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Generalization of Giaccardi Inequality for Isotonic Linear Functionals

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Abstract. In this article, Giaccardi inequality is generalized for isotonic linear functional with the help of (h, m)-convex functions. As a special case, Petrović inequality for (h, m)-convex functions is established for isotonic linear functional. The results obtained by setting different isotonic linear functionals along with different values of h and m are discussed. Results are also extended for different time scales given in the literature.

1. Introduction and Preliminaries

The theory of convexity holds an important position in mathematics and has been the focus of intensive research for more than a century. In recent years, theory of convexity has received special attention by many researchers because of its importance in different fields of pure and applied sciences.

G. Toader [32] made significant contributions to this field by introducing m-convex functions. In [33], S. Varošanec introduced the definition of h-convex functions, which further expanded the scope of convexity in research. More recently, M. E. Ozdemir et. al. [27] took this research one step further by introducing (h, m)-convex functions and deriving several important Hermite-Hadamard type inequalities. One can get the convex functions, P-functions [25], s-convex functions in second sense [6], Godunova-Levin functions [16], and s-Godunova-Levin functions in the second sense [23] by taking $m = 1$ and $h(\xi) = \left\{\xi, 1, \xi^s, \frac{1}{\xi}, \frac{1}{\xi^s}\right\}$ for $\xi, s \in (0, 1)$ respectively. Also, h-convex [33], m-convex [32] and (s, m)-convex functions in the second sense [14] can be obtained by taking $m = 1$, $h(\xi) = \xi$ and $h(\xi) = \xi^s$ respectively.

In the literature, many scientists and mathematicians considered convex functions and isotonic linear functional to produce interesting and useful results. The older examples include generalization of a famous Jensen inequality for isotonic linear functional given by Jessen [18] in 1931. In [3, p. 537], P. R. Beesack and J. E. Pečarić introduced the isotonic linear functional and generalized the Jensen inequality for convex functions. They also find some general complementary inequalities for such isotonic linear functional. S. S. Dragomir gave the refinement of Hadamard's inequality for isotonic linear functionals in [11, Theorem 2.3]. G. S. Yang and H. L. Wu [34, Theroem 2.1] defined a new result involving isotonic linear functionals by comparing with the result given by W. H. Yang in [35]. In [12, Theorem 4], for isotonic linear functional, S. S. Dragomir gave the Jensen's inequality for $m - \varphi$-convex and $M - \varphi$-convex functions. A Grüss type inequality for normalized isotonic linear functional is given in [13]. Some integral inequalities involving

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isotonic linear functional have been defined by M. Anwar et. al. in [2]. D. Chen et. al. gave the Giaccardi inequality for s-convex functions for isotonic linear functional in [7, Theorem 2].

Many examples of isotonic linear functional, say $\mathcal{I}$, are provided in [2]. Examples which are very common and simple included:

$$\mathcal{I}(\varphi(\omega)) = \sum_{n\in E} \sigma_n \varphi_n,$$

when E is a subset of {1,2,3, ...} with all $\sigma_n > 0$ and

$$\mathcal{I}(\varphi(\omega)) = \int_E \varphi d\omega,$$

when ω is a positive measure on some suitable set E.

Before discussing the motivation of the paper it is important to review some basic definitions.

Definition 1.1. *[32] A function $\mathcal{K} : [0, b) \to \mathbb{R}$, $b > 0$ is m-convex, if*

$$\mathcal{K}(\xi\omega + (1-\xi)w) \leq \xi\mathcal{K}(\omega) + m(1-\xi)\mathcal{K}(w), \forall \omega, w \in [0, b), m, \xi \in [0, 1]. \tag{1}$$

Definition 1.2. *[27] Let $h : (0, 1) \subseteq J \to \mathbb{R}$ be a non-negative functions. A function $\mathcal{K} : [0, b) \to \mathbb{R}$, $b > 0$ is (h, m)-convex, $m \in [0, 1]$, if*

$$\mathcal{K}(\xi\omega + m(1-\xi)w) \leq h(\xi)\mathcal{K}(\omega) + mh(1-\xi)\mathcal{K}(w), \ \forall \omega, w \in [0, b), \xi \in (0, 1). \tag{2}$$

Definition 1.3. *[29] Let E be a nonempty set and $\mathfrak{L}(E)$ be a class of real-valued functions $\mathcal{K} : E \to \mathbb{R}$ having the properties:*

L1: If $\mathcal{K}, \varphi \in \mathfrak{L}(E)$, then $\omega\mathcal{K} + w\varphi \in \mathfrak{L}(E)$ for all $\omega, w \in \mathbb{R}$;

L2: $1 \in \mathfrak{L}(E)$, that is, if $\mathcal{K}(\omega) = 1$ for $\omega \in E$, then $\mathcal{K} \in \mathfrak{L}(E)$.

An isotonic linear functional is a functional $A : \mathfrak{L}(E) \to \mathbb{R}^+$ that satisfies the following axioms

A1: $\mathcal{I}(\omega\mathcal{K} + w\varphi) = \omega\mathcal{I}(\mathcal{K}) + w\mathcal{I}(\varphi)$ for $\mathcal{K}, \varphi \in \mathfrak{L}(E), \omega, w \in \mathbb{R}$;

A2: If $\mathcal{K} \in \mathfrak{L}(E), \mathcal{K}(\omega) \geq 0$ on E, then $\mathcal{I}(\mathcal{K}) \geq 0$.

The functional version of Giaccardi inequality proved by Beesack and Pečarić [3, Theorem 12] is given in the following theorem.

Theorem 1.4. *Let $\mathcal{I}$ be an isotonic linear functional defined on $\mathfrak{L}(E)$, $\omega_0 \in \mathbb{R}$ and $\varphi \in \mathfrak{L}(E)$ such that $\omega_0 \neq \mathcal{I}(\varphi(\omega))$ and*

$$(\mathcal{I}(\varphi(\omega)) - \varphi(\omega))(\varphi(\omega) - \omega_0) \geq 0, \ \ \forall \omega \in E. \tag{3}$$

If a function $\mathcal{K}$ is convex on $[\omega_0, \mathcal{I}(\varphi(\omega))]$ or on $[\mathcal{I}(\varphi(\omega)), \omega_0]$ and $\mathcal{K}(\varphi) \in \mathfrak{L}(E)$, then

$$\mathcal{I}(\mathcal{K}(\varphi)) \leq \{[\mathcal{I}(\varphi(\omega)) - \mathcal{I}(1)\omega_0]\mathcal{K}(\mathcal{I}(\varphi(\omega))) + [\mathcal{I}(1) - 1]\mathcal{I}(\varphi(\omega))\mathcal{K}(\omega_0)\}/[\mathcal{I}(\varphi(\omega)) - \omega_0]. \tag{4}$$

In this paper, we obtain a generalization of Theorem 1.4 for m-convex and (h, m)-convex functions. We then discuss different variants of this generalization for specific values of the isotonic linear functional, the function h, and the number m, referring to already published results in the literature. To make this paper reader friendly, throughout the paper, we assume that $\mathcal{I}$ be an isotonic linear functional defined on $\mathfrak{L}(E)$, $\varphi \in \mathfrak{L}(E)$ and $\omega_0 \in \mathbb{R}$ such that $\omega_0 \neq \mathcal{I}(\varphi(\omega))$. Also, (Ω, Λ, ρ) is a measurable space, where $\rho(\Omega)$ is positive finite measure and $h : (0, 1) \subseteq J \to \mathbb{R}$ be a positive multiplicative function.

2. Results for *m*-convex functions

In this section, we derive the functional version of Giaccardi and Petrović's inequalities with the help of m-convex functions. These inequalities are also derived in discrete and integral forms as a special case of isotonic linear functional.

Here we present a lemma that establishes an important result regarding the properties of m-convex functions.

Lemma 2.1. *Let $\mathcal{K} : [0,\infty) \to \mathbb{R}$ be an m-convex function. Then for $u, v, w \in [0,\infty), m \in (0,1], u < v < w$, one has*

$$(w-v)\mathcal{K}(u) - (w-u)\mathcal{K}(v) + m(v-u)\mathcal{K}\left(\frac{w}{m}\right) \geq 0. \tag{5}$$

Proof. Since $\mathcal{K}$ is an m-convex function, so by taking $\tau = \frac{p}{p+q}$ in (1.1), one has

$$\mathcal{K}\left(\frac{p}{p+q}u + m\frac{q}{p+q}v\right) \leq \left(\frac{p}{p+q}\right)\mathcal{K}(u) + m\left(1-\left(\frac{p}{p+q}\right)\right)\mathcal{K}(v). \tag{6}$$

Let $p = w - v$, $q = v - u$ and $v = \frac{w}{m}$. Then

$$\mathcal{K}(v) \leq \left(\frac{w-v}{w-u}\right)\mathcal{K}(u) + m\left(\frac{v-u}{w-u}\right)\mathcal{K}\left(\frac{w}{m}\right).$$

Since $w - u > 0$, one can multiply $(w-u)$ on both sides of the above inequality to get the required result. □

Remark 2.2. *The result for convex function given in [29, p. 02] can be obtained by setting $m = 1$, in* (5).

The following theorem establishes the Giaccardi inequality for isotonic linear functional involving m-convex functions.

Theorem 2.3. *Let the condition* (3) *is satisfied. If $\mathcal{K} : [0,\infty) \to \mathbb{R}$ be an m-convex function, then*

$$\begin{aligned}\mathcal{I}(\mathcal{K}(\varphi)) \leq \min\Bigg\{&\frac{m\left(\mathcal{I}(\varphi(\omega)) - \mathcal{I}(1)\omega_0\right)}{\mathcal{I}(\varphi(\omega)) - \omega_0}\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right) + \frac{\mathcal{I}(\varphi(\omega))(\mathcal{I}(1)-1)}{\mathcal{I}(\varphi(\omega)) - \omega_0}\mathcal{K}(\omega_0),\\ &\frac{m\mathcal{I}(\varphi(\omega))(\mathcal{I}(1)-1)}{\mathcal{I}(\varphi(\omega)) - \omega_0}\mathcal{K}\left(\frac{\omega_0}{m}\right) + \frac{\mathcal{I}\left(\mathcal{I}(\varphi(\omega)) - \mathcal{I}(1)\omega_0\right)}{\mathcal{I}(\varphi(\omega)) - \omega_0}\mathcal{K}(\mathcal{I}(\varphi(\omega)))\Bigg\}.\end{aligned} \tag{7}$$

Proof. We can determine from condition (3)

$$\omega_0 \leq \varphi(\omega) \leq \mathcal{I}(\varphi(\omega)) \text{ for all } \omega \in E \tag{8}$$

or

$$\mathcal{I}(\varphi(\omega)) \leq \varphi(\omega) \leq \omega_0 \text{ for all } \omega \in E. \tag{9}$$

First, we solve it for condition (8). As $\mathcal{K}$ be an m-convex function, one can set $u = \omega_0$, $v = \varphi(\omega)$ and $w = \mathcal{I}(\varphi(\omega))$ in Lemma 2.1 to get

$$\left(\mathcal{I}(\varphi(\omega)) - \varphi(\omega)\right)\mathcal{K}(\omega_0) - \left(\mathcal{I}(\varphi(\omega)) - \omega_0\right)\mathcal{K}\left(\varphi(\omega)\right) + m\left(\varphi(\omega) - \omega_0\right)\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right) \geq 0.$$

Divide the above inequality by $\mathcal{I}(\varphi(\omega)) - \omega_0 \geq 0$, one has

$$\frac{m(\varphi(\omega) - \omega_0)}{\mathcal{I}(\varphi(\omega)) - \omega_0}\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right) + \frac{\mathcal{I}(\varphi(\omega)) - \varphi(\omega)}{\mathcal{I}(\varphi(\omega)) - \omega_0}\mathcal{K}(\omega_0) - \mathcal{K}\left(\varphi(\omega)\right) \geq 0.$$

Using the property of the isotonic linear functional stated in clause (A2) , one has

$$\mathcal{I}\left(\frac{m(\varphi(\omega)-\omega_0)}{\mathcal{I}(\varphi(\omega))-\omega_0}\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right)+\frac{\mathcal{I}(\varphi(\omega))-\varphi(\omega)}{\mathcal{I}(\varphi(\omega))-\omega_0}\mathcal{K}(\omega_0)-\mathcal{K}(\varphi(\omega))\right)\geq 0.$$

Since m, $\mathcal{I}(\varphi(\omega))-\omega_0$, $\mathcal{K}(\omega_0)$ and $\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right)$ are reals, by using clause (A1) of the isotonic linear functional, one has

$$\frac{m\left(\mathcal{I}(\varphi(\omega))-\mathcal{I}(1)\omega_0\right)}{\mathcal{I}(\varphi(\omega))-\omega_0}\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right)+\frac{\mathcal{I}(\varphi(\omega))(\mathcal{I}(1)-1)}{\mathcal{I}(\varphi(\omega))-\omega_0}\mathcal{K}(\omega_0)-\mathcal{I}\left(\mathcal{K}(\varphi)\right)\geq 0. \tag{10}$$

In a similar way, for condition (9), one can take $u=\mathcal{I}(\varphi(\omega))$, $v=\varphi(\omega)$ and $w=\omega_0$ in Lemma 2.1 to get

$$\frac{m\mathcal{I}(\varphi(\omega))(\mathcal{I}(1)-1)}{\mathcal{I}(\varphi(\omega))-\omega_0}\mathcal{K}\left(\frac{\omega_0}{m}\right)+\frac{\mathcal{I}\left(\mathcal{I}(\varphi(\omega))-\mathcal{I}(1)\omega_0\right)}{\mathcal{I}(\varphi(\omega))-\omega_0}\mathcal{K}(\mathcal{I}(\varphi(\omega)))-\mathcal{I}\left(\mathcal{K}(\varphi)\right)\geq 0. \tag{11}$$

From the inequalities (10) and (11), one gets the required result. □

Remark 2.4. *By setting $m=1$ in* (7), *one gets Theorem 1.4.*

Petrović inequality for isotonic linear functionals is given in the following corollary for m-convex functions.

Corollary 2.5. *Let $\mathcal{I}(\Psi(\omega))\geq\Psi(x)$ or $\Psi(\omega)\geq\mathcal{I}(\Psi(\omega))$, $\forall\omega\in E$. Also let the condition* (3) *is satisfied. If $\mathcal{K}:[0,\infty)\to\mathbb{R}$ be an m-convex function, then*

$$\begin{aligned}\mathcal{I}(\mathcal{K}(\varphi))\leq\min\Bigg\{&\frac{m\left(\mathcal{I}(\varphi(\omega))\right)}{\mathcal{I}(\varphi(\omega))}\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right)+\frac{\mathcal{I}(\varphi(\omega))(\mathcal{I}(1)-1)}{\mathcal{I}(\varphi(\omega))}\mathcal{K}(0),\\ &\frac{m\mathcal{I}(\varphi(\omega))(\mathcal{I}(1)-1)}{\mathcal{I}(\varphi(\omega))}\mathcal{K}(0)+\frac{\mathcal{I}\left(\mathcal{I}(\varphi(\omega))\right)}{\mathcal{I}(\varphi(\omega))}\mathcal{K}(\mathcal{I}(\varphi(\omega)))\Bigg\}.\end{aligned} \tag{12}$$

Proof. The required result can be obtained by taking $\omega_0=0$ in (7). □

Remark 2.6. *By taking $m=1$ in (12), one gets the Petrović inequality for isotonic linear functional via convex functions.*

The following theorem consists of the Giaccardi inequality for m-convex functions. It is the generalization of a result proved by M. K. Bakula et. al. [20, Theorem 3.3].

Theorem 2.7. *Let ω be a non-negative,* $\mathbf{q}$ *be positive n-tuples and $\omega_0,\tilde{\omega}_n:=\sum_{i=1}^{n}q_i\omega_i\in[0,\infty)$ such that*

$$(\omega_i-\omega_0)(\tilde{\omega}_n-\omega_i)\geq 0 \text{ and } \tilde{\omega}_n\neq\omega_0. \tag{13}$$

If $\mathcal{K}:[0,\infty)\to\mathbb{R}$ be an m-convex function, then

$$\sum_{i=1}^{n}q_i\mathcal{K}(\omega_i)\leq\min\left\{m\mathcal{S}\mathcal{K}\left(\frac{\tilde{\omega}_n}{m}\right)+\mathcal{T}\mathcal{K}(\omega_0),\mathcal{S}\mathcal{K}(\tilde{\omega}_n)+m\mathcal{T}\mathcal{K}\left(\frac{\omega_0}{m}\right)\right\}, \tag{14}$$

where

$$\mathcal{S}=\frac{\sum_{i=1}^{n}q_i(\omega_i-\omega_0)}{\tilde{\omega}_n-\omega_0} \text{ and } \mathcal{T}=\frac{\sum_{i=1}^{n}q_i(\tilde{\omega}_n-\omega_i)}{\tilde{\omega}_n-\omega_0}.$$

Proof. Let $\mathcal{I}(\varphi(\omega))=\sum_{i\in E}q_i\varphi(i)=\sum_{i=1}^{n}q_i\omega_i$ and $\mathfrak{L}(E)=\{\varphi:E\to[0,\infty)\mid\varphi(i)=\omega_i,i\in E\}$, where $E=\{1,2,...,n\}$. Then isotonic linear functional A satisfy conditions A1, A2 and $\mathfrak{L}(E)$ satisfy conditions L1, L2 of Definition 1.3. The criteria stated in (13) and (14) are satisfied by replacing the aforementioned values of $\mathcal{I}$ and φ in Theorem 2.3. □

Remark 2.8. *By taking $m = 1$ in (14), one gets the well known Giaccardi inequality [28, Theorem 1.1].*

Petrović's inequality for m-convex function is given in the following corollary. M. K. Bakula et. al. proved a similar result in [20].

Corollary 2.9. *Let the conditions of Theorem 2.7 are satisfied with minor changes such as $\tilde{\omega}_n \geq \omega_j$ or $\omega_j \geq \tilde{\omega}_n$ for $j = 1, ..., n$. Then*

$$\sum_{i=1}^{n} q_i \mathcal{K}(\omega_i) \leq \min\left\{ m\mathcal{K}\left(\frac{\tilde{\omega}_n}{m}\right) + \left(\sum_{i=1}^{n} q_i - 1\right)\mathcal{K}(0), \mathcal{K}(\tilde{\omega}_n) + m\left(\sum_{i=1}^{n} q_i - 1\right)\mathcal{K}(0) \right\}. \tag{15}$$

Proof. By taking $\omega_0 = 0$ in (14), one gets the required result. □

Remark 2.10. *One can get the classical Petrović's inequality [29] by setting $m = 1$ in Corollary 2.9.*

The following theorem presents the integral form of Theorem 2.3.

Theorem 2.11. *Let $\varphi : \Omega \to [0, \infty)$ be a measurable function and $\omega_0, \int_\Omega \varphi(\omega)d\sigma \in [0, \infty)$ such that $\int_\Omega \varphi(\omega)d\sigma \neq \omega_0$ and*

$$(\varphi(\omega) - \omega_0)\left(\int_\Omega \varphi(\omega)d\sigma - \varphi(\omega)\right) \geq 0. \tag{16}$$

If $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be an m-convex function, then

$$\begin{aligned} \int_\Omega \mathcal{K}(\varphi)d\sigma \leq \min\Bigg\{ & m\mathcal{C}\mathcal{K}\left(\frac{1}{m}\int_\Omega \varphi(\omega)d\sigma\right) + \mathcal{D}\mathcal{K}(\omega_0), \\ & \mathcal{C}\mathcal{K}\left(\int_\Omega \varphi(\omega)d\sigma\right) + m\mathcal{D}\mathcal{K}\left(\frac{\omega_0}{m}\right)\Bigg\}, \end{aligned} \tag{17}$$

where

$$\mathcal{C} = \frac{\int_\Omega (\varphi(\omega) - \omega_0)\, d\sigma}{\int_\Omega \varphi(\omega)d\sigma - \omega_0} \quad \text{and} \quad \mathcal{D} = \frac{\int_\Omega \left(\int_\Omega \varphi(\omega)d\sigma - \varphi(\omega)\right) d\sigma}{\int_\Omega \varphi(\omega)d\sigma - \omega_0}.$$

Proof. Assume that $\mathcal{I}(\varphi(\omega)) = \int_\Omega \varphi(\omega)d\sigma$ and

$$\mathfrak{L}(E) = \left\{ \varphi : \Omega \to [0, \infty) \mid \int_\Omega \varphi(\omega)d\sigma \text{ exists} \right\}, \text{ where } E = \Omega.$$

Then $\mathcal{I}(\varphi(\omega))$ and $\mathfrak{L}(E)$ satisfies conditions A1, A2 and L1, L2 of Definition 1.3. Substituting the above values of $\mathcal{I}(\varphi(\omega))$ and $\mathfrak{L}(E)$ in Theorem 2.3, one has the required conditions and result. □

In the following corollary, the integral version of Corollary 2.5 for m-convex function is given.

Corollary 2.12. *Suppose that the conditions in Theorem 2.11 are valid with minor changes as follows:*

$$\int_\Omega \varphi(\omega)d\sigma \geq \varphi(\omega) \text{ or } \int_\Omega \varphi(\omega)d\sigma \leq \varphi(\omega) \text{ for } \omega \in \Omega. \tag{18}$$

If $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be an m-convex function, then

$$\begin{aligned} \int_\Omega \mathcal{K}(\varphi)d\sigma \leq \min\Bigg\{ & m\mathcal{K}\left(\frac{1}{m}\int_\Omega \varphi(\omega)d\sigma\right) + \left(\int_\Omega d\sigma - 1\right)\mathcal{K}(0), \\ & \mathcal{K}\left(\int_\Omega \varphi(\omega)d\sigma\right) + m\left(\int_\Omega d\sigma - 1\right)\mathcal{K}(0)\Bigg\}. \end{aligned} \tag{19}$$

Proof. By taking $\omega_0 = 0$ in Theorem 2.11, one gets (19). □

Remark 2.13. *By taking $m = 1$ in Corollary 2.12, gives the result given in [7, Corollary, 6].*

3. Results for (h, m)-convex functions

In this section, the functional version of the Giaccardi inequality for (h, m)-convex functions is derived. The particular case of Giaccardi inequality known as Petrović's inequality is derived for (h, m)-convex functions. On the other hand, as a special case of isotonic linear functional, discrete and integral versions of these inequalities are also derived, but with less conditions imposed on h as compared to the results given in [30].

To present the key findings of this section, following definition and result are very important.

Definition 3.1. *A function $h : (0,1) \subseteq J \to \mathbb{R}$ is super-multiplicative, if*

$$h(z_1 z_2) \geq h(z_1)h(z_2), \quad \forall z_1, z_2 \in J, \text{ and } z_1, z_2 \geq 1. \tag{20}$$

If inequality (20) *is reversed, then h is said to be sub-multiplicative function.*
If the equality holds in (20)*, then h is said to be a multiplicative function.*

In order to establish the primary findings of our research, the following lemma is an important step. That is significant since it offers a crucial insight that is required for the validation of our major claim.

Lemma 3.2. *Let $\mathcal{K} : [0,\infty) \to \mathbb{R}$ be an (h, m)-convex function. Then for $u, v, w \in [0,\infty), m \in (0,1], u < v < w$ such that $w - u, w - v, v - u \in J$, the following inequality holds:*

$$h(w-v)\mathcal{K}(u) - h(w-u)\mathcal{K}(v) + mh(v-u)\mathcal{K}\left(\frac{w}{m}\right) \geq 0. \tag{21}$$

Proof. Let $\mathcal{K}$ is an (h, m)-convex function, and $u, v, w \in [0,\infty)$ be numbers which satisfy the assumptions of the Lemma 3.2. Then $\frac{w-\omega}{w-u} \in (0,1) \subseteq J$, $\frac{\omega-u}{w-u} \in (0,1) \subseteq J$ and $\frac{w-\omega}{w-u} + \frac{\omega-u}{w-u} = 1$.

Since $\mathcal{K}$ is an (h, m)-convex function, so by taking $\tau = \frac{p}{p+q}$ in (2), one has

$$\mathcal{K}\left(\frac{p}{p+q}u + m\frac{q}{p+q}\omega\right) \leq h\left(\frac{p}{p+q}\right)\mathcal{K}(u) + mh\left(1 - \left(\frac{p}{p+q}\right)\right)\mathcal{K}(\omega). \tag{22}$$

Let $p = w - \omega$, $q = \omega - u$ and $\omega = \frac{w}{m}$. Then

$$\begin{aligned}\mathcal{K}\left(\frac{p}{p+q}u + m\frac{q}{p+q}\omega\right) &= \mathcal{K}\left(\frac{(w-\omega)u + m(\omega-u)\frac{w}{m}}{w-\omega+\omega-u}\right)\\ &= \mathcal{K}(\omega)\end{aligned}$$

and

$$h\left(\frac{p}{p+q}\right)\mathcal{K}(u) + mh\left(\frac{q}{p+q}\right)\mathcal{K}(\omega) = h\left(\frac{w-\omega}{w-u}\right)\mathcal{K}(u) + mh\left(\frac{\omega-u}{w-u}\right)\mathcal{K}\left(\frac{w}{m}\right).$$

Putting these values in (22), one has

$$\mathcal{K}(\omega) \leq h\left(\frac{w-\omega}{w-u}\right)\mathcal{K}(u) + mh\left(\frac{\omega-u}{w-u}\right)\mathcal{K}\left(\frac{w}{m}\right).$$

Since h is a positive multiplicative function. Then

$$\mathcal{K}(\omega) \leq \frac{h(w-\omega)}{h(w-u)}\mathcal{K}(u) + m\frac{h(\omega-u)}{h(w-u)}\mathcal{K}\left(\frac{w}{m}\right).$$

Multiplying the above inequality by $h(w-u)$, the desired result can be obtained. □

Remark 3.3. *To get the result given by S. Varošanic [33, Proposition 16], take $m = 1$ in (21).*

The following theorem establishes the Giaccardi inequality for isotonic linear functional involving (h, m)-convex functions.

Theorem 3.4. *Let the conditions of Theorem 1.4 be satisfied. If $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be an (h, m)-convex function, then*

$$\mathcal{I}(\mathcal{K}(\varphi)) \leq \min\left\{ \frac{m\mathcal{I}(h(\varphi(\omega) - \omega_0))}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}\left(\frac{\mathcal{I}(\varphi)}{m}\right) + \frac{\mathcal{I}(h(\mathcal{I}(\varphi) - \varphi(\omega)))}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}(\omega_0), \right. \left. \frac{\mathcal{I}(h(\varphi(\omega) - \omega_0))}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}(\mathcal{I}(\varphi)) + m\frac{\mathcal{I}(h(\mathcal{I}(\varphi) - \varphi(\omega)))}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}\left(\frac{\omega_0}{m}\right) \right\}. \tag{23}$$

Proof. Since h is a positive multiplicative function and $\mathcal{K}$ is an (h, m)-convex function. As shown in the proof of Theorem 2.3, satisfying condition (3) yields both (8) and (9). First, we solve it for condition (8). One can set $u = \omega_0$, $v = \varphi(\omega)$ and $w = \mathcal{I}(\varphi(\omega))$ in Lemma 3.2 to get the following inequality.

$$h\left(\mathcal{I}(\varphi) - \varphi(\omega)\right)\mathcal{K}(\omega_0) - h\left(\mathcal{I}(\varphi) - \omega_0\right)\mathcal{K}\left(\varphi(\omega)\right) + mh\left(\varphi(\omega) - \omega_0\right)\mathcal{K}\left(\frac{\mathcal{I}(\varphi)}{m}\right) \geq 0.$$

A function h is given to be positive, so we can write

$$\frac{mh(\varphi(\omega) - \omega_0)}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}\left(\frac{\mathcal{I}(\varphi)}{m}\right) + \frac{h(\mathcal{I}(\varphi) - \varphi(\omega))}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}(\omega_0) - \mathcal{K}(\varphi) \geq 0.$$

As the right hand side of above inequality is in $\mathfrak{L}(E)$, one can apply the property of the isotonic linear functional given in clause (A2) to obtain

$$\mathcal{I}\left(\frac{mh(\varphi(\omega) - \omega_0)}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}\left(\frac{\mathcal{I}(\varphi)}{m}\right) + \frac{h\mathcal{I}(\varphi) - \varphi(\omega)}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}(\omega_0) - \mathcal{K}(\varphi)\right) \geq 0.$$

Since m, $h(\mathcal{I}(\varphi(\omega)) - \omega_0)$, $\mathcal{K}(\omega_0)$ and $\mathcal{K}\left(\frac{\mathcal{I}(\varphi(\omega))}{m}\right)$ are reals, the isotonic linear functional clause (A1) can be used to obtain the following inequality:

$$\frac{m\mathcal{I}\left(h(\varphi(\omega) - \omega_0)\right)}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}\left(\frac{\mathcal{I}(\varphi)}{m}\right) + \frac{\mathcal{I}\left(h(\mathcal{I}(\varphi) - \varphi(\omega))\right)}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}(\omega_0) - \mathcal{I}\left(\mathcal{K}(\varphi)\right) \geq 0. \tag{24}$$

In a similar way, for condition (9), one can set $u = \mathcal{I}(\varphi(\omega))$, $v = \varphi(\omega)$ and $w = \omega_0$ in Lemma 3.2 to get required result.

$$\frac{m\mathcal{I}\left(h(\mathcal{I}(\varphi) - \varphi(\omega))\right)}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}\left(\frac{\omega_0}{m}\right) + \frac{\mathcal{I}\left(h(\varphi(\omega) - \omega_0)\right)}{h(\mathcal{I}(\varphi) - \omega_0)}\mathcal{K}(\mathcal{I}(\varphi)) - \mathcal{I}\left(\mathcal{K}(\varphi)\right) \geq 0. \tag{25}$$

From (24) and (25), one gets the required result. □

Remark 3.5. *For different values of h and m, one has the following results:*

1. *A similar result for s-convex function given in [7, Theorem 2] can be obtained by taking $h(\varphi) = (\varphi)^s$ and $m = 1$ in (23).*
2. *Result for h-convex function given by Rehman et. al. [31] can be obtained by taking $m = 1$ in (23).*
3. *Setting $h(\varphi) = (\varphi)^s$ in (23) gives the result for (s, m)-convex functions.*

A Petrović inequality for isotonic linear functional for (h, m)-convex functions is given in the following corollary.

Corollary 3.6. *Let the conditions of Theorem 3.4 be satisfied with minar changes as $\tilde{\omega}_n \geq \omega_j$ or $\omega_j \geq \tilde{\omega}_n$ for $j = 1, ..., n$. Then*

$$\mathcal{I}(\mathcal{K}(\varphi)) \leq \min\left\{ \frac{m\mathcal{I}(h(\varphi))}{h(\mathcal{I}(\varphi))}\mathcal{K}\left(\frac{\mathcal{I}(\varphi)}{m}\right) + \frac{\mathcal{I}(h(\mathcal{I}(\varphi) - \varphi(\omega)))}{h(\mathcal{I}(\varphi))}\mathcal{K}(0), \right. \left. \frac{\mathcal{I}(h(\varphi))}{h(\mathcal{I}(\varphi))}\mathcal{K}(\mathcal{I}(\varphi)) + m\frac{\mathcal{I}(h(\mathcal{I}(\varphi) - \varphi(\omega)))}{h(\mathcal{I}(\varphi))}\mathcal{K}(0) \right\}. \tag{26}$$

Proof. It is easy to see that if we take $\omega_0 = 0$ in Theorem 3.4, then the condition (3) becomes the condition of this corollary and inequality (23) becomes our required result. □

A Giaccardi inequality for (h, m)-convex function is given in the following theorem. A similar result has been proved with different conditions by M. Andrić and J. E. Pečarić in [1, Theorem 4.2].

Theorem 3.7. *Let the conditions of Theorem 2.7 are valid. If $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be an (h, m)-convex function, then*

$$\sum_{i=1}^{n} q_i \mathcal{K}(\omega_i) \leq \min\left\{m\mathcal{I}\mathcal{K}\left(\frac{\tilde{\omega}_n}{m}\right) + \mathcal{J}\mathcal{K}(\omega_0), \mathcal{I}\mathcal{K}(\tilde{\omega}_n) + m\mathcal{J}\mathcal{K}\left(\frac{\omega_0}{m}\right)\right\}, \tag{27}$$

where

$$\mathcal{I} = \frac{\sum_{i=1}^{n} q_i h(\omega_i - \omega_0)}{h(\tilde{\omega}_n - \omega_0)} \quad \text{and} \quad \mathcal{J} = \frac{\sum_{i=1}^{n} q_i h(\tilde{\omega}_n - \omega_i)}{h(\tilde{\omega}_n - \omega_0)}. \tag{28}$$

Proof. Assume that $\mathfrak{L}(E) = \{\varphi : E \to [0, \infty) \mid \varphi(i) = \omega_i, i \in E\}$, where $E = \{1, 2, ..., n\}$, and

$$\mathcal{I}(\varphi(\omega)) = \sum_{i \in E} q_i \varphi(i) = \sum_{i=1}^{n} q_i \omega_i.$$

Then $\mathfrak{L}(E)$ satisfy conditions L1, L2 and $\mathcal{I}$ satisfy conditions A1, A2 of Definition 1.3. When $\mathcal{I}$ and φ are replaced with the aforementioned values in Theorem 2.3, then

$$(\varphi(\omega) - \omega_0)\mathcal{I}(\varphi(\omega)) - \varphi(\omega) = (\omega_i - \omega_0)(\tilde{\omega}_n - \omega_i) \geq 0$$

and

$$\mathcal{I}(\varphi(\omega)) = \sum_{i=1}^{n} q_i \omega_i \neq \omega_0.$$

Finally, inequality (27) becomes our required result. □

Remark 3.8. *For different values of h and m in (27), one has the following results:*

1. *A result for (s, m)-convex function can be obtained by taking $h(\varphi) = (\varphi)^s$.*
2. *Setting $m = 1$ gives the result for h-convex function given in [31].*
3. *By taking $h(\varphi) = (\varphi)^s$ and $m = 1$, one gets the result for s-convex function given in [7, Theorem 3].*

The Petrović inequality for (h, m)-convex functions is presented in the following corollary.

Corollary 3.9. *Let* $\mathbf{v} \in [0, a]$ *or* $[a, 0])$, $\mathbf{q}$ *be positive numbers such that*

$$\tilde{\omega}_n \geq \omega_j \text{ or } \omega_j \geq \tilde{\omega}_n \text{ for } j = 1, 2, ..., n, \tag{29}$$

where $\tilde{\omega}_n := \sum_{i=1}^{n} q_i \omega_i \neq 0$. If $\mathcal{K} : [0, \infty) \to \mathbb{R}$ be an (h, m)-convex function, then

$$\sum_{i=1}^{n} q_i \mathcal{K}(\omega_i) \leq \min\left\{m\frac{\sum_{i=1}^{n} q_i h(\omega_i)}{h(\tilde{\omega}_n)}\mathcal{K}\left(\frac{\tilde{\omega}_n}{m}\right) + \frac{\sum_{i=1}^{n} q_i h(\tilde{\omega}_n - \omega_i)}{h(\tilde{\omega}_n)}\mathcal{K}(0), \right.$$
$$\left. \frac{\sum_{i=1}^{n} q_i h(\omega_i)}{h(\tilde{\omega}_n)}\mathcal{K}(\tilde{\omega}_n) + m\frac{\sum_{i=1}^{n} q_i h(\tilde{\omega}_n - \omega_i)}{h(\tilde{\omega}_n)}\mathcal{K}(0)\right\}.$$

Proof. Put $\omega_0 = 0$ in Theorem 3.7 to get the required results. □

The integral analogues of Theorem 3.4 is given in the following theorem.

Theorem 3.10. *If $\mathcal{K}:[0,\infty)\to\mathbb{R}$ is an (h,m)-convex function and the conditions given in Theorem 2.11 are satisfied, then*

$$\begin{aligned}\int_\Omega \mathcal{K}(\varphi)d\sigma \leq \min\Bigg\{& mC\mathcal{K}\left(\frac{1}{m}\int_\Omega \varphi(\omega)d\sigma\right)+\mathcal{D}\mathcal{K}(\omega_0),\\ & C\mathcal{K}\left(\int_\Omega \varphi(\omega)d\sigma\right)+m\mathcal{D}\mathcal{K}\left(\frac{\omega_0}{m}\right)\Bigg\},\end{aligned}\tag{30}$$

where

$$C=\frac{\int_\Omega h\left(\varphi(\omega)-\omega_0\right)d\sigma}{h\left(\int_\Omega \varphi(\omega)d\sigma-\omega_0\right)} \quad \text{and} \quad \mathcal{D}=\frac{\int_\Omega h\left(\int_\Omega \varphi(\omega)d\sigma-\varphi(\omega)\right)d\sigma}{h\left(\int_\Omega \varphi(\omega)d\sigma-\omega_0\right)}.$$

Proof. Assume that $E=\Omega$ and

$$\mathfrak{L}(E)=\left\{\varphi:\Omega\to[0,\infty)\mid \int_\Omega \varphi(\omega)d\sigma \text{ exists}\right\},$$

then $\mathfrak{L}(E)$ satisfies conditions L1, L2 of Definition 1.3. If we take

$$\mathcal{I}(\varphi(\omega))=\int_\Omega \varphi(\omega)d\sigma,$$

then $\mathcal{I}$ satisfies conditions A1, A2 of Definition 1.3. Substituting above values of $\mathcal{I}(\varphi(\omega))$ in Theorem 3.4, we get

$$(\varphi(\omega)-\omega_0)\mathcal{I}(\varphi(\omega))-\varphi(\omega)=(\varphi(\omega)-\omega_0)\left(\int_\Omega \varphi(\omega)d\sigma-\varphi(\omega)\right)\geq 0$$

and

$$\mathcal{I}(\varphi(\omega))=\int_\Omega \varphi(\omega)d\sigma\neq\omega_0.$$

Finally, an inequality (23) becomes (30) as our required result. □

The following corollary provides integral analogues of the well-known Petrović inequality for (h,m)-convex functions.

Corollary 3.11. *Let the conditions given in Corollary 2.12 be satisfied. If $\mathcal{K}:[0,\infty)\to\mathbb{R}$ be an (h,m)-convex function, then*

$$\begin{aligned}\int_\Omega \mathcal{K}(\varphi)d\sigma \leq \min\Bigg\{& m\frac{\int_\Omega h\left(\varphi(\omega)\right)d\sigma}{h\left(\int_\Omega \varphi(\omega)d\sigma\right)}\mathcal{K}\left(\frac{1}{m}\int_\Omega \varphi(\omega)d\sigma\right)\\ &+\frac{\int_\Omega h\left(\int_\Omega \varphi(\omega)d\sigma-\varphi(\omega)\right)d\sigma}{h\left(\int_\Omega \varphi(\omega)d\sigma\right)}\mathcal{K}(0),\\ &\frac{\int_\Omega h\left(\varphi(\omega)\right)d\sigma}{h\left(\int_\Omega \varphi(\omega)d\sigma\right)}\mathcal{K}\left(\int_\Omega \varphi(\omega)d\sigma\right)+m\frac{\int_\Omega h\left(\int_\Omega \varphi(\omega)d\sigma-\varphi(\omega)\right)d\sigma}{h\left(\int_\Omega \varphi(\omega)d\sigma\right)}\mathcal{K}(0)\Bigg\}.\end{aligned}$$

Proof. By taking $\omega_0=0$ in Theorem 3.10, One gets the required result. □

Remark 3.12. *All results given in this section can be obtained for h-convex [31], (s,m)-convex, and s-convex [**?**] functions by taking $m=1$, $h(\varphi)=(\varphi)^s$, and $h(\varphi)=(\varphi)^s$ and $m=1$ respectively.*

4. Applications on time scale calculus

Using a strong framework that incorporates both discrete and continuous times is necessary to model some global concerns. In order to get a sense of and sufficient understanding of the illusive differences between discrete and continuous times, it is normal to consider whether it is possible to present a structure that allows us to integrate both dynamical systems concurrently. S. Hilger first discussed the theory of time scales in his Doctorate thesis [17]. The main purpose of dynamic theory on time scales is to build channels between continuous and discrete situations. It is as archaic as classical calculus, but because of its applications in physics, computer networking, control theory, and fluid dynamics, it has recently become more significant. Scholars working in the field of pure mathematics have combined the discrete and continuous inequalities by using time scale calculus. In recent years, research has been conducted to unify and magnify integral inequalities utilizing fresh concepts and methods to time scales, see [8, 10, 19, 24, 26].

A time scale is any nonempty closed subset $\mathbb{T} \subseteq \mathbb{R}$. Along this paper $\mathbb{T}$ will denote a time scale and, I be an interval of $\mathbb{R}$, $I_{\mathbb{T}} = I \cap \mathbb{T}$, a time scale subset. Unless other issue is specified, $I_{\mathbb{T}} \neq \emptyset$.

Here we define the (h, m)-convex function on time scale as follows:

Definition 4.1. *Let $h : (0, 1) \subseteq J_{\mathbb{T}} \to \mathbb{R}$ with $h \not\equiv 0$. A non-negative function $\mathcal{K} : [0, b)_{\mathbb{T}} \to \mathbb{R}$, $b > 0$ is (h, m)-convex on $[0, b)_{\mathbb{T}}$, if*

$$\mathcal{K}(\xi\omega + (1-\xi)w) \leq h(\xi)\mathcal{K}(\omega) + mh(1-\xi)\mathcal{K}(w), \tag{31}$$

where $\omega, w \in [0, b)_{\mathbb{T}}$ and $m, t \in [0, 1]$ provided that $\xi\omega + (1-\xi)w \in [0, b)_{\mathbb{T}}$.

Remark 4.2. *For different values of h and m in (31), one has:*

1. *One gets the m-convex functions on time scale defined by T. Lara and E. Rosales [21] by setting h as an identity function.*
2. *To get the convex function on time scale given by Dinu [9], take $m = 1$ and $h(\omega) = \omega$.*
3. *By setting $h(\xi) = 1 = m$, one can get P-function on time scale.*
4. *To get the result for h-convex functions on time scale given by F. Bosede and A. Mogbademu in [5], take $m = 1$ in (31).*
5. *For the particular value $h(\xi) = \xi^s$, one gets the s-convex function in the second sense.*
6. *Taking $h(\xi) = \frac{1}{\xi}$, gives the result for Godunova-Levin function on time scale.*
7. *The s-Godunova-Levin function on time scale can be obtained by putting $h(\xi) = \frac{1}{\xi^s}$.*

As an application, results of the previous sections has been given in this section for particular values of isotonic linear functional in time scale.

Theorem 4.3. *Let ω be a non-negative n-tuples and $\omega_0, \mathbf{v} = \sum_{\omega=c}^{d-1} \varphi(\omega) \in [0, \infty)$ such that*

$$(\omega_i - \omega_0)\left(\sum_{\omega=c}^{d-1} \varphi(\omega) - \omega_i\right) \geq 0. \tag{32}$$

Let $h : (0, 1) \subseteq J_{\mathbb{T}} \to \mathbb{R}$ be a positive multiplicative function and $\mathcal{K} : [0, \infty)_{\mathbb{T}} \to \mathbb{R}$ be an (h, m)-convex function on time scale, then

$$\begin{aligned}\sum_{\omega=c}^{d-1} \mathcal{K}(\varphi(\omega)) \leq \min\Bigg\{ & m\frac{\sum_{\omega=c}^{d-1} h\left(\varphi(\omega) - \omega_0\right)}{h\left(\sum_{\omega=c}^{d-1}\varphi(\omega) - \omega_0\right)}\mathcal{K}\left(\frac{1}{m}\sum_{\omega=c}^{d-1}\varphi(\omega)\right) \\ & + \frac{\sum_{\omega=c}^{d-1} h\left(\sum_{\omega=c}^{d-1}\varphi(\omega) - \varphi(\omega)\right)}{h\left(\sum_{\omega=c}^{d-1}\varphi(\omega) - \omega_0\right)}\mathcal{K}(\omega_0), \\ & \frac{\sum_{\omega=c}^{d-1} h\left(\varphi(\omega) - \omega_0\right)}{h\left(\sum_{\omega=c}^{d-1}\varphi(\omega) - \omega_0\right)}\mathcal{K}\left(\sum_{\omega=c}^{d-1}\varphi(\omega)\right) + m\frac{\sum_{\omega=c}^{d-1} h\left(\sum_{\omega=c}^{d-1}\varphi(\omega) - \varphi(\omega)\right)}{h\left(\sum_{\omega=c}^{d-1}\varphi(\omega) - \omega_0\right)}\mathcal{K}\left(\frac{\omega_0}{m}\right)\Bigg\}.\end{aligned} \tag{33}$$

Proof. *Assume that $\mathfrak{L}(E) = \{\varphi : E \to [0, \infty) \mid \varphi(i) = \omega_i, i \in E\}$, where $E := [c, d-1] \cap \mathbb{Z}$ for $a, b \in \mathbb{T} = \mathbb{Z}$ with $c < d$, and $\mathcal{I}(\varphi(\omega)) = \sum_{\omega=c}^{d-1} \varphi(\omega)$. Then isotonic linear functional A satisfy conditions A1, A2 and $\mathfrak{L}(E)$ satisfy conditions L1, L2 of Definition 1.3. By substituting the aforementioned values for A and $\mathfrak{L}(E)$ in Theorem 3.4, we can derive the necessary conditions and the required result.* □

In a similar way, one can get the results for the particular values under the conditions of Theorem 3.4.

1. Let E be the interval. Consider the function $\mathfrak{L}(E)$ defined on $E := [c, d-u] \cap u\mathbb{Z}$ and $\mathcal{I}(\varphi(\omega)) = u \sum_{\omega=c/u}^{d/u-1} \varphi(\omega)$, for $u > 0$, where c and d are elements of the set $\mathbb{T} = u\mathbb{Z}$ with c being less than d.
2. Consider the function $\mathfrak{L}(E)$ defined on $E := [a, b-h] \cap h\mathbb{Z}$ for $h > 0$ and $\mathcal{I}(\varphi(\omega)) = h \sum_{\omega=c/h}^{d/h-1} \varphi(\omega)$, where c and d are elements of the set $\mathbb{T} = h\mathbb{Z}$ with c being less than d.
3. Consider
$$E = [a, b) \cap \mathbb{T}, \quad \mathfrak{L}(E) = C_{rd}([a, b), \mathbb{R}) \text{ and } \mathcal{I}(\varphi(\omega)) = \int_c^d \varphi(\omega)\Delta\omega,$$
where c and d are elements of the set $\mathbb{T}$ with c being less than d and C_{rd} represents rd-continuous functions [15, Pages 149-159] and the integral is the Cauchy delta time-scale integral [4].
4. Consider
$$E = (c, d] \cap \mathbb{T}, \quad \mathfrak{L}(E) = C_{ld}((c, d], \mathbb{R}) \text{ and } \mathcal{I}(\varphi(\omega)) = \int_c^d \varphi(\omega)\nabla\omega,$$
where c and d are elements of the set $\mathbb{T}$ with c being less than d and C_{ld} represents ld-continuous functions [22] and the integral is the Cauchy nabla time-scale integral [36].
5. Consider
$$\mathfrak{L}(E) = C([c, d], \mathbb{R}) \text{ and } \mathcal{I}(\varphi(\omega)) = \int_c^d \varphi(\omega)d\omega,$$
where c and d are elements of the set $\mathbb{T} = \mathbb{R}$ with c being less than d.
6. Consider
$$E = [c, d] \cap \mathbb{T}, \quad \mathfrak{L}(E) = C([c, d], \mathbb{R}) \text{ and } \mathcal{I}(\varphi(\omega)) = \int_c^d \varphi(\omega)\diamond_\alpha\omega,$$
where c and d are elements of the set $\mathbb{T}$ with c being less than d and the integral is Cauchy α-diamond time-scale integral [2] and $\varphi \in \mathfrak{L}(E)$.
7. Consider
$$E \subset ([c_1, d_1) \cap \mathbb{T}_1) \times ... \times ([c_n, d_n) \cap \mathbb{T}_n)$$
be Jordan Δ-measurable and let L be the set of all bounded Δ-integral functions from E to $\mathbb{R}$ and $\mathcal{I}(\varphi(\omega)) = \int_c^d \varphi(\omega)\Delta\omega$, where the integral is the multiple Riemann delta time-scale integral.
8. Consider
$$E \subset ([c_1, d_1) \cap \mathbb{T}_1) \times ... \times ([c_n, d_n) \cap \mathbb{T}_n)$$
be Lebesgue Δ-measurable and let L be the set of all Δ-measurable functions from E to $\mathbb{R}$ and $\mathcal{I}(\varphi(\omega)) = \int_c^d \varphi(\omega)\Delta\omega$, where the integral is the multiple Lebesgue delta time-scale integral.

Remark 4.4. *i) Under the same conditions given in Theorem 4.3 and its particular cases, result for the Petrović inequality for time-scale $\mathbb{T}$ can be obtained by taking $\omega_0 = 0$ in the inequality (33) and related cases.*

ii) One can get the results for m-convex, (s, m)-convex, s-convex, h-convex and convex functions on the time scale, by taking $h(\omega) = \omega$, $h(\omega) = \omega^s$, $h(\omega) = \omega^s$ and $m = 1$, $m = 1$ and h as the identity function, and $m = 1$, respectively, in the above results on the time scale.

5. Conclusion

The article presented the Giaccardi and Petrovi'c inequalities and their variants for isotonic linear functional based on (h, m)-convexity. Also, the authors derived significant variants of Giaccardi and Petrović inequalities by using popular examples of an isotonic linear functional. Many time-scale integrals are used to derive the Giaccardi and Petrović inequalities on different time scales.

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