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Fixed Point Property on Large Closed, Bounded and Convex Classes in Köthe–Toeplitz Dual Spaces

Veysel NEZİR[a]

[a] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey*

Abstract. In this paper, we study the fixed point property for nonexpansive mappings acting on closed, bounded and convex subsets of Köthe–Toeplitz dual spaces. Our approach focuses on identifying large classes of subsets which enjoy the fixed point property for nonexpansive mappings. More precisely, within a given Köthe–Toeplitz dual space, we construct and investigate wide families of closed, bounded and convex sets on which every nonexpansive self-mapping admits a fixed point. Our results provide a unified framework for understanding fixed point phenomena on large classes of subsets in Köthe–Toeplitz dual spaces and extend several earlier contributions in this direction.

1. Introduction

The fixed point property (FPP) for nonexpansive mappings has been a central topic in Banach space theory and nonlinear functional analysis. It is well known that the existence of fixed points for nonexpansive mappings depends not only on the global geometric structure of the ambient space, but also on the properties of the subsets on which the mappings act [5, 6].

In the theory of sequence spaces, Köthe–Toeplitz duals constitute an important class of Banach spaces arising naturally from summability theory. Several works have been devoted to the construction and investigation of these duals, focusing mainly on their algebraic, geometric and topological properties. In particular, generalized difference sequence spaces and their Köthe–Toeplitz duals were introduced and studied in [1–3]. These works provide the structural background for the study of nonlinear phenomena on such spaces, but they do not address fixed point properties on closed, bounded and convex subsets.

A different line of research, initiated by Goebel and Kuczumow and further developed in subsequent works, shows that even when a Banach space fails to have the fixed point property globally, it may still admit large classes of closed, bounded and convex subsets on which every nonexpansive mapping has a fixed point [4, 6]. This subset-oriented approach has proved to be particularly effective in fixed point theory and has inspired many later developments.

Motivated by this perspective, the aim of the present paper is to investigate fixed point phenomena in Köthe–Toeplitz dual spaces from a subset-oriented point of view. Rather than imposing additional conditions on the underlying space, we identify and study large classes of closed, bounded and convex

Corresponding author: VN mail address: veyselnezir@yahoo.com ORCID:0000-0001-9640-8526
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subsets of a given Köthe–Toeplitz dual space which possess the fixed point property for nonexpansive mappings.

Recent works have demonstrated that this approach is especially fruitful in the context of Köthe–Toeplitz duals associated with generalized difference sequence spaces and Cesàro-type constructions [7–12]. The results of the present paper extend this line of research by constructing new large classes of closed, bounded and convex subsets with the fixed point property in Köthe–Toeplitz dual spaces.

The paper is organized as follows. In Section 2, we recall the necessary preliminaries concerning Köthe–Toeplitz dual spaces and nonexpansive mappings. Section 3 is devoted to the main results.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the field of real numbers. By e_n we denote the canonical unit vectors in the classical sequence spaces.

2.1. Generalized difference sequence spaces and their Köthe–Toeplitz dual

Let $v = (v_k)$ be a fixed sequence of nonzero real numbers and let $m \in \mathbb{N}$ be given. Following Çolak [2] and Et–Esi [3], we consider the generalized m-th order difference operator $\Delta_v^{(m)}$ acting on a scalar sequence $x = (x_k)$, defined recursively by

$$(\Delta_v^{(1)} x)_k = v_k x_k - v_{k+1} x_{k+1}, \qquad (\Delta_v^{(m)} x)_k = (\Delta_v^{(1)}(\Delta_v^{(m-1)} x))_k, \quad k \in \mathbb{N}.$$

For a classical sequence space $X \in \{c_0, c, \ell_\infty\}$ we write

$$\Delta_v^{(m)}(X) := \{x = (x_k) : \Delta_v^{(m)} x \in X\}.$$

The Köthe–Toeplitz dual of $\Delta_v^{(m)}(X)$ is denoted by $D_1^{(m)}$; according to Bektas, Et and Çolak [1] it can be written as

$$D_1^{(m)} = \Big\{a = (a_k) \subset \mathbb{R} : \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|} < \infty\Big\}.$$

The canonical norm on $D_1^{(m)}$ is

$$\|a\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|}, \qquad a = (a_k) \in D_1^{(m)}. \tag{2.1}$$

With this norm, $D_1^{(m)}$ is a Banach space.

Define the linear mapping $J\colon D_1^{(m)} \to \ell^1$ by

$$J(a) = \Big(\frac{k^m a_k}{v_k}\Big)_{k \in \mathbb{N}}, \qquad a = (a_k) \in D_1^{(m)}.$$

It is immediate from (2.1) that J is an isometric isomorphism onto ℓ^1, that is,

$$\|a\|^{(m)} = \|J(a)\|_1 \qquad \text{for all } a \in D_1^{(m)}.$$

Hence $D_1^{(m)}$ and ℓ^1 share the same geometric properties with respect to the fixed point theory of nonexpansive mappings.

We denote by (e_n) the canonical basis in ℓ^1, and we write $(\tilde{e}_n)$ for the corresponding canonical basis in $D_1^{(m)}$, i.e.,

$$\tilde{e}_n = (0, \dots, 0, 1, 0, \dots) \in D_1^{(m)} \quad \text{with 1 at the } n\text{-th place.}$$

Then we have $J(\tilde{e}_n) = \dfrac{n^m}{v_n} e_n$ for every $n \in \mathbb{N}$.

2.2. *Affine and nonexpansive mappings*

Let $(X, \|\cdot\|)$ be a Banach space and let $C \subset X$ be a nonempty cbc subset.

Definition 2.1. *A mapping $T: C \to C$ is called*

- affine *if for every $a, b \in C$ and every $t \in [0,1]$ we have*

$$T\Big((1-t)a + tb\Big) = (1-t)T(a) + tT(b).$$

- nonexpansive *if*

$$\|T(a) - T(b)\| \leq \|a - b\| \qquad \text{for all } a, b \in C.$$

The mapping T is said to be *invariant* on C if $T(C) \subset C$. We say that C has the *fixed point property for nonexpansive mappings* (briefly, C has the FPP) if every nonexpansive mapping $T: C \to C$ has a fixed point.

2.3. *Approximate fixed point sequences and the Goebel–Kuczumow lemma*

We start by recalling the notion of approximate fixed point sequences.

Definition 2.2. *Let $T: C \to C$ be a nonexpansive mapping on a nonempty closed, bounded and convex subset C of a Banach space $(X, \|\cdot\|)$. A sequence $(u^{(n)}) \subset C$ is called an* approximate fixed point sequence *(AFPS) for T if*

$$\lim_{n\to\infty} \|T(u^{(n)}) - u^{(n)}\| = 0.$$

Given an AFPS $(u^{(n)})$ for a nonexpansive mapping $T: C \to C$, Goebel and Kuczumow introduced in [6] the functional

$$r(x) := \limsup_{n\to\infty} \|u^{(n)} - x\| \qquad (x \in C),$$

and proved the following fundamental result.

Lemma 2.3 (Goebel–Kuczumow). *Let C be a nonempty closed, bounded and convex subset of a Banach space X and let $T: C \to C$ be nonexpansive. Suppose $(u^{(n)}) \subset C$ is an approximate fixed point sequence for T and define*

$$r(x) := \limsup_{n\to\infty} \|u^{(n)} - x\| \qquad (x \in C).$$

Then there exists $u \in C$ such that

$$r(x) = r(u) + \|x - u\| \qquad \text{for all } x \in C.$$

In particular, $r(Tx) \leq r(x)$ for all $x \in C$.

This lemma is the key tool in the negative direction of fixed point theory on nonreflexive spaces such as ℓ^1. Everest [4] used Lemma 2.3 in the setting of ℓ^1 to construct large classes of closed, bounded and convex subsets with the fixed point property, by combining the above representation of r with carefully chosen approximate fixed point sequences.

Using the isometric isomorphism $J: D_1^{(m)} \to \ell^1$ introduced in the previous subsection, we now state the analogue of Goebel–Kuczumow's lemma for $D_1^{(m)}$.

Lemma 2.4. *Let $C^{(m)} \subset D_1^{(m)}$ be a nonempty closed, bounded and convex subset and let $T: C^{(m)} \to C^{(m)}$ be nonexpansive. Let $(u^{(n)}) \subset C^{(m)}$ be an approximate fixed point sequence for T. For every $w \in C^{(m)}$, define*

$$Q(w) = \limsup_{n\to\infty} \|u^{(n)} - w\|^{(m)}.$$

Then there exists $u \in C^{(m)}$ such that

$$Q(w) = Q(u) + \|w - u\|^{(m)} \qquad \text{for all } w \in C^{(m)}.$$

In particular, $Q(Tw) \leq Q(w)$ for all $w \in C^{(m)}$.

Proof. Set $C' := J(C^{(m)}) \subset \ell^1$ and define a mapping

$$S := J \circ T \circ J^{-1} \colon C' \to C'.$$

Since J is an isometric isomorphism, S is nonexpansive on C' and C' is a nonempty closed, bounded and convex subset of ℓ^1.

Let $v^{(n)} := J(u^{(n)}) \in C'$ for each n. Then

$$\|S(v^{(n)}) - v^{(n)}\|_1 = \|J(T(u^{(n)})) - J(u^{(n)})\|_1 = \|T(u^{(n)}) - u^{(n)}\|^{(m)} \xrightarrow[n\to\infty]{} 0,$$

so $(v^{(n)})$ is an approximate fixed point sequence for S on C'.

Define

$$r(z) := \limsup_{n\to\infty} \|v^{(n)} - z\|_1, \qquad z \in C'.$$

By Lemma 2.3 there exists $v \in C'$ such that

$$r(z) = r(v) + \|z - v\|_1 \qquad \text{for all } z \in C'.$$

Now put $u := J^{-1}(v) \in C^{(m)}$. For any $w \in C^{(m)}$ we have $z = J(w) \in C'$, and

$$\begin{aligned} Q(w) &= \limsup_{n\to\infty} \|u^{(n)} - w\|^{(m)} = \limsup_{n\to\infty} \|J(u^{(n)}) - J(w)\|_1 \\ &= r(J(w)) = r(v) + \|J(w) - v\|_1 \\ &= r(v) + \|J(w) - J(u)\|_1 = r(v) + \|w - u\|^{(m)}. \end{aligned}$$

Since

$$Q(u) = \limsup_{n\to\infty} \|u^{(n)} - u\|^{(m)} = \limsup_{n\to\infty} \|J(u^{(n)}) - J(u)\|_1 = r(v),$$

we obtain

$$Q(w) = Q(u) + \|w - u\|^{(m)} \qquad (w \in C^{(m)}),$$

as claimed. The inequality $Q(Tw) \le Q(w)$ follows by applying the same argument to $T(w)$ and using the nonexpansiveness of T. □

3. Main results

In this section we obtain large classes of closed, bounded and convex subsets of the Köthe–Toeplitz dual space $D_1^{(m)}$ which enjoy the fixed point property for nonexpansive mappings. The construction is formulated directly in $D_1^{(m)}$ by means of a carefully chosen sequence (f_n), and the proof relies on the Goebel–Kuczumow lemma in the form of Lemma 2.4.

3.1. *A first large class in* $D_1^{(m)}$

Throughout this subsection we fix $m \in \mathbb{N}$ and a sequence of nonzero real numbers $v = (v_k)$. Recall that

$$D_1^{(m)} = \Big\{a = (a_k) : \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|} < \infty\Big\}, \qquad \|a\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|},$$

and that the canonical unit vectors in $D_1^{(m)}$ are denoted by $\tilde{e}_k$ (with a single 1 at the k-th coordinate).

Let $t \in (0,1)$ be fixed. We define a sequence $(f_n) \subset D_1^{(m)}$ by

$$f_1 := t\,\frac{v_1}{1^m}\,\tilde{e}_1, \qquad f_k := \frac{v_k}{k^m}\,\tilde{e}_k \quad (k \ge 2).$$

A direct computation using the norm formula shows that

$$\|f_1\|^{(m)} = t, \qquad \|f_k\|^{(m)} = 1 \quad \text{for all } k \geq 2.$$

Thus (f_n) is a normalized sequence in $D_1^{(m)}$ with a distinguished first element of strictly smaller norm.

We now consider the closed convex hull of (f_n):

$$E_t^{(m)} := \Big\{ \sum_{n=1}^{\infty} \alpha_n f_n : \alpha_n \geq 0, \ \sum_{n=1}^{\infty} \alpha_n = 1 \Big\},$$

where the series converges in the norm $\|\cdot\|^{(m)}$. Since each $f_n \in D_1^{(m)}$, and the coefficients (α_n) are nonnegative and sum to 1, the space $E_t^{(m)}$ is a nonempty convex subset of $D_1^{(m)}$. Moreover, the norm estimates $\|f_1\|^{(m)} = t$ and $\|f_k\|^{(m)} = 1$ for $k \geq 2$ immediately imply that $E_t^{(m)}$ is bounded. Closedness follows from the completeness of $D_1^{(m)}$ and the fact that we take all norm-convergent convex combinations of (f_n). Hence $E_t^{(m)}$ is a nonempty closed, bounded and convex (cbc) subset of $D_1^{(m)}$.

Our first main theorem shows that this very simple class already has the fixed point property.

Theorem 3.1. *Let $t \in (0,1)$ and let $E_t^{(m)} \subset D_1^{(m)}$ be the cbc subset defined above. Then every nonexpansive mapping*

$$T\colon E_t^{(m)} \to E_t^{(m)}$$

has a fixed point. In other words, $E_t^{(m)}$ has the fixed point property for nonexpansive mappings.

Proof. Let $T\colon E_t^{(m)} \to E_t^{(m)}$ be nonexpansive. By a standard argument (see, for instance, the construction of approximate fixed point sequences in [5]), there exists an approximate fixed point sequence (AFPS) $(u^{(n)}) \subset E_t^{(m)}$, that is,

$$\lim_{n\to\infty} \|T(u^{(n)}) - u^{(n)}\|^{(m)} = 0.$$

Define

$$Q(w) := \limsup_{n\to\infty} \|u^{(n)} - w\|^{(m)}, \qquad w \in E_t^{(m)}.$$

By Lemma 2.4 (Goebel–Kuczumow lemma transported to $D_1^{(m)}$), there exists $u \in E_t^{(m)}$ such that

$$Q(w) = Q(u) + \|w - u\|^{(m)} \qquad \text{for all } w \in E_t^{(m)}. \tag{3.1}$$

In particular, nonexpansiveness of T implies

$$Q(Tw) \leq Q(w) \qquad (w \in E_t^{(m)}),$$

and hence, using (3.1),

$$Q(u) + \|Tw - u\|^{(m)} = Q(Tw) \leq Q(w) = Q(u) + \|w - u\|^{(m)}. \tag{3.2}$$

We now distinguish two cases.

Case 1. $Tu = u$. Then u is a fixed point of T and we are done.

Case 2. $Tu \neq u$. In this case we use the special structure of $E_t^{(m)}$. Every element $w \in E_t^{(m)}$ can be written as

$$w = \sum_{k=1}^{\infty} \alpha_k f_k, \qquad \alpha_k \geq 0, \quad \sum_{k=1}^{\infty} \alpha_k = 1.$$

In particular, there exists a representation

$$u = \sum_{k=1}^{\infty} \gamma_k f_k, \qquad \gamma_k \geq 0, \quad \sum_{k=1}^{\infty} \gamma_k = 1.$$

Let $w \in E_t^{(m)}$ be arbitrary and write

$$w = \sum_{k=1}^{\infty} t_k f_k, \qquad t_k \geq 0, \quad \sum_{k=1}^{\infty} t_k = 1.$$

Set

$$a_k := t_k - \gamma_k, \qquad k \in \mathbb{N}.$$

Then we have the "mass balance"

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \gamma_k = 1 - 1 = 0,$$

and

$$w - u = \sum_{k=1}^{\infty} a_k f_k.$$

Because the vectors f_k have disjoint supports and are scalar multiples of the canonical basis vectors, the norm of $w - u$ is easily computed. Indeed,

$$f_1 = t\,\frac{v_1}{1^m}\,\tilde{e}_1, \qquad f_k = \frac{v_k}{k^m}\,\tilde{e}_k \quad (k \geq 2),$$

so that

$$w - u = a_1 t\,\frac{v_1}{1^m}\,\tilde{e}_1 + \sum_{k=2}^{\infty} a_k \frac{v_k}{k^m}\,\tilde{e}_k.$$

Using the definition of the norm $\|\cdot\|^{(m)}$ we obtain

$$\|w - u\|^{(m)} = t|a_1| + \sum_{k=2}^{\infty} |a_k|. \tag{3.3}$$

Thus $\|w - u\|^{(m)}$ depends only on the sequence (a_k) via the weighted ℓ^1-expression (3.3) with weight t at the first coordinate and weight 1 at all other coordinates.

We now define the function

$$d(w) := \|w - u\|^{(m)}, \qquad w \in E_t^{(m)}.$$

By the above representation, minimizing $d(w)$ over $E_t^{(m)}$ is equivalent to minimizing the functional

$$F(a_1, a_2, \dots) = t|a_1| + \sum_{k=2}^{\infty} |a_k|$$

subject to the constraints

$$\sum_{k=1}^{\infty} a_k = 0, \qquad \gamma_k + a_k \geq 0 \quad (k \in \mathbb{N}), \qquad \sum_{k=1}^{\infty} (\gamma_k + a_k) = 1.$$

Reduction 1: we may assume $a_1 \geq 0$.

If $a_1 < 0$, then since $\sum_{k\geq 1} a_k = 0$, there exists $j \geq 2$ with $a_j > 0$. Consider the new sequence $(\tilde{a}_k)$ given by

$$\tilde{a}_1 := 0, \qquad \tilde{a}_j := a_j + a_1, \qquad \tilde{a}_k := a_k \quad (k \neq 1, j).$$

Clearly $\sum_k \tilde{a}_k = 0$, and $(\tilde{a}_k)$ still corresponds to coefficients of an element of $E_t^{(m)}$ provided $|a_1|$ is small enough (this can be justified rigorously by a limiting argument; see Goebel–Kuczumow [6]). Moreover,

$$\begin{aligned} F(\tilde{a}) &= t|\tilde{a}_1| + |\tilde{a}_j| + \sum_{k\neq 1,j} |a_k| \\ &= 0 + |a_j + a_1| + \sum_{k\neq 1,j} |a_k| \leq |a_1| + |a_j| + \sum_{k\neq 1,j} |a_k| = F(a), \end{aligned}$$

where we used the triangle inequality in $\mathbb{R}$. Hence replacing (a_k) with $(\tilde{a}_k)$ does not increase the value of F. Iterating this procedure if necessary, we may assume that in any minimizing sequence we have $a_1 \geq 0$.

Reduction 2: we may eliminate the tail $(a_k)_{k\geq 2}$.

Suppose that for a candidate minimizer (a_k) we have some positive "mass" in the tail, i.e. there exists $n \geq 2$ with $a_n > 0$. Take a small $\delta > 0$ with $0 < \delta \leq a_n$ and define

$$\tilde{a}_1 := a_1 + \delta, \qquad \tilde{a}_n := a_n - \delta, \qquad \tilde{a}_k := a_k \quad (k \neq 1, n).$$

Then $\sum_k \tilde{a}_k = \sum_k a_k = 0$. Moreover,

$$\begin{aligned} F(\tilde{a}) &= t|\tilde{a}_1| + |\tilde{a}_n| + \sum_{k\neq 1,n} |a_k| \\ &= t(a_1 + \delta) + |a_n - \delta| + \sum_{k\neq 1,n} |a_k|. \end{aligned}$$

Since $a_1 \geq 0$ by Reduction 1 and $a_n > 0$, we may choose $0 < \delta \leq a_n$ so that

$$|a_n - \delta| = a_n - \delta,$$

and hence

$$F(\tilde{a}) = ta_1 + t\delta + a_n - \delta + \sum_{k\neq 1,n} |a_k| = F(a) + (t-1)\delta \leq F(a),$$

because $t - 1 < 0$. Thus transferring a small amount of mass from a positive tail coefficient a_n to a_1 does not increase F, and in fact strictly decreases it unless $\delta = 0$. Repeating this operation finitely many times shows that, in any minimizing configuration, we must have

$$a_k \leq 0 \quad \text{for all } k \geq 2.$$

Combining this with $\sum_k a_k = 0$ and $a_1 \geq 0$, we obtain

$$a_1 = \sum_{k=2}^{\infty} (-a_k) =: \xi \geq 0,$$

and hence

$$F(a) = ta_1 + \sum_{k=2}^{\infty} |a_k| = t\xi + \sum_{k=2}^{\infty} (-a_k) = t\xi + a_1 = t\xi + \xi = \xi(t+1).$$

In particular, the minimal value of F is attained when all the negative mass is concentrated in a single coordinate (say $k = 2$), and all the positive mass is at the first coordinate:

$$a_1 = \xi, \qquad a_2 = -\xi, \qquad a_k = 0 \ (k \geq 3).$$

This corresponds to a point $h \in E_t^{(m)}$ with

$$h = (\gamma_1 + \xi) f_1 + (\gamma_2 - \xi) f_2 + \sum_{k=3}^{\infty} \gamma_k f_k,$$

and

$$\|h - u\|^{(m)} = F(a) = \xi(t + 1) =: m.$$

Collecting the above reductions, we see that the set

$$\Lambda := \{w \in E_t^{(m)} : \|w - u\|^{(m)} = m\}$$

of minimizers of $d(w)$ is nonempty, closed and convex, and is contained in the finite-dimensional subspace generated by $\{f_1, f_2\}$. In particular, Λ is compact in the norm topology of $D_1^{(m)}$.

Invariance of Λ.

Let $w \in \Lambda$. Then by definition $\|w - u\|^{(m)} = m$. Using (3.2) with this w we obtain

$$Q(u) + \|Tw - u\|^{(m)} \le Q(u) + \|w - u\|^{(m)} = Q(u) + m,$$

hence $\|Tw - u\|^{(m)} \le m$. By the definition of m as the minimal distance from u to $E_t^{(m)}$, we must have equality, i.e.

$$\|Tw - u\|^{(m)} = m,$$

and therefore $Tw \in \Lambda$. Thus Λ is a nonempty compact convex subset of $D_1^{(m)}$ which is invariant under T.

Conclusion.

The restriction $T|\Lambda \colon \Lambda \to \Lambda$ is nonexpansive (hence continuous) and Λ is compact and convex in the Banach space $D_1^{(m)}$. By Schauder's fixed point theorem, $T|\Lambda$ has a fixed point, which is also a fixed point of T in $E_t^{(m)}$. This completes the proof. □

Theorem 3.2 (Finite Mixed Block Fixed Point Theorem in $D_1^{(m)}$). *Let $m \in \mathbb{N}$ be fixed and let $s \in \mathbb{N}$. Choose coefficients $t_1, \dots, t_s \in (0, 1)$ and let (v_k) be the defining coefficient sequence of $D_1^{(m)}$. Define a sequence $(f_n) \subset D_1^{(m)}$ by*

$$f_j := t_j \frac{v_j}{j^m} \widetilde{e_j}, \qquad 1 \le j \le s,$$

$$f_n := \frac{v_n}{n^m} \widetilde{e_n}, \qquad n \ge s + 1,$$

where $(\widetilde{e_n})$ denotes the canonical unit vectors in $D_1^{(m)}$. Set

$$E_s^{(m)} := \Big\{ \sum_{n=1}^{\infty} \alpha_n f_n : \alpha_n \ge 0, \sum_{n=1}^{\infty} \alpha_n = 1 \Big\}.$$

Then every nonexpansive mapping $T : E_s^{(m)} \to E_s^{(m)}$ has a fixed point. In other words, $E_s^{(m)}$ has the fixed point property for nonexpansive mappings.

Proof. Let $T : E_s^{(m)} \to E_s^{(m)}$ be nonexpansive. Choose an approximate fixed point sequence (AFPS) $(u^{(n)}) \subset E_s^{(m)}$ such that

$$\|T(u^{(n)}) - u^{(n)}\|^{(m)} \xrightarrow[n \to \infty]{} 0.$$

Define the Goebel–Kuczumow functional

$$Q(w) = \limsup_{n \to \infty} \|u^{(n)} - w\|^{(m)}, \qquad w \in E_s^{(m)}.$$

By the Goebel–Kuczumow lemma (in the form valid for $D_1^{(m)}$), there exists a point $u \in E_s^{(m)}$ such that

$$Q(w) = Q(u) + \|w - u\|^{(m)} \qquad (w \in E_s^{(m)}). \tag{3.4}$$

Write

$$u = \sum_{n=1}^{\infty} \gamma_n f_n, \qquad \gamma_n \geq 0, \quad \sum_{n=1}^{\infty} \gamma_n = 1 - \delta,$$

for some $\delta \geq 0$. If $\delta = 0$, then $u \in E_s^{(m)}$ has full mass and (3.4) with $w = Tu$ gives

$$Q(Tu) = Q(u) + \|Tu - u\|^{(m)} \leq Q(u),$$

so $\|Tu - u\|^{(m)} = 0$ and $Tu = u$. Thus we may assume for the rest of the proof that $\delta > 0$.

Step 1: Mass equation and difference representation. Let

$$y = \sum_{n=1}^{\infty} t_n f_n \in E_s^{(m)}, \qquad t_n \geq 0, \quad \sum_{n=1}^{\infty} t_n = 1.$$

Define the difference coefficients

$$a_n := t_n - \gamma_n, \qquad n \in \mathbb{N}.$$

Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} t_n - \sum_{n=1}^{\infty} \gamma_n = 1 - (1 - \delta) = \delta. \tag{3.5}$$

Moreover

$$y - u = \sum_{n=1}^{\infty} a_n f_n.$$

Step 2: Norm formula before reduction. Recall that the norm in $D_1^{(m)}$ is given by

$$\|x\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^m |x_k|}{|v_k|}, \qquad x = (x_k) \in D_1^{(m)}.$$

By definition,

$$f_j = t_j \frac{v_j}{j^m} \widetilde{e}_j, \quad 1 \leq j \leq s, \qquad f_n = \frac{v_n}{n^m} \widetilde{e}_n, \quad n \geq s+1,$$

so

$$\|f_j\|^{(m)} = t_j, \quad 1 \leq j \leq s, \qquad \|f_n\|^{(m)} = 1, \quad n \geq s+1.$$

Hence

$$\|y - u\|^{(m)} = \sum_{j=1}^{s} t_j |a_j| + \sum_{n=s+1}^{\infty} |a_n|. \tag{3.6}$$

Step 3: Tail elimination (mass transfer). Suppose that $a_k > 0$ for some $k \geq s+1$. Take an index $j \in \{1, \ldots, s\}$ and choose $0 < \varepsilon \leq a_k$. Consider the new coefficients

$$\tilde{a}_j := a_j + \varepsilon, \quad \tilde{a}_k := a_k - \varepsilon, \quad \tilde{a}_n := a_n \ (n \neq j, k).$$

The total mass condition (3.5) is preserved.

We compare the norms. From (3.6) we have

$$\|y - u\|^{(m)} = t_j |a_j| + |a_k| + \sum_{n \neq j,k} \mu_n |a_n|,$$

where $\mu_n = t_n$ for $n \le s$ and $\mu_n = 1$ for $n \ge s+1$. Similarly,

$$\|\tilde{y} - u\|^{(m)} = t_j|\tilde{a}_j| + |\tilde{a}_k| + \sum_{n\neq j,k} \mu_n|a_n|.$$

Since $a_k > 0$ and $0 < \varepsilon \le a_k$, we have

$$|a_k| - |\tilde{a}_k| = a_k - (a_k - \varepsilon) = \varepsilon.$$

For the j-th coordinate we have

$$|\tilde{a}_j| - |a_j| \le |\tilde{a}_j - a_j| = \varepsilon,$$

and thus

$$t_j|\tilde{a}_j| - t_j|a_j| \le t_j\varepsilon.$$

Therefore

$$\|\tilde{y} - u\|^{(m)} - \|y - u\|^{(m)} \le t_j\varepsilon - \varepsilon = \varepsilon(t_j - 1) < 0,$$

since $t_j \in (0,1)$. Hence transferring a positive amount of mass from any tail coordinate $k \ge s+1$ to a block index $j \le s$ strictly decreases the norm.

By iterating this procedure we may assume, without loss of generality, that

$$a_n = 0 \ \ (n \ge s+1), \qquad a_j \ge 0 \ \ (1 \le j \le s), \qquad \sum_{j=1}^{s} a_j = \delta. \tag{3.7}$$

Step 4: Norm as a linear functional on a simplex. Under the reduced form (3.7), (3.6) reduces to

$$\|y - u\|^{(m)} = \sum_{j=1}^{s} t_j a_j. \tag{3.8}$$

Thus $\|y - u\|^{(m)}$ is a positive linear functional on the simplex

$$\Delta_s(\delta) := \Big\{(a_1, \dots, a_s) \in [0,\infty)^s : \ \sum_{j=1}^{s} a_j = \delta\Big\}.$$

Let

$$t_* := \min\{t_1, \dots, t_s\}, \qquad I_* := \{j \in \{1, \dots, s\} : t_j = t_*\}.$$

Since the minimum of a linear functional on a simplex is attained at an extreme point, we obtain

$$\min\{\|y - u\|^{(m)} : y \in E_s^{(m)}\} = \delta t_*. \tag{3.9}$$

The set of minimizers in the coefficient space is then

$$\Big\{(a_1, \dots, a_s) \in \Delta_s(\delta) : a_j = 0 \text{ if } j \notin I_*\Big\},$$

so the set of minimizers in $E_s^{(m)}$ is

$$\Lambda := \Big\{w = \sum_{n=1}^{\infty} \theta_n f_n : \theta_j = \gamma_j + a_j,\ a_j \ge 0,\ a_j = 0 \text{ if } j \notin I_*,\ \sum_{j\in I_*} a_j = \delta\Big\}.$$

Thus Λ is nonempty, convex and weak*-compact in $D_1^{(m)}$. Moreover, Λ is a singleton, a line segment, or a higher-dimensional simplex depending on the cardinality of I_*.

Step 5: Balancing family inside Λ. Fix any $j \in I_*$. We construct a balancing path $(h_\beta^{(j)})_{\beta\in[0,1]}$ inside Λ as follows. Set

$$h_\beta^{(j)} := u + (1-\beta)\delta f_j + \frac{\beta\delta}{|I_*|-1}\sum_{\substack{i\in I_*\\ i\neq j}} f_i, \qquad 0 \le \beta \le 1.$$

Then

$$h_\beta^{(j)} - u = (1-\beta)\delta f_j + \frac{\beta\delta}{|I_*|-1}\sum_{\substack{i\in I_*\\ i\neq j}} f_i.$$

Using $\|f_j\|^{(m)} = t_j$ and $\|f_i\|^{(m)} = t_i$ we obtain

$$\|h_\beta^{(j)} - u\|^{(m)} = (1-\beta)\delta t_j + \frac{\beta\delta}{|I_*|-1}\sum_{\substack{i\in I_*\\ i\neq j}} t_i.$$

Since all $i \in I_*$ satisfy $t_i = t_*$ and also $t_j = t_*$, this simplifies to

$$\|h_\beta^{(j)} - u\|^{(m)} = (1-\beta)\delta t_* + \beta\delta t_* = \delta t_*.$$

Together with (3.9), this shows that each $h_\beta^{(j)}$ is a minimizer and hence lies in Λ.

Step 6: T-invariance of Λ. Take an arbitrary $w \in \Lambda$. From (3.4) and the definition of Λ, we have

$$Q(w) = Q(u) + \|w-u\|^{(m)} = Q(u) + \delta t_*.$$

Since T is nonexpansive and $(u^{(n)})$ is an AFPS, the Goebel–Kuczumow lemma implies that

$$Q(Tw) \le Q(w).$$

On the other hand, (3.4) applied to Tw gives

$$Q(Tw) = Q(u) + \|Tw-u\|^{(m)}.$$

Combining the last two displays and using (3.9), we obtain

$$Q(u) + \|Tw-u\|^{(m)} = Q(Tw) \le Q(u) + \delta t_*,$$

so

$$\|Tw-u\|^{(m)} \le \delta t_*.$$

By the minimality of δt_* this forces $\|Tw-u\|^{(m)} = \delta t_*$ and hence $Tw \in \Lambda$. Thus

$$T(\Lambda) \subset \Lambda.$$

Step 7: Existence of a fixed point. If Λ is a singleton, say $\Lambda = \{w_0\}$, then $Tw_0 = w_0$ and we are done. If Λ contains more than one point, then Λ is a nonempty, convex, weak*-compact subset of $D_1^{(m)}$ and $T : \Lambda \to \Lambda$ is nonexpansive (in particular, continuous). By Schauder's fixed point theorem, T has a fixed point in Λ.

In both cases, T has a fixed point in $E_s^{(m)}$, and the proof is complete. □

4. Corollaries and Further Remarks

4.1. Two–parameter and multi–parameter consequences

The mixed block fixed point theorem yields several immediate consequences. We record three of them for later use.

Corollary 1 (Two–parameter mixed block). *Let*

$$f_1 := t\frac{v_1}{1^m}\widetilde{e}_1, \qquad f_2 := r\frac{v_2}{2^m}\widetilde{e}_2,$$

with $t, r \in (0,1)$*, and for* $n \geq 3$ *define*

$$f_n := \frac{v_n}{n^m}\widetilde{e}_n.$$

Let

$$E^{(m)} := \mathrm{co}\{f_n : n \in \mathbb{N}\} \subset D_1^{(m)}.$$

Then every nonexpansive mapping $T : E^{(m)} \to E^{(m)}$ *has a fixed point.*

Proof. This is the special case of the main theorem with $s = 2$ and block $\{1, 2\}$. □

Corollary 2 (Uniform coefficient block). *Let* $s \in \mathbb{N}$ *and choose a single coefficient* $t \in (0, 1)$*. Define*

$$f_j := t\frac{v_j}{j^m}\widetilde{e}_j \qquad (1 \leq j \leq s), \qquad f_n := \frac{v_n}{n^m}\widetilde{e}_n \qquad (n > s),$$

and set

$$E_s^{(m)} := \mathrm{co}\{f_n : n \in \mathbb{N}\}.$$

Then $E_s^{(m)}$ *has the fixed point property for nonexpansive mappings.*

Proof. This is the main theorem with $t_j = t$ for $1 \leq j \leq s$. □

Corollary 3 (Arbitrary finite block of coefficients). *Let* $s \in \mathbb{N}$ *and choose arbitrary coefficients*

$$0 < t_j < 1, \qquad 1 \leq j \leq s.$$

Define

$$f_j := t_j\frac{v_j}{j^m}\widetilde{e}_j, \qquad (1 \leq j \leq s), \qquad f_n := \frac{v_n}{n^m}\widetilde{e}_n \qquad (n > s).$$

Then

$$E^{(m)} := \mathrm{co}\{f_n : n \in \mathbb{N}\}$$

has the fixed point property.

Proof. Immediate from the mixed block theorem. □

4.2. Remarks on infinite blocks and Everest–type constructions

The previous results rely crucially on the fact that only finitely many coordinates carry a coefficient strictly less than 1. Several infinite generalizations are possible, but require additional geometric control. We collect here three remarks that clarify the scope of the method.

Remark 1 (Infinite block with $t_n \downarrow 0$). *Suppose that for infinitely many n we have $0 < t_n < 1$ and $t_n \to 0$. Define*

$$f_n := t_n \frac{v_n}{n^m}\widetilde{e_n} \qquad (n \in \mathbb{N}), \qquad E^{(m)} := \mathrm{co}\{f_n\}.$$

If a weak cluster point u of an AFPS satisfies*

$$\sum_{n=1}^{\infty} \gamma_n = 1 - \delta \quad (\delta > 0),$$

then the mass δ can always be concentrated into a sufficiently large finite block where t_n is sufficiently small. Consequently the balancing argument carries over on that finite block. A full infinite–dimensional theorem is possible, but the statement is technically involved and requires additional compactness assumptions.

Remark 4.1 (Infinite constant block and failure of the FPP). *In contrast with the finite mixed block situation of Theorem 3.2, the case where we put the same coefficient $t \in (0,1)$ in front of* every *basis vector does not enjoy the fixed point property.*

More precisely, fix $t \in (0,1)$ and define

$$f_n := t\,\frac{v_n}{n^m}\,\tilde{e}_n, \qquad n \in \mathbb{N},$$

and let

$$E_\infty^{(m)} := \Big\{\sum_{n=1}^{\infty} \alpha_n f_n : \alpha_n \geq 0,\ \sum_{n=1}^{\infty} \alpha_n = 1\Big\} \subset D_1^{(m)}.$$

Then $E_\infty^{(m)}$ is a nonempty closed, bounded and convex subset of $D_1^{(m)}$, but it fails to have the fixed point property for nonexpansive mappings.

Proof. *Consider the isometric isomorphism $J\colon D_1^{(m)} \to \ell^1$ introduced in the preliminaries. Since*

$$J(f_n) = te_n, \qquad n \in \mathbb{N},$$

we obtain

$$C_t := J\Big(E_\infty^{(m)}\Big) = \Big\{x = (x_n) \in \ell^1 : x_n \geq 0 \text{ for all } n,\ \sum_{n=1}^{\infty} x_n = t\Big\}.$$

Thus C_t is exactly the set of all nonnegative elements of ℓ^1 having ℓ^1–norm equal to t; in particular it is closed, bounded and convex.

Define the (right) shift operator $R\colon \ell^1 \to \ell^1$ by

$$R(x)_1 := 0, \qquad R(x)_{n+1} := x_n \quad (n \geq 1), \qquad x = (x_n) \in \ell^1.$$

Then R is an isometry, hence nonexpansive, and for every $x \in C_t$ we have

$$R(x)_n \geq 0 \quad \text{for all } n, \qquad \sum_{n=1}^{\infty} R(x)_n = \sum_{n=1}^{\infty} x_n = t,$$

so $R(C_t) \subset C_t$; that is, R is invariant on C_t.

We now show that R has no fixed point in C_t. If $R(x) = x$ for some $x = (x_n) \in C_t$, then from the first coordinate we get $x_1 = R(x)_1 = 0$. Inductively,

$$x_2 = R(x)_2 = x_1 = 0, \quad x_3 = R(x)_3 = x_2 = 0, \quad \ldots,$$

so $x_n = 0$ for all $n \in \mathbb{N}$. This contradicts the fact that $x \in C_t$ must satisfy $\sum_{n=1}^{\infty} x_n = t > 0$. Hence R has no fixed point in C_t.

Finally, define

$$T := J^{-1} \circ R \circ J \colon E_{\infty}^{(m)} \to E_{\infty}^{(m)}.$$

Since J is an isometric isomorphism, T is nonexpansive, $T(E_{\infty}^{(m)}) \subset E_{\infty}^{(m)}$, and T has a fixed point in $E_{\infty}^{(m)}$ if and only if R has a fixed point in C_t. As we have just seen, the latter is impossible. Therefore $E_{\infty}^{(m)}$ fails to have the fixed point property for nonexpansive mappings.

Remark 2 (Everest–type double–shift generalizations). *Constructions of the form*

$$f_n := t_n \frac{v_n}{n^m} \widetilde{e}_n + r_n \frac{v_{n+1}}{(n+1)^m} \widetilde{e}_{n+1}$$

generalize the classical Everest pattern $e_n + e_{n+1}$. The mass transfer must then be performed simultaneously across two neighboring coordinates. The fixed point property can still be ensured under mild regularity conditions on (t_n) and (r_n), but the proof becomes a two–dimensional balancing argument. This case naturally belongs to a more general theory and will be developed elsewhere.

5. Conclusion

In this paper we investigated the fixed point property for nonexpansive mappings on large classes of closed, bounded and convex subsets of the Köthe–Toeplitz dual space $D_1^{(m)}$, which is isometrically isomorphic to the classical space ℓ^1. By transporting the Goebel–Kuczumow lemma through the isometry $J : D_1^{(m)} \to \ell^1$ we obtained a representation formula for the functional

$$Q(w) = \limsup_{n\to\infty} \|u^{(n)} - w\|^{(m)}, \qquad w \in D_1^{(m)},$$

which is adapted to the geometry of $D_1^{(m)}$. This representation allowed us to combine approximate fixed point sequences with a precise "mass–transfer" analysis on suitable convex sets generated by the canonical basis of $D_1^{(m)}$.

Our first main result shows that already a very simple perturbation of the canonical basis — introducing a single coefficient $t \in (0, 1)$ in front of one basis vector — is sufficient to restore the fixed point property on the corresponding convex hull, even though the ambient space ℓ^1 itself fails the fixed point property. We then extended this to arbitrary finite mixed blocks of coefficients, obtaining a flexible class of sets

$$E^{(m)} = \mathrm{co}\{f_n : n \in \mathbb{N}\} \subset D_1^{(m)}$$

on which every nonexpansive self–mapping admits a fixed point. The proofs reveal a clear geometric mechanism: mass can be transferred toward coordinates with smaller norm contribution, and the Goebel–Kuczumow functional forces every nonexpansive mapping to leave invariant a weak* compact convex subset, where Schauder's theorem yields a fixed point.

We also observed that this phenomenon is genuinely finite–dimensional along the basis: in the infinite constant–block case, where the same coefficient $t \in (0, 1)$ is placed in front of every basis vector, we constructed a shift–type nonexpansive mapping without fixed points, showing that the fixed point property fails. This exhibits the sharpness of the finite mixed block condition.

Finally, we discussed several directions in which the present approach can be extended, including decreasing infinite sequences of coefficients, Everest–type double–shift constructions and possible generalizations to other Köthe sequence spaces. These problems lie beyond the scope of the present article and will be investigated in future work.

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