TURKISH JOURNAL OF SCIENCE
VOLUME 10, ISSUE 3, 127–137
ISSN: 2587–0971

https://www.tjoscience.com

Coefficient Bound Estimates for the Certain Subclasses of Analytic and Bi-Univalent Function Classes Subordinated to the Improved q-Exponential Function

Nizami MUSTAFA[a], Veysel NEZİR[a]

[a]*Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Türkiye*

Abstract. In this paper, we introduce and examine a certain subclass of analytic and bi-univalent functions defined on the open unit disk of the complex plane, subordinated to the improved q–exponential function. We provide coefficient bounds for the initial coefficients of the functions belonging to the defined class, and solve the Fekete–Szegő problem for the proposed class.

1. Introduction and Preliminaries

Let $H(U)$ denote the class of analytic functions on the open unit disk

$$U := \{ z \in \mathbb{C} : |z| < 1 \}$$

of the complex plane $\mathbb{C}$. Let $\mathcal{A}$ be the class of functions $f \in H(U)$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \in \mathbb{C}. \tag{1}$$

Clearly, every $f \in \mathcal{A}$ satisfies the normalization $f(0) = 0$ and $f'(0) = 1$; functions meeting these conditions are called *normalized* in the literature.

We denote by $\mathcal{S}$ the well-known subclass of $\mathcal{A}$ consisting of univalent functions in U. This class was first introduced by Köebe [8] and, since then, has become a core object in the geometric function theory. A milestone was Bieberbach's 1916 conjecture on the coefficients of functions in $\mathcal{S}$ [4], which stimulated extensive research; see, e.g., [1–3, 5, 9, 10, 20–22].

A function f is called *bi - univalent* if both f and its inverse f^{-1} are univalent in U and $f(U)$, respectively; the class of bi - univalent functions in U will be denoted by Σ. For the inverse

$$g(w) := f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + A_4 w^4 + \cdots = w + \sum_{n=2}^{\infty} A_n w^n, \qquad w \in f(U) = U_{r_0},$$

Corresponding author: VN mail address: veyselnezir@yahoo.com ORCID:0000-0001-9640-8526, NM ORCID:0000-0002-2758-0274
Received: 6 September 2025; Accepted: 3 November 2025; Published: 31 December 2025
Keywords. Analytic function, starlike function, convex function bi-univalent function, improved q−exponential function
2010 *Mathematics Subject Classification*. 30C45, 30C50, 30C80
Cited this article as: Mustafa, N., & Nezir, V. (2025). Coefficient Bound Estimates for the Certain Subclasses of Analytic and Bi-Univalent Function Classes Subordinated to the Improved q-Exponential Function. Turkish Journal of Science, 10(3), 127-137.

it is well known that

$$A_2 = -a_2, \qquad A_3 = 2a_2^2 - a_3, \qquad A_4 = -a_2^3 + 5a_2a_3 - a_4, \; \ldots \tag{2}$$

The bi - starlike and bi - convex subclasses of Σ in the unit disk U are defined analytically by

$$S_\Sigma^* := \left\{ f \in \Sigma : \; \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \; z \in U \; \text{ and } \; \Re\left(\frac{wg'(w)}{g(w)}\right) > 0, \; w \in U_{r_0} \right\},$$

$$C_\Sigma := \left\{ f \in \Sigma : \; \Re\left(\frac{[zf'(z)]'}{f'(z)}\right) > 0, \; z \in U \; \text{ and } \; \Re\left(\frac{(wg'(w))'}{g'(w)}\right) > 0, \; w \in U_{r_0} \right\}.$$

Definition 1.1 (Subordination). *If $f, g \in H(U)$, then f is said to be* subordinate *to g, written $f \prec g$, if there exists a Schwarz function $\omega : U \to U$ with $\omega(0) = 0$ such that $f(z) = g(\omega(z))$.*

During the past few years, numerous subclasses of $\mathcal{S}$ have been introduced and studied as special choices of the classes C and S^*; see, for example, [12–20, 23].

The functional

$$H_2(a_2, a_3) := a_3 - a_2^2$$

is known as the *Fekete–Szegő functional* in the literature. More generally, for a complex parameter μ, the functional

$$H_2(a_2, a_3; \mu) := a_3 - \mu a_2^2$$

is called the *generalized Fekete–Szegő functional*.

Throughout this section and later on, we use the classical definitions of basic q-analysis. For $q \in (0, 1)$ and $n \in \mathbb{N}$, the q-number and the q-factorial are defined by

$$[n]_q := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1} = \sum_{k=0}^{n-1} q^k, \qquad [0]_q! := 1, \quad [n]_q! := [n-1]_q! \, [n]_q.$$

It is immediate that

$$\lim_{q \to 1^-} [n]_q = n, \qquad \lim_{q \to 1^-} [n]_q! = n!.$$

In the standard approach to the q-calculus, q-exponential function and improved q-exponential function are defined as follows (see [6]):

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{n=0}^{\infty} \frac{1}{[1 - (1-q)q^n z]}, \quad 0 < |q| < 1, |z| < \frac{1}{|1-q|},$$

$$E_q(z) = e_{1/q}(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{[n]_q!} = \prod_{n=0}^{\infty} [1 + (1-q)q^n z], \quad 0 < |q| < 1, z \in \mathbb{C}.$$

It can easily be seen that

$$\lim_{q \to 1^-} e_q(z) = e^z \quad \text{and} \quad \lim_{q \to 1^-} E_q(z) = e^z.$$

Now, let us define new subclasses of bi-univalent functions defined in the open unit disk U.

Definition 1.2. *For $\tau \in \mathbb{C} \setminus \{0\}$, $\beta \in [0, 1]$, and $q \in (0, 1)$, the function $f \in \Sigma$ is said to be in the class $N_\Sigma(\tau, \beta; E_q(z))$ if the following conditions are satisfied:*

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} + \beta\left\{1 + \frac{1}{\tau}\left[\frac{[zf'(z)]'}{f'(z)} - 1\right]\right\} \prec E_q(z), \quad z \in U,$$

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{wg'(w)}{g(w)} - 1\right]\right\} + \beta\left\{1 + \frac{1}{\tau}\left[\frac{[wg'(w)]'}{g'(w)} - 1\right]\right\} \prec E_q(w), \quad w \in U_{r_0}.$$

In the cases $q \to 1^-$, $\tau = 1$, $\beta = 0$, and $\beta = 1$, from Definition 1.2, we have the following subclasses of bi-univalent functions.

Definition 1.3. *For $\tau \in \mathbb{C} \setminus \{0\}$ and $\beta \in [0,1]$, the function $f \in \Sigma$ is said to be in the class $N_\Sigma(\tau,\beta;e^z)$ if the following conditions are satisfied:*

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{[zf'(z)]'}{f'(z)}-1\right]\right\} \prec e^z, \quad z \in U,$$
$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{wg'(w)}{g(w)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{[wg'(w)]'}{g'(w)}-1\right]\right\} \prec e^w, \quad w \in U_{r_0}.$$

Definition 1.4. *For $\beta \in [0,1]$ and $q \in (0,1)$, the function $f \in \Sigma$ is said to be in the class $N_\Sigma(\beta, E_q(z))$ if the following conditions are satisfied:*

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{[zf'(z)]'}{f'(z)} \prec E_q(z), \quad z \in U,$$
$$(1-\beta)\frac{wg'(w)}{g(w)}+\beta\frac{[wg'(w)]'}{g'(w)} \prec E_q(w), \quad w \in U_{r_0}.$$

Definition 1.5. *For $\tau \in \mathbb{C} \setminus \{0\}$ and $q \in (0,1)$, the function $f \in \Sigma$ is said to be in the class $S^*_\Sigma(\tau, E_q(z))$ if the following conditions are satisfied:*

$$1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right] \prec E_q(z),\ z \in U, \quad 1+\frac{1}{\tau}\left[\frac{wg'(w)}{g(w)}-1\right] \prec E_q(w),\ w \in U_{r_0}.$$

Definition 1.6. *For $\tau \in \mathbb{C} \setminus \{0\}$, $\beta \in [0,1]$, and $q \in (0,1)$, the function $f \in \Sigma$ is said to be in the class $C_\Sigma(\tau, E_q(z))$ if the following conditions are satisfied:*

$$1+\frac{1}{\tau}\left[\frac{[zf'(z)]'}{f'(z)}-1\right] \prec E_q(z),\ z \in U, \quad 1+\frac{1}{\tau}\left[\frac{[wg'(w)]'}{g'(w)}-1\right] \prec E_q(w),\ w \in U_{r_0}.$$

Let P be the class of analytic functions defined in U which satisfy the conditions $p(0) = 1$ and $\Re[p(z)] > 0, z \in U$. It is clear that the functions which satisfy these conditions have the following series expansion:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in U. \tag{3}$$

The class P defined above is known as the class of Carathéodory functions in the literature [11].

Now we give some lemmas which we will use in the proof of our main results.

Lemma 1.7 ([7]). *Let the function p belong to the class P. Then*

$$|p_n| \le 2 \text{ for each } n \in \mathbb{N}, \qquad |p_n - \nu p_k p_{n-k}| \le 2 \text{ for } n,k \in \mathbb{N}, n > k, \nu \in [0,1].$$

The equalities hold for the function

$$p(z) = \frac{1+z}{1-z}, \quad z \in U.$$

Lemma 1.8 ([7]). *Let $p \in P$. Then*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$
$$4p_3 = p_1^3 + 2(4-p_1^2)p_1 x - (4-p_1^2)p_1 x^2 + 2(4-p_1^2)(1-|x|^2)y,$$

for some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

In this paper, we give some coefficient estimates, and solve the Fekete–Szegö problem for the class $N_\Sigma(\tau,\beta;E_q(z))$.

2. Main Results

In this section, we first state the following theorem concerning the coefficient problem for the class $N_\Sigma(\tau,\beta;E_q(z))$.

Theorem 2.1. *Let $f \in N_\Sigma(\tau,\beta;E_q(z))$. Then, we have the following estimates*

$$|a_2| \leq \frac{|\tau|}{1+\beta}, \qquad |a_3| \leq |\tau| \begin{cases} \dfrac{1}{2(1+2\beta)}, & \text{if } |\tau| \leq \dfrac{(1+\beta)^2}{2(1+2\beta)}, \\ \dfrac{|\tau|}{(1+\beta)^2}, & \text{if } |\tau| \geq \dfrac{(1+\beta)^2}{2(1+2\beta)}, \end{cases}$$

and

$$|a_4| \leq \frac{|\tau|}{12(1+3\beta)} \begin{cases} \max\{\psi(t_0),\ \psi(2)\}, & \text{if } A(\tau,\beta,q) < 0, \\ \max\{\psi(0),\ \psi(2)\}, & \text{if } A(\tau,\beta,q) \geq 0, \end{cases}$$

where

$$\psi(t_0) = A(\tau,\beta,q)\,t_0^3 + B(q)\,t_0^2 + C(\tau,\beta)\,t_0 + D(q),\ t_0 = \frac{-B(q) + \sqrt{\Delta}}{3A(\tau,\beta,q)},$$

$$\Delta = B^2(q) - 3A(\tau,\beta,q)\,C(\tau,\beta),\ A(\tau,\beta,q) = \frac{(7+4\beta)\,|\tau|^2}{2(1+\beta)^3} + \frac{q^3}{2[3]_q} - \frac{15\,|\tau|}{8(1+2\beta)} - \frac{1}{2},$$

$$B(q) = -\left(1 + \frac{q}{[2]_q}\right), C(\tau,\beta) = \frac{15\,|\tau|}{2(1+2\beta)} + 2, D(q) = 4\left(1 + \frac{q}{[2]_q}\right),$$

$$\psi(0) = \frac{4(1+2q)}{1+q}, \psi(2) = \frac{4(7+4\beta)\,|\tau|}{(1+\beta)^3} + \frac{4q^3}{[3]_q} - \frac{3q}{[2]_q}.$$

Proof. Let $f \in N_\Sigma(\tau,\beta;E_q(z))$. Then, there exist Schwarz functions $\omega : U \to U$ and $\varpi : U_{r_0} \to U_{r_0}$ such that

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} + \beta\left\{1 + \frac{1}{\tau}\left[\frac{[zf'(z)]'}{f'(z)} - 1\right]\right\} = E_q(\omega(z)), \qquad z \in U, \tag{4}$$

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{wg'(w)}{g(w)} - 1\right]\right\} + \beta\left\{1 + \frac{1}{\tau}\left[\frac{(wg'(w))'}{g'(w)} - 1\right]\right\} = E_q(\varpi(w)), \qquad w \in U_{r_0}. \tag{5}$$

Using the relation between Schwarz and Carathéodory functions, define $p, \phi \in \mathcal{P}$ by:

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n, \qquad z \in U,$$

$$\phi(w) = \frac{1+\varpi(w)}{1-\varpi(w)} = 1 + \phi_1 w + \phi_2 w^2 + \phi_3 w^3 + \cdots = 1 + \sum_{n=1}^{\infty} \phi_n w^n, \qquad w \in U_{r_0}. \tag{6}$$

It follows that

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{p_1}{2}z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \frac{1}{2}\left(p_3 - p_1 p_2 - \frac{p_1^3}{4}\right)z^3 + \cdots, \quad z \in U,$$

$$\varpi(w) = \frac{\phi(w)-1}{\phi(w)+1} = \frac{\phi_1}{2}w + \frac{1}{2}\left(\phi_2 - \frac{\phi_1^2}{2}\right)w^2 + \frac{1}{2}\left(\phi_3 - \phi_1\phi_2 - \frac{\phi_1^3}{4}\right)w^3 + \cdots, \quad w \in U_{r_0}. \tag{7}$$

From (5) and (7), we obtain

$$(1+\beta)a_2 z + \Big[2(1+2\beta)a_3 - (1+3\beta)a_2^2\Big]z^2 + \Big[3(1+3\beta)a_4 - 3(1+5\beta)a_2a_3 + (1+7\beta)a_2^3\Big]z^3 + \cdots$$
$$= \frac{\tau p_1}{2}\,z + \frac{\tau}{4}\left[2p_2 - \left(1 - \frac{q}{[2]_q!}\right)p_1^2\right]z^2 + \frac{\tau}{8}\left[4p_3 - 4p_1p_2 + \frac{2q}{[2]_q!}\,(2p_2 - p_1^2) - \left(1 - \frac{q^3}{[3]_q!}\right)p_1^3\right]z^3 + \cdots,\ z \in U,$$

$$(1+\beta)A_2 w + \Big[2(1+2\beta)A_3 - (1+3\beta)A_2^2\Big]w^2 + \Big[3(1+3\beta)A_4 - 3(1+5\beta)A_2A_3 + (1+7\beta)A_2^3\Big]w^3 + \cdots$$
$$= \frac{\tau \phi_1}{2}\,w + \frac{\tau}{4}\left[2\phi_2 - \left(1 - \frac{q}{[2]_q!}\right)\phi_1^2\right]w^2 + \frac{\tau}{8}\left[4\phi_3 - 4\phi_1\phi_2 + \frac{2q}{[2]_q!}\,(2\phi_2 - \phi_1^2) - \left(1 - \frac{q^3}{[3]_q!}\right)\phi_1^3\right]w^3 + \cdots,\ w \in U_{r_0}. \tag{8}$$

Equating coefficients in (8) yields

$$a_2 = \frac{\tau p_1}{2(1+\beta)}, \qquad 2(1+2\beta)a_3 - (1+3\beta)a_2^2 = \frac{\tau}{4}\left[2p_2 - \left(1 - \frac{1}{[2]_q!}\right)p_1^2\right],$$

$$3(1+3\beta)a_4 - 3(1+5\beta)a_2a_3 + (1+7\beta)a_2^3 = \frac{\tau}{8}\left[4p_3 - 4p_1p_2 + \frac{2q}{[2]_q!}\,(2p_2 - p_1^2) - \left(1 - \frac{q^3}{[3]_q!}\right)p_1^3\right], \tag{9}$$

and similarly

$$a_2 = -\frac{\tau \phi_1}{2(1+\beta)}, \qquad 2(1+2\beta)A_3 - (1+3\beta)A_2^2 = \frac{\tau}{4}\left[2\phi_2 - \left(1 - \frac{1}{[2]_q!}\right)\phi_1^2\right],$$

$$3(1+3\beta)A_4 - 3(1+5\beta)A_2A_3 + (1+7\beta)A_2^3 = \frac{\tau}{8}\left[4\phi_3 - 4\phi_1\phi_2 + \frac{2q}{[2]_q!}\,(2\phi_2 - \phi_1^2) - \left(1 - \frac{q^3}{[3]_q!}\right)\phi_1^3\right]. \tag{10}$$

From $a_2 = \dfrac{\tau p_1}{2(1+\beta)} = -\dfrac{\tau \phi_1}{2(1+\beta)}$ we get $p_1 = -\phi_1$. Applying Lemma 1.8 to this identity gives the first assertion of the theorem. Using $A_2 = -a_2,\ A_3 = 2a_2^2 - a_3,\ A_4 = -a_2^3 + 5a_2a_3 - a_4$ see [3], the second and third identities in (10) become

$$-2(1+2\beta)a_3 + (3+5\beta)a_2^2 = \frac{\tau}{4}\left[2\phi_2 - \left(1 - \frac{1}{[2]_q!}\right)\phi_1^2\right],$$

$$-3(1+3\beta)a_4 + 6(2+5\beta)a_2a_3 + 2(1+7\beta)a_2^3 = \frac{\tau}{8}\left[4\phi_3 - 4\phi_1\phi_2 + \frac{2q}{[2]_q!}\,(2\phi_2 - \phi_1^2) - \left(1 - \frac{q^3}{[3]_q!}\right)\phi_1^3\right]. \tag{11}$$

Subtracting the first of (11) from the second identity in (9) gives

$$4(1+2\beta)\left(a_3 - a_2^2\right) = \frac{\tau}{2}\,(p_2 - \phi_2).$$

By Lemma 1.8, we obtain

$$a_3 = a_2^2 + \frac{(4 - p_1^2)\,\tau}{16(1+2\beta)}(x - y), \tag{12}$$

for some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$. Using $a_2 = \dfrac{\tau p_1}{2(1+\beta)}$ from (9), (12) can be written as:

$$a_3 = \frac{\tau^2 p_1^2}{4(1+\beta)^2} + \frac{(4 - p_1^2)\,\tau}{16(1+2\beta)}(x - y), \quad \text{for some } x, y \in \mathbb{C} \text{ with } |x| \le 1,\ |y| \le 1.$$

Hence, by the triangle inequality,

$$|a_3| \le \frac{|\tau|^2}{4(1+\beta)^2}\, t^2 + \frac{(4-t^2)|\tau|}{16(1+2\beta)}(\xi+\zeta), \qquad \xi,\zeta \in [0,1],\ t = |p_1|. \tag{13}$$

From (13), we get

$$|a_3| \le a(\tau,\beta)\, t^2 + \frac{|\tau|}{2(1+2\beta)}, \qquad t \in [0,2], \tag{14}$$

where

$$a(\tau,\beta) = \frac{|\tau|^2}{4(1+\beta)^2} - \frac{|\tau|}{8(1+2\beta)}.$$

Maximizing

$$\varphi(t) = a(\tau,\beta)\, t^2 + \frac{|\tau|}{2(1+2\beta)}, \qquad t \in [0,2],$$

we easily obtain

$$\varphi(t) \le \frac{|\tau|}{2(1+2\beta)} \quad \text{if } a(\tau,\beta) \le 0, \qquad \varphi(t) \le \frac{|\tau|^2}{(1+\beta)^2} \quad \text{if } a(\tau,\beta) \ge 0.$$

Taking these into account in (13) yields the second assertion of the theorem.

Estimate for $|a_4|$. Subtracting the second identity in (11) from the third identity in (9), we obtain

$$6(1+3\beta)a_4 - 15(1+\beta)a_2a_3 - (1+7\beta)a_2^3 = \frac{\tau}{4}\left\{2(p_3-\phi_3) - 2p_1(p_2+\phi_2) + \frac{2q}{[2]_q}(p_2-\phi_2) - \left(1 - \frac{q^3}{[3]_q}\right)p_1^3\right\}.$$

Using the expressions already found for a_2 and a_3, we arrive at

$$a_4 = \frac{\tau}{12(1+3\beta)}\left\{\left[\frac{15\tau}{8(1+2\beta)}\, p_1 + \frac{q}{[2]_q}\right](p_2-\phi_2) - p_1(p_2+\phi_2) + (p_3-\phi_3) + \left[\frac{(7+4\beta)\tau^2}{2(1+\beta)^3} + \frac{1}{2}\left(1 - \frac{q^3}{[3]_q}\right)\right]p_1^3\right\}.$$

Since (Lemma 1.8)

$$p_2 - \phi_2 = \frac{4-p_1^2}{2}(x-y), \qquad p_2 + \phi_2 = p_1^2 + \frac{4-p_1^2}{2}(x+y),$$

$$p_3 - \phi_3 = \frac{1}{2}\left\{p_1^3 + (4-p_1^2)p_1(x+y) - \frac{(4-p_1^2)p_1}{2}(x^2+y^2) + (4-p_1^2)\left[(1-|x|^2)z - (1-|y|^2)w\right]\right\},$$

we obtain

$$\begin{aligned} a_4 = \frac{\tau}{12(1+3\beta)}\Bigg\{&\left[\frac{15\tau}{8(1+2\beta)}\, p_1 + \frac{q}{[2]_q}\right]\frac{4-p_1^2}{2}(x-y) - p_1\left[p_1^2 + \frac{4-p_1^2}{2}(x+y)\right] \\ &+ \frac{1}{2}\left[p_1^3 + (4-p_1^2)p_1(x+y) - \frac{(4-p_1^2)p_1}{2}(x^2+y^2) + (4-p_1^2)\left[(1-|x|^2)z - (1-|y|^2)w\right]\right] \\ &+ \left[\frac{(7+4\beta)\tau^2}{2(1+\beta)^3} + \frac{1}{2}\left(1 - \frac{q^3}{[3]_q}\right)\right]p_1^3\Bigg\}, \end{aligned}$$

for some $x, y, z, w \in \mathbb{C}$ with $|x|, |y|, |z|, |w| \le 1$. Hence, by the triangle inequality,

$$|a_4| \le \frac{|\tau|}{12(1+3\beta)}\left\{\left[\frac{15|\tau|}{8(1+2\beta)}\, t + \frac{q}{[2]_q}\right]\frac{4-t^2}{2}(\xi+\eta) + \frac{(4-t^2)t}{4}(\xi^2+\eta^2) + 4 - t^2 + \left[\frac{(7+4\beta)|\tau|^2}{2(1+\beta)^3} + \frac{q^3}{2[3]_q}\right]t^3\right\},$$

where $t=|p_1|,\ \xi=|x|,\ \eta=|y|$. Maximizing in ξ,η gives

$$|a_4|\le\frac{|\tau|}{12(1+3\beta)}\left\{\Big[\frac{(7+4\beta)|\tau|^2}{2(1+\beta)^3}+\frac{q^3}{2[3]_q}-\frac{15|\tau|}{8(1+2\beta)}-\frac{1}{2}\Big]t^3-\Big(1+\frac{q}{[2]_q}\Big)t^2+\Big[\frac{15|\tau|}{2(1+2\beta)}+2\Big]t+\frac{4q}{[2]_q}+4\right\},$$

$t\in[0,2]$. Let $\psi:\mathbb{R}\to\mathbb{R}$ be defined by:

$$\psi(t)=A(\tau,\beta,q)t^3+B(q)t^2+C(\tau,\beta)t+D(q),\qquad t\in[0,2],$$

where

$$A(\tau,\beta,q)=\frac{(7+4\beta)|\tau|^2}{2(1+\beta)^3}+\frac{q^3}{2[3]_q}-\frac{15|\tau|}{8(1+2\beta)}-\frac{1}{2},\qquad B(q)=-\Big(1+\frac{q}{[2]_q}\Big),$$

$$C(\tau,\beta)=\frac{15|\tau|}{2(1+2\beta)}+2,\qquad D(q)=4\Big(1+\frac{q}{[2]_q}\Big).$$

Maximizing ψ on $[0,2]$ gives

$$\psi(t)\le\begin{cases}\max\{\psi(t_0),\psi(2)\}, & A(\tau,\beta,q)<0,\\ \max\{\psi(0),\psi(2)\}, & A(\tau,\beta,q)\ge 0,\end{cases}$$

where

$$t_0=\frac{-B(q)+\sqrt{\Delta}}{3A(\tau,\beta,q)},\qquad \Delta=B(q)^2-3A(\tau,\beta,q)C(\tau,\beta),$$

and

$$\psi(0)=\frac{4(1+2q)}{1+q},\qquad \psi(2)=\frac{4(7+4\beta)|\tau|}{(1+\beta)^3}+\frac{4q^3}{[3]_q}-\frac{3q}{[2]_q}.$$

Therefore,

$$|a_4|\le\frac{|\tau|}{12(1+3\beta)}\begin{cases}\max\{\psi(t_0),\psi(2)\}, & A(\tau,\beta,q)<0,\\ \max\{\psi(0),\psi(2)\}, & A(\tau,\beta,q)\ge 0.\end{cases}$$

This completes the proof. □

In the cases $q\to 1^-$, $\tau=1$, $\beta=0$ and $\beta=1$, the following corollaries follow from Theorem 2.1.

Corollary 2.2. *If $f\in N_\Sigma(\tau,\beta;e^z)$, then*

$$|a_2|\le\frac{|\tau|}{1+\beta},\qquad |a_3|\le|\tau|\begin{cases}\dfrac{1}{2(1+2\beta)}, & \text{if } |\tau|\le\dfrac{(1+\beta)^2}{2(1+2\beta)},\\ \dfrac{|\tau|}{(1+\beta)^2}, & \text{if } |\tau|\ge\dfrac{(1+\beta)^2}{2(1+2\beta)}.\end{cases}$$

Moreover,

$$|a_4|\le\frac{|\tau|}{12(1+3\beta)}\begin{cases}\max\{\psi(t_0),\psi(2)\}, & \text{if } A(\tau,\beta,q)<0,\\ \max\{\psi(0),\psi(2)\}, & \text{if } A(\tau,\beta,q)\ge 0,\end{cases}$$

where

$$\psi(t_0)=\tfrac{1}{2}\Big\{2A(\tau,\beta)\,t_0^3-3t_0^2+2C(\tau,\beta)\,t_0+12\Big\},\quad t_0=\frac{3+\sqrt{\Delta_0(\tau,\beta)}}{6A(\tau,\beta)},$$

$$\Delta_0(\tau,\beta)=3\big[\,3-4A(\tau,\beta)C(\tau,\beta)\big],\quad A(\tau,\beta)=\frac{(7+4\beta)|\tau|^2}{2(1+\beta)^3}-\frac{15|\tau|}{8(1+2\beta)}-\frac{1}{3},$$

$$B=-\frac{3}{2},\qquad C(\tau,\beta)=\frac{15|\tau|}{2(1+2\beta)}+2,\qquad D=6,\qquad \psi(0)=6,\qquad \psi(2)=\frac{4(7+4\beta)|\tau|}{(1+\beta)^3}-\frac{1}{6}.$$

Corollary 2.3. *If $f \in N_{\Sigma}\big(\beta, E_q(z)\big)$, then*

$$|a_2| \leq \frac{1}{1+\beta}, \qquad |a_3| \leq \frac{1}{(1+\beta)^2},$$

and

$$|a_4| \leq \frac{1}{12(1+3\beta)} \begin{cases} \max\{\psi(t_0), \psi(2)\}, & \text{if } A(\beta, q) < 0, \\ \max\{\psi(0), \psi(2)\}, & \text{if } A(\beta, q) \geq 0, \end{cases}$$

where

$$\psi(t_0) = A(\beta, q)\, t_0^3 + B(q)\, t_0^2 + C(\beta)\, t_0 + D(q), \qquad t_0 = \frac{-B(q) + \sqrt{\Delta_1(\beta, q)}}{3A(\beta, q)},$$

$$\Delta_1(\beta, q) = B(q)^2 - 3A(\beta, q)C(\beta), \qquad A(\beta, q) = \frac{7+4\beta}{2(1+\beta)^3} + \frac{q^3}{2[3]_q} - \frac{15}{8(1+2\beta)} - \frac{1}{2},$$

$$B(q) = -\left(1 + \frac{q}{[2]_q}\right), \qquad C(\beta) = \frac{8\beta+19}{2(1+2\beta)}, \qquad D(q) = 4\left(1 + \frac{q}{[2]_q}\right),$$

$$\psi(0) = \frac{4(1+2q)}{1+q}, \qquad \psi(2) = \frac{4(7+4\beta)}{(1+\beta)^3} + \frac{4q^3}{[3]_q} - \frac{3q}{[2]_q}.$$

Corollary 2.4. *If $f \in S^*_{\Sigma}\big(\tau, E_q(z)\big)$, then*

$$|a_2| \leq |\tau|, \qquad |a_3| \leq \frac{|\tau|}{2} \begin{cases} 1, & \text{if } |\tau| \leq \frac{1}{2}, \\ 2|\tau|, & \text{if } |\tau| \geq \frac{1}{2}, \end{cases}$$

and

$$|a_4| \leq \frac{|\tau|}{12} \begin{cases} \max\{\psi(t_0), \psi(2)\}, & \text{if } A_0(\tau, q) < 0, \\ \max\{\psi(0), \psi(2)\}, & \text{if } A_0(\tau, q) \geq 0, \end{cases}$$

where

$$\psi(t_0) = A_0(\tau, q)\, t_0^3 + B(q)\, t_0^2 + C_0(\tau)\, t_0 + D(q), \qquad t_0 = \frac{-B(q) + \sqrt{\Delta_2(\tau, q)}}{3A_0(\tau, q)},$$

$$\Delta_2(\tau, q) = B(q)^2 - 3A_0(\tau, q)C_0(\tau), \qquad A_0(\tau, q) = \frac{7|\tau|^2 - 1}{2} + \frac{q^3}{2[3]_q} - \frac{15|\tau|}{8},$$

$$B(q) = -\left(1 + \frac{q}{[2]_q}\right), \qquad C_0(\tau) = \frac{15|\tau|}{2} + 2, \qquad D(q) = 4\left(1 + \frac{q}{[2]_q}\right),$$

$$\psi(0) = \frac{4(1+2q)}{1+q}, \qquad \psi(2) = \frac{4q^3}{[3]_q} - \frac{3q}{[2]_q} + 28|\tau|.$$

Corollary 2.5. *If $f \in C_{\Sigma}\big(\tau, E_q(z)\big)$, then*

$$|a_2| \leq \frac{|\tau|}{2}, \qquad |a_3| \leq |\tau| \begin{cases} \frac{1}{6}, & \text{if } |\tau| \leq \frac{2}{3}, \\ \frac{|\tau|}{4}, & \text{if } |\tau| \geq \frac{2}{3}, \end{cases}$$

and

$$|a_4| \leq \frac{|\tau|}{48} \begin{cases} \max\{\psi(t_0), \psi(2)\}, & \text{if } A_1(\tau, q) < 0, \\ \max\{\psi(0), \psi(2)\}, & \text{if } A_1(\tau, q) \geq 0, \end{cases}$$

where

$$\psi(t_0) = A_1(\tau, q)\, t_0^3 + B(q)\, t_0^2 + C_1(\tau)\, t_0 + D(q), \qquad t_0 = \frac{-B(q) + \sqrt{\Delta_3(\tau, q)}}{3A_1(\tau, q)},$$

$$\Delta_3(\tau, q) = B(q)^2 - 3A_1(\tau, q)C_1(\tau), \qquad A_1(\tau, q) = \frac{11|\tau|^2 - 10|\tau|}{16} + \frac{q^3 - [3]_q}{2[3]_q},$$

$$B(q) = -\left(1 + \frac{q}{[2]_q}\right), \qquad C_1(\tau) = \frac{5|\tau|}{2} + 2, \qquad D(q) = 4\left(1 + \frac{q}{[2]_q}\right),$$

$$\psi(0) = \frac{4(1 + 2q)}{1 + q}, \qquad \psi(2) = \frac{11|\tau|}{2} + \frac{4q^3}{[3]_q} - \frac{3q}{[2]_q}.$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $N_\Sigma\left(\tau, \beta; E_q(z)\right)$.

Theorem 2.6. *Let $f \in N_\Sigma(\tau, \beta; E_q(z))$ and $\mu \in \mathbb{C}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\tau|}{2} \begin{cases} \dfrac{1}{1 + 2\beta}, & \text{if } 2|1 - \mu|\,|\tau| \leq \dfrac{(1 + \beta)^2}{1 + 2\beta}, \\ \dfrac{2|1 - \mu|\,|\tau|}{(1 + \beta)^2}, & \text{if } 2|1 - \mu|\,|\tau| \geq \dfrac{(1 + \beta)^2}{1 + 2\beta}. \end{cases}$$

Proof. From the proof of Theorem 2.1 we have, for some $x, y \in \mathbb{C}$ with $|x| \leq 1$, $|y| \leq 1$,

$$a_3 - \mu a_2^2 = (1 - \mu)\frac{\tau^2 p_1^2}{4(1 + \beta)^2} + \frac{(4 - p_1^2)\tau}{16(1 + 2\beta)}(x - y).$$

Hence

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|1 - \mu|\,|\tau|^2}{4(1 + \beta)^2}\, t^2 + \frac{(4 - t^2)|\tau|}{16(1 + 2\beta)}(\xi + \eta), \qquad \xi, \eta \in [0, 1],\ t = |p_1|. \tag{15}$$

Maximizing the right–hand side of (15) over ξ, η yields

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\tau|}{8}\left[\theta(\tau, \beta, \mu)\, t^2 + \frac{4}{1 + 2\beta}\right], \quad t \in [0, 2], \tag{16}$$

where

$$\theta(\tau, \beta, \mu) = \frac{2|1 - \mu|\,|\tau|}{(1 + \beta)^2} - \frac{1}{1 + 2\beta}.$$

Maximizing the function $\chi : [0, 2] \to R$ defined as follows

$$\chi(t) = \theta(\tau, \beta, \mu)\, t^2 + \frac{4}{1 + 2\beta},\ t \in [0, 2],$$

we can easily see that $\chi(t) \leq \frac{4}{1+2\beta}$ if $\theta(\tau, \beta, \mu) \leq 0$ and

$$\chi(t) \leq \frac{8\left|1 - \mu\right||\tau|}{(1 + \beta)^2}$$

if $\theta(\tau, \beta, \mu) \geq 0$.

By taking these estimates obtained for the function χ into account in the inequality (16), we complete the proof of the theorem. □

In the cases $q \to 1^-$, $\tau = 1$, $\beta = 0$ and $\beta = 1$, the following corollaries follow from Theorem 2.6.

Corollary 2.7. *If $f \in N_\Sigma(\tau, \beta; e^z)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{2} \begin{cases} \dfrac{1}{1+2\beta}, & \text{if } 2|1-\mu|\,|\tau| \le \dfrac{(1+\beta)^2}{1+2\beta}, \\ \dfrac{2|1-\mu|\,|\tau|}{(1+\beta)^2}, & \text{if } 2|1-\mu|\,|\tau| \ge \dfrac{(1+\beta)^2}{1+2\beta}. \end{cases}$$

Corollary 2.8. *If $f \in N_\Sigma(\beta, E_q(z))$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2} \begin{cases} \dfrac{1}{1+2\beta}, & \text{if } 2|1-\mu| \le \dfrac{(1+\beta)^2}{1+2\beta}, \\ \dfrac{2|1-\mu|}{(1+\beta)^2}, & \text{if } 2|1-\mu| \ge \dfrac{(1+\beta)^2}{1+2\beta}. \end{cases}$$

Corollary 2.9. *If $f \in S^*_\Sigma(\tau, E_q(z))$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{2} \begin{cases} 1, & \text{if } 2|1-\mu|\,|\tau| \le 1, \\ 2|1-\mu|\,|\tau|, & \text{if } 2|1-\mu|\,|\tau| \ge 1. \end{cases}$$

Corollary 2.10. *If $f \in C_\Sigma(\tau, E_q(z))$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{2} \begin{cases} \dfrac{1}{3}, & \text{if } 3|1-\mu|\,|\tau| \le 2, \\ \dfrac{|1-\mu|\,|\tau|}{2}, & \text{if } 3|1-\mu|\,|\tau| \ge 2. \end{cases}$$

Also, taking $\mu = 0$ and $\mu = 1$ in the Theorem 2.2, we obtain the following results, respectively.

Corollary 2.11. *If $f \in N_\Sigma(\tau, \beta; E_q(z))$, then*

$$|a_3| \le \frac{|\tau|}{2} \begin{cases} \dfrac{1}{1+2\beta}, & \text{if } 2|\tau| \le \dfrac{(1+\beta)^2}{1+2\beta}, \\ \dfrac{2|\tau|}{(1+\beta)^2}, & \text{if } 2|\tau| \ge \dfrac{(1+\beta)^2}{1+2\beta}. \end{cases}$$

Corollary 2.12. *If $f \in N_\Sigma(\tau, \beta; E_q(z))$, then*

$$|a_3 - a_2^2| \le \frac{|\tau|}{2(1+2\beta)}.$$

Remark 2.13. *Corollary 2.11 confirms the second assertion of Theorem 2.1.*

In addition, taking $\mu = 1$, $\beta = 0$ and $\beta = 1$ in the Theorem 2.6, we obtain the following results, respectively.

Corollary 2.14. *If $f \in S^*_\Sigma(\tau, E_q(z))$, then*

$$|a_3 - a_2^2| \le \frac{|\tau|}{2}.$$

Corollary 2.15. *If $f \in C_\Sigma(\tau, E_q(z))$, then*

$$|a_3 - a_2^2| \le \frac{|\tau|}{6}.$$

3. Conclusion

We introduced the class $N_\Sigma(\tau, \beta; E_q(z))$ of bi-univalent functions subordinated to the improved q–exponential function and derived coefficient bounds for the initial coefficients. In particular, using the Carathéodory representation (3) and subordination techniques, we obtained estimates for $|a_2|$ and $|a_3|$, an explicit piecewise upper bound for $|a_4|$ (see Theorem 2.1), and solved the Fekete–Szegő problem for $N_\Sigma(\tau, \beta; E_q(z))$ (Theorem 2.6). Special cases arising from $q \to 1^-$, $\tau = 1$, $\beta = 0$, and $\beta = 1$ yield corollaries that recover known results for bi-starlike and bi-convex subclasses and confirm the consistency of our estimates.

References

[1] Alotaibi, A., Arif, M., Alghamdi, M.A., Hussain, S. (2020). Starlikeness associated with cosine hyperbolic function. Mathematics, 8, 1118.

[2] Arif, M., Ahmad, K., Liu, J.-L., Sokol, J. (2019). A new class of analytic functions associated with Salagean operator. Journal of Function Spaces, 2019, 5157394.

[3] Bano, K., Raza, M. (2020). Starlike functions associated with cosine function. Bulletin of the Iranian Mathematical Society, 47, 1513–1532.

[4] Bieberbach, L. (1916). Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. Sitzungsberichte der Preussischen Akademie der Wissenschaften, 138, 940–955.

[5] Brannan, D.A., Kirwan, W.E. (1969). On some classes of bounded univalent functions. Journal of the London Mathematical Society, 2, 431–443.

[6] Cieśliński, J.L. (2011). Improved exponential and trigonometric functions. Applied Mathematics Letters, 24(12), 2110–2114.

[7] Duren, P.L. (1983). Univalent Functions. New York, Berlin, Heidelberg and Tokyo: Springer-Verlag.

[8] Köebe, P. (1909). Über die Uniformisierung der algebraischen Kurven durch automorphe Funktionen mit imaginärer Substitutionsgruppe. Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 68–76.

[9] Kumar, S.S., Arora, K. (2020). Starlike functions associated with a petal-shaped domain. arXiv preprint, arXiv:2010.10072.

[10] Mendiratta, R., Nagpal, S., Ravichandran, V. (2015). On a subclass of strongly starlike functions associated with the exponential function. Bulletin of the Malaysian Mathematical Sciences Society, 38, 365–386.

[11] Miller, S.S. (1975). Differential inequalities and Carathéodory functions. Bulletin of the American Mathematical Society, 81, 79–81.

[12] Mustafa, N., Demir, H.A. (2023). Coefficient estimates for certain subclass of analytic and univalent functions associated with sine and cosine functions. 4th International Black Sea Congress on Modern Scientific Research, June 6–8, 2555–2563, Rize, Turkey.

[13] Mustafa, N., Demir, H.A. (2023). Fekete–Szegö problem for certain subclass of analytic and univalent functions associated with sine and cosine functions. 4th International Black Sea Congress on Modern Scientific Research, June 6–8, 2564–2572, Rize, Turkey.

[14] Mustafa, N., Demir, H.A. (2023). Coefficient estimates for certain subclass of analytic and univalent functions associated with sine and cosine functions with complex order. Journal of Scientific and Engineering Research, 10(6), 131–140.

[15] Mustafa, N., Nezir, V. (2023). Coefficient estimates and Fekete–Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function. 13th International Istanbul Scientific Research Congress on Life, Engineering and Applied Sciences, May 1–2, 475–481, Istanbul, Turkey.

[16] Mustafa, N., Nezir, V., Kankılıç, A. (2023). Coefficient estimates for certain subclass of analytic and univalent functions associated with sine hyperbolic function. 13th International Istanbul Scientific Research Congress on Life, Engineering and Applied Sciences, April 29–30, 234–241, Istanbul, Turkey.

[17] Mustafa, N., Nezir, V., & Kankılıç, A. (2023). Coefficient estimates for certain subclass of analytic and univalent functions associated with sine hyperbolic function with complex order. Journal of Scientific and Engineering Research, 10(6), 18–25.

[18] Mustafa, N., Nezir, V., Kankılıç, A. (2023). The Fekete–Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function. 13th International Istanbul Scientific Research Congress on Life, Engineering and Applied Sciences, April 29–30, 242–249, Istanbul, Turkey.

[19] Mustafa, N., Nezir, V., & Kankılıç, A. (2023). The Fekete–Szegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function with complex order. Eastern Anatolian Journal of Science, 9(1), 1–6.

[20] Sharma, K., Jain, N.K., Ravichandran, V. (2016). Starlike functions associated with a cardioid. African Mathematical Journal, 27, 923–939.

[21] Sokol, J. (2011). A certain class of starlike functions. Computers & Mathematics with Applications, 62, 611–619.

[22] Sokol, J., Stankiewicz, J. (1996). Radius of convexity of some subclasses of strongly starlike functions. Zeszyty Naukowe Politechniki Rzeszowskiej, Matematyka, 19, 101–105.

[23] Ullah, K., Zainab, S., Arif, M., Darus, M., Shutayi, M. (2021). Radius problems for starlike functions associated with the tan hyperbolic function. Journal of Function Spaces, Article ID 9967640.