hybrid sequences of the Tribonacci and Tribonacci-Lucas numbers, which were defined by third-order recurrence relations, were introduced. Moreover, the Binet formulas, generating functions, and various properties of these derived hybrid sequences were obtained.

On the other hand, polynomial sequences defined by recurrence relations using hybrid numbers have been considered, and their new generalizations have been introduced and called hybrinomial sequences. Furthermore, the identies and formulas of these hybrinomial sequences, including the Binet formulas, generating functions and various other properties, have been presented. In [21–24], hybrinomial sequences of the Fibonacci, Lucas, Pell, generalized Fibonacci-Pell, Horadam polynomials, which were defined by second-order recurrence relations, were presented. In particular, in [25], these obtained results were generalized and introduced a new generalization of the Fibonacci type and the Lucas type hybrid numbers and polynomials and called generalized Fibonacci hybrinomials. In [26], hybrinomial sequences of the Tribonacci and Tribonacci-Lucas polynomials, which were defined by third-order recurrence relations, were presented and derived these hybrinomials by the matrices.

The aim of this study is to extend hybrid numbers to the class of bivariate polynomials defined by recurrence relations, and to present a new generalization of recurrence sequences by generalizing all results for hybrid numbers and hybrinomials related to numbers and polynomials defined by recurrence relations. Therefore, in this study, the bivariate Pell and bivariate Pell-Lucas hybrinomials are introduced and the Binet formulas, generating functions and well-known properties such as Catalan's identity, Cassini's identity, d'Ocagne's identity, Honsberger's identity are presented for these hybrinomials.

2. Bivariate Pell and Bivariate Pell-Lucas Hybrinomials

In this section, we introduce the definitions of the bivariate Pell and bivariate Pell-Lucas hybrinomials using hybrid numbers and the bivariate Pell and bivariate Pell-Lucas polynomials. We derive the Binet formulas and generating functions for both the bivariate Pell hybrinomials and the bivariate Pell-Lucas hybrinomials. Moreover, we explore their matrix representations.

Definition 2.1. *The nth bivariate Pell hybrinomial, $P_nH(x,y)$ and the bivariate Pell-Lucas hybrinomial, $Q_nH(x,y)$ are defined by*

$$P_nH(x,y) = P_n(x,y) + P_{n+1}(x,y)\mathbf{i} + P_{n+2}(x,y)\varepsilon + P_{n+3}(x,y)\mathbf{h} \quad (5)$$

and

$$Q_nH(x,y) = Q_n(x,y) + Q_{n+1}(x,y)\,\mathbf{i} + Q_{n+2}(x,y)\boldsymbol{\varepsilon} + Q_{n+3}(x,y)\,\mathbf{h} \tag{6}$$

where $P_n(x,y)$ *and* $Q_n(x,y)$ *are the nth bivariate Pell polynomial and the nth bivariate Pell-Lucas polynomial, respectively. Here hybrid units* $\mathbf{i}$, $\boldsymbol{\varepsilon}$, $\mathbf{h}$ *satisfy the equations* $\mathbf{i}^2 = -1$, $\boldsymbol{\varepsilon}^2 = 0$, $\mathbf{h}^2 = 1$, $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$.

Using the Definition 2.1, we obtain the bivariate Pell hybrinomial sequence, $\{P_nH(x,y)\}_{n\geq 0}$, and the bivariate Pell-Lucas hybrinomial sequence, $\{Q_nH(x,y)\}_{n\geq 0}$. Additionally, these sequences can also be generated from recurrence relations given by the following theorem.

Theorem 2.2. *For* $n \geq 2$, *the recurrence relations of the bivariate Pell hybrinomial sequence and the bivariate Pell-Lucas hybrinomial sequence are*

$$P_nH(x,y) = 2xyP_{n-1}H(x,y) + yP_{n-2}H(x,y) \tag{7}$$

$$Q_nH(x,y) = 2xyQ_{n-1}H(x,y) + yQ_{n-2}H(x,y) \tag{8}$$

with

$$\begin{aligned} P_0H(x,y) &= \mathbf{i} + (2xy)\,\boldsymbol{\varepsilon} + \left(4x^2y^2 + y\right)\mathbf{h}, \\ P_1H(x,y) &= 1 + (2xy)\mathbf{i} + \left(4x^2y^2 + y\right)\boldsymbol{\varepsilon} + (8x^3y^3 + 4xy^2)\mathbf{h} \end{aligned}$$

and

$$\begin{aligned} Q_0H(x,y) &= 2 + (2xy)\,\mathbf{i} + \left(4x^2y^2 + 2y\right)\boldsymbol{\varepsilon} + (8x^3y^3 + 6xy^2)\mathbf{h}, \\ Q_1H(x,y) &= 2xy + \left(4x^2y^2 + 2y\right)\mathbf{i} + \left(8x^3y^3 + 6xy^2\right)\boldsymbol{\varepsilon} + \left(16x^4y^4 + 16x^2y^3 + 2y^2\right)\mathbf{h} \end{aligned}$$

respectively.

Proof. First, let us prove the equation (7). Using the equations (1) and (5), we obtain

$$\begin{aligned} 2xyP_{n-1}H(x,y) + yP_{n-2}H(x) &= 2xy\left(P_{n-1}(x,y) + P_n(x,y)\,\mathbf{i} + P_{n+1}(x,y)\,\boldsymbol{\varepsilon} + P_{n+2}(x,y)\,\mathbf{h}\right) \\ &+ y\left(P_{n-2}(x,y) + P_{n-1}(x,y)\,\mathbf{i} + P_n(x,y)\,\boldsymbol{\varepsilon} + P_{n+1}(x,y)\,\mathbf{h}\right) \\ &= 2xyP_{n-1}(x,y) + yP_{n-2}(x,y) + \left(2xyP_n(x,y) + yP_{n-1}(x,y)\right)\mathbf{i} \\ &+ \left(2xyP_{n+1}(x,y) + yP_n(x,y)\right)\boldsymbol{\varepsilon} + \left(2xyP_{n+2}(x,y) + yP_{n+1}(x,y)\right)\mathbf{h} \\ &= P_n(x,y) + P_{n+1}(x,y)\,\mathbf{i} + P_{n+2}(x,y)\,\boldsymbol{\varepsilon} + P_{n+3}(x,y)\,\mathbf{h} \\ &= P_nH(x,y). \end{aligned}$$

Similarly, let us now prove the equation (8). Using the equations (2) and (6), we obtain

$$\begin{aligned} 2xyQ_{n-1}H(x,y) + yQ_{n-2}H(x) &= 2xy\left(Q_{n-1}(x,y) + Q_n(x,y)\,\mathbf{i} + Q_{n+1}(x,y)\,\boldsymbol{\varepsilon} + Q_{n+2}(x,y)\,\mathbf{h}\right) \\ &+ y\left(Q_{n-2}(x,y) + Q_{n-1}(x,y)\,\mathbf{i} + Q_n(x,y)\,\boldsymbol{\varepsilon} + Q_{n+1}(x,y)\,\mathbf{h}\right) \\ &= 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y) + \left(2xyQ_n(x,y) + yQ_{n-1}(x,y)\right)\mathbf{i} \\ &+ \left(2xyQ_{n+1}(x,y) + yQ_n(x,y)\right)\boldsymbol{\varepsilon} + \left(2xyQ_{n+2}(x,y) + yQ_{n+1}(x,y)\right)\mathbf{h} \\ &= Q_n(x,y) + Q_{n+1}(x,y)\,\mathbf{i} + Q_{n+2}(x,y)\,\boldsymbol{\varepsilon} + Q_{n+3}(x,y)\,\mathbf{h} \\ &= Q_nH(x,y). \end{aligned}$$

Thus, the proof is completed.

2.1. Binet Formulas and Generating Functions of the Hybrinomials $P_n(x, y)$ and $Q_n(x, y)$

In this section, we present the Binet formulas and generating functions for the bivariate Pell and bivariate Pell-Lucas hybrinomials.

The Binet formula, which is one of the general formulas that allows the determination of the nth term of a sequence without the need to know all previous terms, is given by the following theorem for both the nth bivariate Pell hybrinomial and the nth bivariate Pell-Lucas hybrinomial.

Theorem 2.3. *For $n \geq 0$, the Binet formulas for the bivariate Pell and bivariate Pell-Lucas hybrinomials are given by*

$$P_nH(x,y) = \frac{\hat{\sigma}(x,y)\sigma^n(x,y) - \hat{\rho}(x,y)\rho^n(x,y)}{\sigma(x,y) - \rho(x,y)} \tag{9}$$

and

$$Q_nH(x,y) = \hat{\sigma}(x,y)\sigma^n(x,y) + \hat{\rho}(x,y)\rho^n(x,y) \tag{10}$$

where $\hat{\sigma}(x,y) = 1 + \sigma(x,y)\mathbf{i} + \sigma^2(x,y)\boldsymbol{\varepsilon} + \sigma^3(x,y)\mathbf{h}$, $\hat{\rho}(x,y) = 1 + \rho(x,y)\mathbf{i} + \rho^2(x,y)\boldsymbol{\varepsilon} + \rho^3(x,y)\mathbf{h}$ *and* $\sigma(x,y)$, $\rho(x,y)$ *are the roots of the quadratic equation* $t^2 - 2xyt - y = 0$, *respectively.*

Proof. First, let us prove the equation (9). Using the equations (3) and (5), we have

$$\begin{aligned}
P_nH(x,y) &= P_n(x,y) + P_{n+1}(x,y)\mathbf{i} + P_{n+2}(x,y)\boldsymbol{\varepsilon} + P_{n+3}(x,y)\mathbf{h} \\
&= \frac{\sigma^n(x,y) - \rho^n(x,y)}{\sigma(x,y) - \rho(x,y)} + \left(\frac{\sigma^{n+1}(x,y) - \rho^{n+1}(x,y)}{\sigma(x,y) - \rho(x,y)}\right)\mathbf{i} \\
&+ \left(\frac{\sigma^{n+2}(x,y) - \rho^{n+2}(x,y)}{\sigma(x,y) - \rho(x,y)}\right)\boldsymbol{\varepsilon} + \left(\frac{\sigma^{n+3}(x,y) - \rho^{n+3}(x,y)}{\sigma(x,y) - \rho(x,y)}\right)\mathbf{h} \\
&= \frac{\left(1 + \sigma(x,y)\mathbf{i} + \sigma^2(x,y)\boldsymbol{\varepsilon} + \sigma^3(x,y)\mathbf{h}\right)\sigma^n(x,y)}{\sigma(x,y) - \rho(x,y)} \\
&- \frac{\left(1 + \rho(x,y)\mathbf{i} + \rho^2(x,y)\boldsymbol{\varepsilon} + \rho^3(x,y)\mathbf{h}\right)\rho^n(x,y)}{\sigma(x,y) - \rho(x,y)} \\
&= \frac{\hat{\sigma}(x,y)\sigma^n(x,y) - \hat{\rho}(x,y)\rho^n(x,y)}{\sigma(x,y) - \rho(x,y)}
\end{aligned}$$

and now using the equations (4) and (6) to prove the equation (10), we have

$$\begin{aligned}
Q_nH(x,y) &= Q_n(x,y) + Q_{n+1}(x,y)\mathbf{i} + Q_{n+2}(x,y)\boldsymbol{\varepsilon} + Q_{n+3}(x,y)\mathbf{h} \\
&= \sigma^n(x,y) + \rho^n(x,y) + \left(\sigma^{n+1}(x,y) + \rho^{n+1}(x,y)\right)\mathbf{i} \\
&+ \left(\sigma^{n+2}(x,y) + \rho^{n+2}(x,y)\right)\boldsymbol{\varepsilon} + \left(\sigma^{n+3}(x,y) + \rho^{n+3}(x,y)\right)\mathbf{h} \\
&= \left(1 + \sigma(x,y)\mathbf{i} + \sigma^2(x,y)\boldsymbol{\varepsilon} + \sigma^3(x,y)\mathbf{h}\right)\sigma^n(x,y) \\
&+ \left(1 + \rho(x,y)\mathbf{i} + \rho^2(x,y)\boldsymbol{\varepsilon} + \rho^3(x,y)\mathbf{h}\right)\rho^n(x,y) \\
&= \hat{\sigma}(x,y)\sigma^n(x,y) + \hat{\rho}(x,y)\rho^n(x,y)
\end{aligned}$$

where $\hat{\sigma}(x,y) = 1 + \sigma(x,y)\mathbf{i} + \sigma^2(x,y)\boldsymbol{\varepsilon} + \sigma^3(x,y)\mathbf{h}$ and $\hat{\rho}(x,y) = 1 + \rho(x,y)\mathbf{i} + \rho^2(x,y)\boldsymbol{\varepsilon} + \rho^3(x,y)\mathbf{h}$. So, the proof is completed.

We now present the generating functions of the bivariate Pell and bivariate Pell-Lucas hybrinomials by the following theorem.

Theorem 2.4. *The generating functions of the bivariate Pell and bivariate Pell-Lucas hybrinomials are*

$$p(t) = \sum_{n=0}^{\infty} P_nH(x,y)t^n = \frac{P_0H(x,y) + P_1H(x,y)t - 2xyP_0H(x,y)t}{1-2xyt-yt^2} \tag{11}$$

and

$$q(t) = \sum_{n=0}^{\infty} Q_nH(x,y)t^n = \frac{Q_0H(x,y) + Q_1H(x,y)t - 2xyQ_0H(x,y)t}{1-2xyt-yt^2} \tag{12}$$

where $P_nH(x,y)$ *and* $Q_nH(x,y)$ *are the nth bivariate Pell hybrinomial and the nth bivariate Pell-Lucas hybrinomial, respectively.*

Proof. First, let us prove the equation (11). Let $p(t) = \sum_{n=0}^{\infty} P_nH(x,y)t^n$ be the generating function of the bivariate Pell hybrinomials. Using the equation (7), we have

$$\begin{aligned} p(t) &= \sum_{n=0}^{\infty} P_nH(x,y)t^n \\ &= P_0H(x,y) + P_1H(x,y)t + \sum_{n=2}^{\infty} P_nH(x,y)t^n \\ &= P_0H(x,y) + P_1H(x,y)t + \sum_{n=2}^{\infty} (2xyP_{n-1}H(x,y) + yP_{n-2}H(x,y))t^n \\ &= P_0H(x,y) + P_1H(x,y)t + 2xyt\sum_{n=0}^{\infty} P_nH(x,y)t^n - 2xyP_0H(x,y)t + yt^2\sum_{n=0}^{\infty} P_nH(x,y)t^n \\ &= P_0H(x,y) + P_1H(x,y)t + 2xytp(t) - 2xyP_0H(x,y)t + yt^2p(t) \end{aligned}$$

and we obtain that

$$\left(1-2xyt-yt^2\right)p(t) = P_0H(x,y) + P_1H(x,y)t - 2xyP_0H(x,y)t.$$

Then, the generating function of the bivariate Pell hybrinomials is

$$p(t) = \frac{P_0H(x,y) + P_1H(x,y)t - 2xyP_0H(x,y)t}{1-2xyt-yt^2}.$$

Now, let us prove the equation (12). Let $q(t) = \sum_{n=0}^{\infty} Q_nH(x,y)t^n$ be the generating function of the bivariate Pell-Lucas hybrinomials. Using the equation (8), we have

$$\begin{aligned} q(t) &= \sum_{n=0}^{\infty} Q_nH(x,y)t^n \\ &= Q_0H(x,y) + Q_1H(x,y)t + \sum_{n=2}^{\infty} Q_nH(x,y)t^n \\ &= Q_0H(x,y) + Q_1H(x,y)t + \sum_{n=2}^{\infty} (2xyQ_{n-1}H(x,y) + yQ_{n-2}H(x,y))t^n \\ &= Q_0H(x,y) + Q_1H(x,y)t + 2xyt\sum_{n=0}^{\infty} Q_nH(x,y)t^n - 2xyQ_0H(x,y)t + yt^2\sum_{n=0}^{\infty} Q_nH(x,y)t^n \\ &= Q_0H(x,y) + Q_1H(x,y)t + 2xytq(t) - 2xyQ_0H(x,y)t + yt^2q(t) \end{aligned}$$

and we obtain that

$$\left(1-2xyt-yt^2\right)q(t) = Q_0H(x,y)+Q_1H(x,y)t-2xyQ_0H(x,y)t.$$

Then, the generating function of the bivariate Pell-Lucas hybrinomials is

$$q(t) = \frac{Q_0H(x,y)+Q_1H(x,y)t-2xyQ_0H(x,y)t}{1-2xyt-yt^2}.$$

Hence, the proof is completed.

2.2. *Matrix Representations and Identities of the Hybrinomials $P_n(x,y)$ and $Q_n(x,y)$*

In this section, we derive the matrix representations of the bivariate Pell and bivariate Pell-Lucas hybrinomials. Then, we present several interesting identities, including Catalan's identity, Cassini's identity, d'Ocagne's identity and Honsberger's identity for these hybrinomials.

The following theorem presents the matrix representations of the bivariate Pell and bivariate Pell-Lucas hybrinomials.

Theorem 2.5. *Let $n \geq 0$ be an integer. Then,*

$$\begin{bmatrix} P_{n+2}H(x,y) & P_{n+1}H(x,y) \\ P_{n+1}H(x,y) & P_nH(x,y) \end{bmatrix} = \begin{bmatrix} P_2H(x,y) & P_1H(x,y) \\ P_1H(x,y) & P_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^n \tag{13}$$

and

$$\begin{bmatrix} Q_{n+2}H(x,y) & Q_{n+1}H(x,y) \\ Q_{n+1}H(x,y) & Q_nH(x,y) \end{bmatrix} = \begin{bmatrix} Q_2H(x,y) & Q_1H(x,y) \\ Q_1H(x,y) & Q_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^n \tag{14}$$

where $P_nH(x,y)$ and $Q_nH(x,y)$ are the nth bivariate Pell hybrinomial and the nth bivariate Pell-Lucas hybrinomial, respectively.

Proof. By using induction on n. First, let us prove equation (13). If $n = 0$, the result is immediately clear. Let us assume that the result holds for any $n \geq 0$ and therefore,

$$\begin{bmatrix} P_{n+2}H(x,y) & P_{n+1}H(x,y) \\ P_{n+1}H(x,y) & P_nH(x,y) \end{bmatrix} = \begin{bmatrix} P_2H(x,y) & P_1H(x,y) \\ P_1H(x,y) & P_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^n.$$

We now aim to prove that the equation holds for the integer $n+1$. In other words,

$$\begin{bmatrix} P_{n+3}H(x,y) & P_{n+2}H(x,y) \\ P_{n+2}H(x,y) & P_{n+1}H(x,y) \end{bmatrix} = \begin{bmatrix} P_2H(x,y) & P_1H(x,y) \\ P_1H(x,y) & P_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^{n+1}.$$

Thus, using the induction hypothesis and the equation (7), we obtain

$$\begin{aligned}
\begin{bmatrix} P_2H(x,y) & P_1H(x,y) \\ P_1H(x,y) & P_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} P_2H(x,y) & P_1H(x,y) \\ P_1H(x,y) & P_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^n \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \\
&= \begin{bmatrix} P_{n+2}H(x,y) & P_{n+1}H(x,y) \\ P_{n+1}H(x,y) & P_nH(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2xyP_{n+2}H(x,y)+yP_{n+1}H(x,y) & P_{n+2}H(x,y) \\ 2xyP_{n+1}H(x,y)+yP_nH(x,y) & P_{n+1}H(x,y) \end{bmatrix} \\
&= \begin{bmatrix} P_{n+3}H(x,y) & P_{n+2}H(x) \\ P_{n+2}H(x,y) & P_{n+1}H(x,y) \end{bmatrix}.
\end{aligned}$$

Now, let us prove equation (14). If $n = 0$, the result is immediately clear. Let us assume that the result holds for any $n \geq 0$ and therefore,

$$\begin{bmatrix} Q_{n+2}H(x,y) & Q_{n+1}H(x,y) \\ Q_{n+1}H(x,y) & Q_nH(x,y) \end{bmatrix} = \begin{bmatrix} Q_2H(x,y) & Q_1H(x,y) \\ Q_1H(x,y) & Q_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^n.$$

We now aim to prove that the equation holds for the integer $n+1$. In other words,

$$\begin{bmatrix} Q_{n+3}H(x,y) & Q_{n+2}H(x,y) \\ Q_{n+2}H(x,y) & Q_{n+1}H(x,y) \end{bmatrix} = \begin{bmatrix} Q_2H(x,y) & Q_1H(x,y) \\ Q_1H(x,y) & Q_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^{n+1}.$$

Thus, using the induction hypothesis and the equation (8), we obtain

$$\begin{aligned} \begin{bmatrix} Q_2H(x,y) & Q_1H(x,y) \\ Q_1H(x,y) & Q_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} Q_2H(x,y) & Q_1H(x,y) \\ Q_1H(x,y) & Q_0H(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}^{n} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \\ &= \begin{bmatrix} Q_{n+2}H(x,y) & Q_{n+1}H(x,y) \\ Q_{n+1}H(x,y) & Q_nH(x,y) \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2xyQ_{n+2}H(x,y) + yQ_{n+1}H(x,y) & Q_{n+2}H(x,y) \\ 2xyQ_{n+1}H(x,y) + yQ_nH(x,y) & Q_{n+1}H(x,y) \end{bmatrix} \\ &= \begin{bmatrix} Q_{n+3}H(x,y) & Q_{n+2}H(x) \\ Q_{n+2}H(x,y) & Q_{n+1}H(x,y) \end{bmatrix}. \end{aligned}$$

So, the proof is completed.

The following theorem gives the Catalan's identities for the bivariate Pell and bivariate Pell-Lucas hybrinomials.

Theorem 2.6 (Catalan's Identity). *For $0 \leq r \leq n$, we have*

$$\begin{aligned} P_{n+r}H(x,y)\,P_{n-r}H(x,y) - (P_nH(x,y))^2 &= \frac{(-y)^n\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,(1-\sigma^r(x,y)\,\rho^{-r}(x,y))\right.}{(\sigma(x,y)-\rho(x,y))^2} \\ &+ \frac{\left.\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,(1-\rho^r(x,y)\,\sigma^{-r}(x,y))\right]}{(\sigma(x,y)-\rho(x,y))^2} \end{aligned} \tag{15}$$

and

$$\begin{aligned} Q_{n+r}H(x,y)\,Q_{n-r}H(x,y) - (Q_nH(x,y))^2 &= (-y)^n\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,(\sigma^r(x,y)\,\rho^{-r}(x,y)-1)\right. \\ &+ \left.\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,(\rho^r(x,y)\,\sigma^{-r}(x,y)-1)\right] \end{aligned} \tag{16}$$

where $P_nH(x,y)$ and $Q_nH(x,y)$ are the nth bivariate Pell hybrinomial and the nth bivariate Pell-Lucas hybrinomial, respectively.

Proof. First, let us prove equation (15). Using the equation (9), we have

$$
\begin{aligned}
&P_{n+r}H(x,y)\,P_{n-r}H(x,y)-\left(P_nH(x,y)\right)^2\\
&\quad=\frac{\left(\hat{\sigma}(x,y)\,\sigma^{n+r}(x,y)-\hat{\rho}(x,y)\,\rho^{n+r}(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^{n-r}(x,y)-\hat{\rho}(x,y)\,\rho^{n-r}(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad-\frac{\left(\hat{\sigma}(x,y)\,\sigma^{n}(x,y)-\hat{\rho}(x,y)\,\rho^{n}(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^{n}(x,y)-\hat{\rho}(x,y)\,\rho^{n}(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad=\frac{\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^{n}(x,y)\,\rho^{n}(x,y)\left(1-\sigma^{r}(x,y)\,\rho^{-r}(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad+\frac{\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^{n}(x,y)\,\sigma^{n}(x,y)\left(1-\rho^{r}(x,y)\,\sigma^{-r}(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad=\frac{(-y)^n\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\left(1-\sigma^{r}(x,y)\,\rho^{-r}(x,y)\right)+\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\left(1-\rho^{r}(x,y)\,\sigma^{-r}(x,y)\right)\right]}{(\sigma(x,y)-\rho(x,y))^2}.
\end{aligned}
$$

Now, let us prove equation (16). Using the equation (10), we have

$$
\begin{aligned}
&Q_{n+r}H(x,y)\,Q_{n-r}H(x,y)-\left(Q_nH(x,y)\right)^2\\
&\quad=\left(\hat{\sigma}(x,y)\,\sigma^{n+r}(x,y)+\hat{\rho}(x,y)\,\rho^{n+r}(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^{n-r}(x,y)+\hat{\rho}(x,y)\,\rho^{n-r}(x,y)\right)\\
&\quad-\left(\hat{\sigma}(x,y)\,\sigma^{n}(x,y)+\hat{\rho}(x,y)\,\rho^{n}(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^{n}(x,y)+\hat{\rho}(x,y)\,\rho^{n}(x,y)\right)\\
&\quad=\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^{n}(x,y)\,\rho^{n}(x,y)\left(\sigma^{r}(x,y)\,\rho^{-r}(x,y)-1\right)\\
&\quad+\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^{n}(x,y)\,\sigma^{n}(x,y)\left(\rho^{r}(x,y)\,\sigma^{-r}(x,y)-1\right)\\
&\quad=(-y)^n\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\left(\sigma^{r}(x,y)\,\rho^{-r}(x,y)-1\right)+\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\left(\rho^{r}(x,y)\,\sigma^{-r}(x,y)-1\right)\right].
\end{aligned}
$$

Thus, the proof is completed.

Note that if we take $r = 1$ in the Theorem 2.6, we obtain the Cassini's identities for the bivariate Pell and bivariate Pell-Lucas hybrinomials. So, we can write following corollary.

Corollary 2.7 (Cassini's Identity). *For $1 \leq n$, we have*

$$
\begin{aligned}
P_{n+1}H(x,y)\,P_{n-1}H(x,y)-\left(P_nH(x,y)\right)^2 &= \frac{(-y)^n\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\left(1-\sigma(x,y)\,\rho^{-1}(x,y)\right)\right.}{(\sigma(x,y)-\rho(x,y))^2}\\
&+\frac{\left.\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\left(1-\rho(x,y)\,\sigma^{-1}(x,y)\right)\right]}{(\sigma(x,y)-\rho(x,y))^2}
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{n+1}H(x,y)\,Q_{n-1}H(x,y)-\left(Q_nH(x,y)\right)^2 &= (-y)^n\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\left(\sigma(x,y)\,\rho^{-1}(x,y)-1\right)\right.\\
&+\left.\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\left(\rho(x,y)\,\sigma^{-1}(x,y)-1\right)\right].
\end{aligned}
$$

The following theorem gives the Honsberger's identities for the bivariate Pell and bivariate Pell-Lucas hybrinomials.

Theorem 2.8 (Honsberger's Identity). *For $n, m \geq 0$, we have*

$$\begin{aligned} P_nH(x,y)\,P_mH(x,y) + P_{n+1}H(x,y)\,P_{m+1}H(x,y) &= \frac{\hat{\sigma}^2(x,y)\,\sigma^{n+m}(x,y)\left(1+\sigma^2(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2} \\ &+ \frac{(y-1)\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^n(x,y)\,\rho^m(x,y)\right.}{(\sigma(x,y)-\rho(x,y))^2} \\ &+ \frac{\left.\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^n(x,y)\,\sigma^m(x,y)\right]}{(\sigma(x,y)-\rho(x,y))^2} \\ &+ \frac{\hat{\rho}^2(x,y)\,\rho^{n+m}(x,y)\left(1+\rho^2(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2} \end{aligned} \tag{17}$$

and

$$\begin{aligned} Q_nH(x,y)\,Q_mH(x,y) + Q_{n+1}H(x,y)\,Q_{m+1}H(x,y) &= \hat{\sigma}^2(x,y)\,\sigma^{n+m}(x,y)\left(1+\sigma^2(x,y)\right) \\ &+ (1-y)\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^n(x,y)\,\rho^m(x,y)\right. \\ &+ \left.\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^n(x,y)\,\sigma^m(x,y)\right] \\ &+ (\hat{\rho}^2(x,y)\,\rho^{n+m}(x,y)\left(1+\rho^2(x,y)\right) \end{aligned} \tag{18}$$

where $P_nH(x,y)$ and $Q_nH(x,y)$ are the nth the bivariate Pell hybrinomial and the nth the bivariate Pell-Lucas hybrinomial, respectively.

Proof. First, let us prove equation (17). Using the equation (9), we have

$$\begin{aligned} &P_nH(x,y)\,P_mH(x,y) + P_{n+1}H(x,y)\,P_{m+1}H(x,y) \\ &= \frac{(\hat{\sigma}(x,y)\,\sigma^n(x,y) - \hat{\rho}(x,y)\,\rho^n(x,y))\,(\hat{\sigma}(x,y)\,\sigma^m(x,y) - \hat{\rho}(x,y)\,\rho^m(x,y))}{(\sigma(x,y)-\rho(x,y))^2} \\ &+ \frac{(\hat{\sigma}(x,y)\,\sigma^{n+1}(x,y) - \hat{\rho}(x,y)\,\rho^{n+1}(x,y))(\hat{\sigma}(x,y)\,\sigma^{m+1}(x,y) - \hat{\rho}(x,y)\,\rho^{m+1}(x,y))}{(\sigma(x,y)-\rho(x,y))^2} \\ &= \frac{\hat{\sigma}^2(x,y)\,\sigma^{n+m}(x,y)\left(1+\sigma^2(x,y)\right) - \hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^n(x,y)\,\rho^m(x,y)\,(1+\sigma(x,y)\,\rho(x,y))}{(\sigma(x,y)-\rho(x,y))^2} \\ &- \frac{\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^n(x,y)\,\sigma^m(x,y)\,(1+\rho(x,y)\,\sigma(x,y)) - \rho^2(x,y)\,\rho^{n+m}(x,y)\left(1+\rho^2(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2} \\ &= \frac{\hat{\sigma}^2(x,y)\,\sigma^{n+m}(x,y)\left(1+\sigma^2(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2} \\ &+ \frac{(y-1)\left[\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^n(x,y)\,\rho^m(x,y) + \hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^n(x,y)\,\sigma^m(x,y)\right]}{(\sigma(x,y)-\rho(x,y))^2} \\ &+ \frac{\hat{\rho}^2(x,y)\,\rho^{n+m}(x,y)\left(1+\rho^2(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2}. \end{aligned}$$

Now, let us prove equation (18). Using the equation (10), we have

$$
\begin{aligned}
&Q_nH(x,y)\,Q_mH(x,y)+Q_{n+1}H(x,y)\,Q_{m+1}H(x,y)\\
&\quad=(\hat{\sigma}(x,y)\,\sigma^n(x,y)+\hat{\rho}(x,y)\,\rho^n(x,y))\,(\hat{\sigma}(x,y)\,\sigma^m(x,y)+\hat{\rho}(x,y)\,\rho^m(x,y))\\
&\quad+(\hat{\sigma}(x,y)\,\sigma^{n+1}(x,y)+\hat{\rho}(x,y)\,\rho^{n+1}(x,y))(\hat{\sigma}(x,y)\,\sigma^{m+1}(x,y)+\hat{\rho}(x,y)\,\rho^{m+1}(x,y))\\
&\quad=\hat{\sigma}^2(x,y)\,\sigma^{n+m}(x,y)\left(1+\sigma^2(x,y)\right)+\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^n(x,y)\,\rho^m(x,y)\,(1+\sigma(x,y)\,\rho(x,y))\\
&\quad+\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^n(x,y)\,\sigma^m(x,y)\,(1+\rho(x,y)\,\sigma(x,y))+\rho^2(x,y)\,\rho^{n+m}(x,y)\left(1+\rho^2(x,y)\right)\\
&\quad=\hat{\sigma}^2(x,y)\,\sigma^{n+m}(x,y)\left(1+\sigma^2(x,y)\right)\\
&\quad+(1-y)\,(\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^n(x,y)\,\rho^m(x,y)+\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^n(x,y)\,\sigma^m(x,y))\\
&\quad+\hat{\rho}^2(x,y)\,\rho^{n+m}(x,y)\left(1+\rho^2(x,y)\right).
\end{aligned}
$$

Therefore, the proof is completed.

The following theorem gives the d'Ocagne's identities for the bivariate Pell and bivariate Pell-Lucas hybrinomials.

Theorem 2.9 (d'Ocagne's Identity). *For $n \le m$, we have*

$$
P_mH(x,y)P_{n+1}H(x,y)-P_{m+1}H(x,y)P_nH(x,y)=\frac{(-y)^n\left[\hat{\sigma}(x,y)\hat{\rho}(x,y)\,\sigma^{m-n}(x,y)-\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\rho^{m-n}(x,y)\right]}{\sigma(x,y)-\rho(x,y)} \quad (19)
$$

and

$$
\begin{aligned}
Q_mH(x,y)\,Q_{n+1}H(x,y)-Q_{m+1}H(x,y)\,Q_nH(x,y) &= (-y)^n\,(\sigma(x,y)-\rho(x,y))\left[-\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^{m-n}(x,y)\right.\\
&+\left.\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\rho^{m-n}(x,y)\right] \quad (20)
\end{aligned}
$$

where $P_nH(x,y)$ and $Q_nH(x,y)$ are the nth the bivariate Pell hybrinomial and the nth the bivariate Pell-Lucas hybrinomial, respectively.

Proof. First, let us prove equation (19). Using the equation (9), we have

$$
\begin{aligned}
&P_mH(x,y)\,P_{n+1}H(x,y)-P_{m+1}H(x,y)\,P_nH(x,y)\\
&\quad=\frac{\left(\hat{\sigma}(x,y)\,\sigma^m(x,y)-\hat{\rho}(x,y)\,\rho^m(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^{n+1}(x,y)-\hat{\rho}(x,y)\,\rho^{n+1}(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad-\frac{\left(\hat{\sigma}(x,y)\,\sigma^{m+1}(x,y)-\hat{\rho}(x,y)\,\rho^{m+1}(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^n(x,y)-\hat{\rho}(x,y)\,\rho^n(x,y)\right)}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad=\frac{\hat{\sigma}(x,y)\hat{\rho}(x,y)\,\sigma^m(x,y)\,\rho^n(x,y)\,(-\rho(x,y)+\sigma(x,y))}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad-\frac{\hat{\rho}(x,y)\hat{\sigma}(x,y)\,\rho^m(x,y)\,\sigma^n(x,y)\,(\sigma(x,y)-\rho(x,y))}{(\sigma(x,y)-\rho(x,y))^2}\\
&\quad=\frac{\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^m(x,y)\,\rho^n(x,y)-\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^m(x,y)\,\sigma^n(x,y)}{\sigma(x,y)-\rho(x,y)}\\
&\quad=\frac{(-y)^n\left[\hat{\sigma}(x,y)\hat{\rho}(x,y)\,\sigma^{m-n}(x,y)-\hat{\rho}(x,y)\,\hat{\sigma}(x,y)\rho^{m-n}(x,y)\right]}{\sigma(x,y)-\rho(x,y)}.
\end{aligned}
$$

Now, let us prove equation (20). Using the equation (10), we have

$$\begin{aligned}
&Q_mH(x,y)\,Q_{n+1}H(x,y) - Q_{m+1}H(x,y)\,Q_nH(x,y)\\
&= \left(\hat{\sigma}(x,y)\,\sigma^m(x,y) + \hat{\rho}(x,y)\,\rho^m(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^{n+1}(x,y) + \hat{\rho}(x,y)\,\rho^{n+1}(x,y)\right)\\
&- \left(\hat{\sigma}(x,y)\,\sigma^{m+1}(x,y) + \hat{\rho}(x,y)\,\rho^{m+1}(x,y)\right)\left(\hat{\sigma}(x,y)\,\sigma^{n}(x,y) + \hat{\rho}(x,y)\,\rho^{n}(x,y)\right)\\
&= -\hat{\sigma}(x,y)\hat{\rho}(x,y)\,\sigma^m(x,y)\,\rho^n(x,y)\,(-\rho(x,y) + \sigma(x,y))\\
&+\hat{\rho}(x,y)\hat{\sigma}(x,y)\,\rho^m(x,y)\,\sigma^n(x,y)\,(\sigma(x,y) - \rho(x,y))\\
&= (\sigma(x,y) - \rho(x,y)\,[-\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^m(x,y)\,\rho^n(x,y) + \hat{\rho}(x,y)\,\hat{\sigma}(x,y)\,\rho^m(x,y)\,\sigma^n(x,y)]\\
&= (-y)^n\,(\sigma(x,y) - \rho(x,y))\left[-\hat{\sigma}(x,y)\,\hat{\rho}(x,y)\,\sigma^{m-n}(x,y) + \hat{\rho}(x,y)\,\hat{\sigma}(x,y)\rho^{m-n}(x,y)\right].
\end{aligned}$$

Hence, the proof is completed.

3. Conclusion and Suggestion

The generalizations and applications of numbers and polynomials defined by recurrence relations have been presented in many ways. In this study, the bivariate Pell and bivariate Pell-Lucas hybrinomials are defined by extending the classes of the bivariate Pell and bivariate Pell-Lucas polynomials and hybrid numbers, which arise from the combination of different algebraic structures. The Binet formulas for both the nth bivariate Pell hybrinomial and the nth bivariate Pell-Lucas hybrinomial were derived, along with matrix representations. Additionally, the generating functions, Catalan's identity, Cassini's identity, d'Ocagne's identity and Honsberger's identity for these hybrinomials.

It would be interesting to explore these hybrinomials further in matrix theory and linear algebra. Additionally, more general formulas for calculating the nth terms and sum formulas for the sequences of these hybrinomials can be investigated.

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