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Gaussian and Mean Curvatures of the Surface Obtained Along T-Pedal Curve of A Given Curve

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Abstract. In this study, the surface was constructed using Frenet vectors along a curve known as T-pedal curve, which is geometric location of the perpendicular projection points from a point not on the curve onto the tangent vector of any curve. Then surfaces containing the T-pedal curve were characterized using marching-scale functions by employing Frenet vectors along the T-pedal curve. Finally, the geometric properties, Gaussian and Mean curvatures of these obtained surfaces are calculated.

1. Introduction

Curve theory and surface curvatures are extensively in differential geometry textbooks. Many studies exist on the problem of finding surfaces, particularly on what is called the inverse problem ([5-7], [10], [12]). The geometric locus of the perpendicular projection points from a point not on the curve onto the tangent (or principal normal) vector of a curve is called a pedal (or contra-pedal) curve. Many studies exist on pedal curves [1-3], [8], [11], [13-15], [17-18]). In this study, surfaces containing the T-pedal curve were investigated under different marching-scale functions. Geometric properties, Gaussian and Mean curvatures of these surfaces were determined.

2. Preliminaries

If the curve $\alpha(t)$ is given with arbitrary parameters, the Frenet vectors T, N, B and curvatures κ, τ then Frenet formulas are shown as following relations, respectively:

$$\begin{aligned} T &= \frac{\alpha'}{\|\alpha'\|} \\ N &= B \wedge T = \frac{(\alpha' \wedge \alpha'') \wedge \alpha'}{\|\alpha' \wedge \alpha''\| \, \|\alpha'\|} \\ B &= \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|} \end{aligned} \tag{1}$$

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The curvature κ and torsion τ are given by:

$$\begin{aligned} \kappa &= \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3} \\ \tau &= \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2} \end{aligned} \tag{2}$$

$$\begin{aligned} T' &= v\kappa N \\ N' &= v(-\kappa T + \tau B) \\ B' &= -v\tau N \end{aligned} \tag{3}$$

where $v = \|\alpha'(t)\|$, ([4], [9]).

Let $M_\alpha \subset E^3$ be a surface and $\hat{n}$ be a normal vector field of M_α.Formulas of the surface's Gauss curvature K_{M_α} and mean curvature H_{M_α} are given by following equalities ([4], [16]):

$$K_{M_\alpha} = \frac{ln - m^2}{EG - F^2} \qquad and \qquad H_{M_\alpha} = \frac{En + Gl - 2Fm}{2(EG - F^2)} \tag{4}$$

We give the following relations with about the expressions l, m, n, E, F, G respectively ([4], [9], [16]).

$$\begin{aligned} l &= < \hat{n}, M_{ss} > \\ m &= < \hat{n}, M_{ts} > \\ n &= < \hat{n}, M_{tt} > \\ E &= < M_s, M_s > \\ F &= < M_s, M_t > \\ G &= < M_t, M_t > \end{aligned} \tag{5}$$

3. Gauss and Mean Curvatures of T-Pedal Curves

Theorem 3.1. *([21]) Let $\alpha(s)$ be a curve non-zero unit speed curve and M_α be the surface of curve $\alpha(s)$. New surfaces belonged to curve $\alpha(s)$ are obtained with each different selection of the marching-scale functions o, p and r from class C^k. The equation of this surface M_α is given by following equality:*

$$M_\alpha(s,t) = \alpha(s) + o(s,t)T(s) + p(s,t)N(s) + r(s,t)B(s) \tag{6}$$

where $L_1 \leq s \leq L_2, T_1 \leq t \leq T_2$.

Definition 3.2. *Let T_α denote the tangent vector of a regular curve in E^2. The geometric locus of the perpendicular projection of points onto a tangent vector from a given point $P \in E^2$ that is not on the curve is called the pedal curve of the curve α ([11], [17], [20]).*

Theorem 3.3. *(([17], [20]) The pedal curve of a regular curve α according to the point $P \in E^2$ is given by the following equality:*

$$\alpha_T(t) = \alpha(t) - \langle P - \alpha(t), T_\alpha \rangle T_\alpha \tag{7}$$

Definition 3.4. *Let T be the tangent vector of a given regular curve in E^3 The geometric locus of the perpendicular projection of points onto a tangent vector from a given point $P \in E^3$ that is not on the curve is called the $T - Pedal$ curve α according to P ([17]).*

Theorem 3.5. *([17], [20]) The equation of $T-$ Pedal curve of a regular curve α is as follows:*

$$\alpha_T(t) = \alpha(t) - \langle P - \alpha(t), T\rangle\, T \tag{8}$$

If the point P is taking as origin, then the equation of $T-$ Pedal curve of a regular curve is obtained by following equality ([17]):

$$\alpha_T(t) = \alpha(t) - \langle \alpha(t), T\rangle\, T \tag{9}$$

Theorem 3.6. *Let $\alpha_T(t)$ be $T-$ Pedal curve of the curve α. Let T_1, N_1, B_1 denote Frenet vectors and κ_1, τ_1 denote curvatures of curve $\alpha_T(t)$. Also, let take f, g and h as deviation functions. The equations of Gauss curvature $K_{M_{\alpha_T}}$ and mean curvature $H_{M_{\alpha_T}}$ belonged to the surface M of Curve $\alpha_T(t)$ are calculated as follows:*

$$K_{M_{\alpha_T}} = \frac{\begin{vmatrix}(y_1)_t & (z_1)_t\\ y_1 & z_1\end{vmatrix} f_t + \begin{vmatrix}(z_1)_t & (x_1)_t\\ z_1 & x_1\end{vmatrix} g_t + \begin{vmatrix}(x_1)_t & (y_1)_t\\ x_1 & y_1\end{vmatrix} h_t}{(x_1^2+y_1^2+z_1^2)(f_t^2+g_t^2+h_t^2) - (x_1 f_t + y_1 g_t + z_1 h_t)^2} \tag{10}$$

$$H_{M_\alpha} = \frac{\begin{aligned}&(f_t^2+g_t^2+h_t^2)\Big(x_2\begin{vmatrix}h_t & g_t\\ z_1 & y_1\end{vmatrix} + y_2\begin{vmatrix}f_t & h_t\\ x_1 & z_1\end{vmatrix} + z_2\begin{vmatrix}g_t & f_t\\ y_1 & x_1\end{vmatrix}\Big)\\ &- (x_1 f_t + y_1 g_t + z_1 h_t)\Big((x_1)_t\begin{vmatrix}h_t & g_t\\ z_1 & y_1\end{vmatrix} + (y_1)_t\begin{vmatrix}f_t & h_t\\ x_1 & z_1\end{vmatrix} + (z_1)_t\begin{vmatrix}g_t & f_t\\ y_1 & x_1\end{vmatrix}\Big)\end{aligned}}{\Big((x_1^2+y_1^2+z_1^2)(f_t^2+g_t^2+h_t^2) - (x_1 f_t + y_1 g_t + z_1 h_t)^2\Big) \times \sqrt{\begin{vmatrix}h_t & g_t\\ z_1 & y_1\end{vmatrix}^2 + \begin{vmatrix}f_t & h_t\\ x_1 & z_1\end{vmatrix}^2 + \begin{vmatrix}g_t & f_t\\ y_1 & x_1\end{vmatrix}^2}} \tag{11}$$

where the coefficients are as follows:

$$\begin{aligned} x_1 &= \mu + f_s - \mu\, g\kappa_1,\\ y_1 &= \mu\, f\kappa_1 + g_s - \mu\, h\tau_1,\\ z_1 &= \mu\, g\tau_1 + h_s. \end{aligned} \tag{12}$$

and

$$\begin{aligned} x_2 &= (x_1)_s - \mu\kappa_1 y_1,\\ y_2 &= \mu\, x_1\kappa_1\, (y_1)_s - \mu\, z_1\tau_1,\\ z_2 &= \mu\, y_1\tau_1 + (z_1)_s\,. \end{aligned} \tag{13}$$

Proof. We know that surfaces belonged the curve $\alpha_t(t)$ can be written as follows:

$$M(s,t) = \alpha_t + f(s,t)T_1(s) + g(s,t)N_1(s) + h(s,t)B_1(s) \tag{14}$$

New surfaces are obtained that pass through the $T-$ *Pedal* curve $\alpha_t(t)$ with each different selection of the marching-scale functions f, g and h from class C^k.

Since the $T-$ *pedal* curve $\alpha_t(t)$ is not unit speed, we should take $T_1 = \frac{\alpha_T'}{\|\alpha_T'\|}$. From here, $\alpha_T' = T_1\|\alpha_T'\|$. Let take $\|\alpha_T'\| = \mu$. So, we have following equalities:

$$\begin{aligned} T_1 &= \mu\kappa_1\\ N_1 &= \mu(-\kappa_1 T_1 + \tau_1 B_1)\\ B_1 &= \mu\tau_1 N_1 \end{aligned} \tag{15}$$

By taking the derivatives of the surface

$$M_s = (\alpha_T' + f_s(s,t)T_1(s) + fT_1'(s,t) + g_s(s,t)N_1(s) + g(s,t)N_1'(s) + h_s(s,tB_1(s) + h(s,t)B_1'(s)) \quad (16)$$

By substituting the relations in (12) into (16) and simplifying, the following expression is obtained:

$$M_s = (\mu + f_s(s,t) - \mu g\kappa_1)T_1(s) + (\mu f\kappa_1 + g_s - \mu h\tau_1)N_1(S) + (\mu g\tau_1 + h_s)B_1(s). \quad (17)$$

Moreover, we compute the partial derivatives of the surface as follows:

$$\begin{aligned} M_s &= x_1T_1 + y_1N_1 + z_1B_1, \\ M_t &= f_tT_1 + g_tN_1 + z_1B_1 \\ M_{ss} &= ((x_1)_s - \mu\kappa_1 y_1)T_1 + (\mu x_1\kappa_1(y_1)_s - \mu z_1\tau_1)N_1(s) + (\mu y_1\tau_1 + (z_1)_s)B_1 \end{aligned} \quad (18)$$

If we substitute the relations given in (13) into equation (18), the following expressions are obtained:

$$\begin{aligned} M_{ss} &= x_2T_1 + y_2N_1 + z_2B_1 \\ M_{st} &= (x_1)_tT_1 + (Y_1)N_1 + (z_1)_tB_1 \\ M_{tt} &= 0. \end{aligned} \quad (19)$$

Let $\hat{N}$ be the normal vector field along the curve α_t of the surface M. The normal vector field $\hat{N}$ is calculated using the following equation:

$$\hat{N} = \frac{(y_1h_t - z_1g_t)T_1 + (z_1f_t - x_1h_t)N_1 + (x_1g_t - y_1f_t)B_1}{\sqrt{(y_1h_t - z_1g_t)^2 + (z_1f_t - x_1h_t)^2 + (x_1g_t - y_1f_t)^2}} \quad (20)$$

We calculated following equalites from the expression (5):

$$\begin{aligned} E &= \langle M_s, M_s\rangle = x_1^2 + y_1^2 + z_1^2, \\ F &= \langle M_s, M_t\rangle = x_1f_t + Y_1g_t + z_1h_t, \\ G &= \langle M_t, M_t\rangle = f_t^2 + g_t^2 + h_t^2. \end{aligned} \quad (21)$$

and

$$\begin{aligned} l &= \left\langle \hat{N}, M_s\right\rangle = \frac{x_2(y_1h_t - z_1g_t) + y_2(z_1f_t - z_1h_t) + z_2(x_1g_t - y_1f_t)}{\sqrt{(y_1h_t - z_1g_t)^2 + (y_2(z_1f_t - z_1h_t)^2 + (z_2(x_1g_t - y_1f_t)^2}}, \\ m &= \left\langle M_{st}, \hat{N}\right\rangle = \frac{(x_1)_t(y_1h_t - z_1g_t) + (y_1)_t(z_1f_t - z_1h_t) + (z_1)_t(z_2(x_1g_t - y_1f_t)}{\sqrt{(y_1h_t - z_1g_t)^2 + (y_2(z_1f_t - z_1h_t)^2 + (z_2(x_1g_t - y_1f_t)^2}}, \\ n &= \left\langle M_{tt}, \hat{N}\right\rangle = 0. \end{aligned} \quad (22)$$

By substituting these relations into equation (4), the Gaussian and mean curvatures of the surface M corresponding to the T-pedal curve $\alpha_T(t)$ are given by $K_{M_{\alpha_T}}$ and $H_{M_{\alpha_T}}$, respectively, as follows:

$$K_{M_{\alpha_T}} = \frac{\begin{vmatrix} (y_1)_t & (z_1)_t \\ y_1 & z_1 \end{vmatrix} f_t + \begin{vmatrix} (z_1)_t & (x_1)_t \\ z_1 & x_1 \end{vmatrix} g_t + \begin{vmatrix} (x_1)_t & (y_1)_t \\ x_1 & y_1 \end{vmatrix} h_t}{(x_1^2 + y_1^2 + z_1^2)(f_t^2 + g_t^2 + h_t^2) - (x_1f_t + y_1g_t + z_1h_t)^2}. \quad (23)$$

$$H_{M_{\alpha_T}} = \frac{(f_t^2 + g_t^2 + h_t^2)\left(x_2\begin{vmatrix} h_t & g_t \\ z_1 & y_1 \end{vmatrix} + y_2\begin{vmatrix} f_t & h_t \\ x_1 & z_1 \end{vmatrix} + z_2\begin{vmatrix} g_t & f_t \\ y_1 & x_1 \end{vmatrix}\right) - (x_1f_t + y_1g_t + z_1h_t)\left((x_1)_t\begin{vmatrix} h_t & g_t \\ z_1 & y_1 \end{vmatrix} + (y_1)_t\begin{vmatrix} f_t & h_t \\ x_1 & z_1 \end{vmatrix} + (z_1)_t\begin{vmatrix} g_t & f_t \\ y_1 & x_1 \end{vmatrix}\right)}{\left((x_1^2 + y_1^2 + z_1^2)(f_t^2 + g_t^2 + h_t^2) - (x_1f_t + y_1g_t + z_1h_t)^2\right) \times \sqrt{\begin{vmatrix} h_t & g_t \\ z_1 & y_1 \end{vmatrix}^2 + \begin{vmatrix} f_t & h_t \\ x_1 & z_1 \end{vmatrix}^2 + \begin{vmatrix} g_t & f_t \\ y_1 & x_1 \end{vmatrix}^2}} \quad (24)$$

□

Example 3.7. *Let us consider the unit speed curve parametrized as* $\alpha(s)=\left(-\frac{1}{2}cos\,\sqrt{2}s,-\frac{1}{2}sin\,\sqrt{2}s,\frac{\sqrt{2}}{2}s\right)$.

Frenet vectors of the curve are as follows:

$$\begin{aligned}T(s)&=\left(\frac{\sqrt{2}}{2}sin\,\sqrt{2}s,-\frac{\sqrt{2}}{2}cos\,\sqrt{2}s,\frac{\sqrt{2}}{2}\right)\\N(s)&=\left(cos\,\sqrt{2}s,sin\,\sqrt{2}s,0\right)\\B(s)&=\left(-\frac{\sqrt{2}}{2}sin\,\sqrt{2}s,\frac{\sqrt{2}}{2}cos\,\sqrt{2}s,\frac{\sqrt{2}}{2}\right).\end{aligned}$$

If $\alpha(t)$ *denote the* $T-$ *Pedal curve of curve and make some algebraic operations, then following relations exist:*

$$\begin{aligned}\alpha_T(s)&=\alpha(s)-\frac{s}{2}T=\left(-\frac{1}{2}cos\,\sqrt{2}s-\frac{s}{2},-\frac{1}{2}sin\,\sqrt{2}s,\frac{\sqrt{2}}{2}s\right),\\\alpha_T'(s)&=\frac{1}{2}\left(T-sN\right),\\\alpha_T''(s)&=\frac{s}{2}\left(T-B\right),\\\alpha_T'''(s)&=\frac{1}{2}\left(T-B\right)+sN,\\\|\alpha_T'(s)\|&=\frac{1}{2}\sqrt{1+s^2},\\\alpha_T'\wedge\alpha_T''&=\frac{s^2}{4}(sT-N-sB),\\\|\alpha_T'\wedge\alpha_T''\|&=\frac{s^2}{4}\sqrt{1+2s^2}.\end{aligned}$$

Frenet elements and curvatures of $T-$ *Pedal curve* α_t *are computed as follows:*

$$\begin{aligned}T_1&=\frac{1}{\sqrt{1+s^2}}(T-sN),\\B_1&=\frac{1}{\sqrt{1+2s^2}}(sT-N+sB),\\N_1&=\frac{1}{\sqrt{1+s^2}\sqrt{1+2s^2}}(sT+sN+(1-s^2)B),\\\kappa_1&=\frac{2s^2\sqrt{1+2s^2}}{(1+s^2)\sqrt{1+s^2}},\\\tau_1&=-\frac{4}{s(1+2s^2)}.\end{aligned}$$

The general equation of the surface M that belonged $T-$ *pedal curve is given by following equality:*

$$M(s,t)=\alpha_T(s)+f(s,t)T_1(s)+g(s,t)N_1(s)+h(s,t)B_1(s) \tag{25}$$

If marching-scale functions are selected as $f(s,t)=s, g(s,t)=t, h(s,t)=st$ *and the equalities (9) are substituted into (10) , then the equation of surface M is computed as follows:*

$$M(s,t)=\left(A_0^{(1)}(s,t),\,A_0^{(2)}(s,t),\,A_0^{(3)}(s,t)\right),$$

$$B_0(s) = \sqrt{\frac{\begin{array}{c}448\sin(\sqrt{2}s)\cos(\sqrt{2}s)^4\sqrt{2} + 64\cos(\sqrt{2}s)^6 \\ -5576\sin(\sqrt{2}s)\cos(\sqrt{2}s)^2\sqrt{2} - 2928\cos(\sqrt{2}s)^4 \\ +10378\sin(\sqrt{2}s)\sqrt{2} + 15089\cos(\sqrt{2}s)^2 - 14725\end{array}}{\begin{array}{c}3840\sin(\sqrt{2}s)\cos(\sqrt{2}s)^5\sqrt{2} + 512\cos(\sqrt{2}s)^7 \\ -47680\sin(\sqrt{2}s)\cos(\sqrt{2}s)^3\sqrt{2} - 25536\cos(\sqrt{2}s)^5 \\ +81340\sin(\sqrt{2}s)\sqrt{2} + 124536\cos(\sqrt{2}s)^3 - 115137\end{array}}}. \tag{26}$$

$$A_0^{(1)}(s,t) = -\frac{1}{2}\cos(\sqrt{2}s) - \frac{1}{2}s + \frac{s\left(\sin(\sqrt{2}s)\sqrt{2} - 1\right)}{\sqrt{5 - 2\sin(\sqrt{2}s)\sqrt{2}}} + \frac{t\left(5\sin(\sqrt{2}s)\sqrt{2} + 4\cos(\sqrt{2}s)^2 - 4\right)}{B_0(s)}. \tag{27}$$

$$A_0^{(1)}(s,t) = -\frac{1}{2}\cos(\sqrt{2}s) - \frac{1}{2}s + \frac{s\left(\sin(\sqrt{2}s)\sqrt{2} - 1\right)}{\sqrt{5 - 2\sin(\sqrt{2}s)\sqrt{2}}} + \frac{t\left(5\sin(\sqrt{2}s)\sqrt{2} + 4\cos(\sqrt{2}s)^2 - 4\right)}{B_0(s)}. \tag{28}$$

$$A_0^{(2)}(s,t) = -\frac{1}{2}\sin(\sqrt{2}s) + \frac{s\cos(\sqrt{2}s)\sqrt{2}}{\sqrt{5 - 2\sin(\sqrt{2}s)\sqrt{2}}} - \frac{t\cos(\sqrt{2}s)\sqrt{2}}{B_0(s)}. \tag{29}$$

$$A_0^{(3)}(s,t) = \frac{1}{2}\sqrt{2}s + \frac{s\sqrt{2}}{\sqrt{5 - 2\sin(\sqrt{2}s)\sqrt{2}}} + \frac{t\sqrt{2}\cos(\sqrt{2}s)}{B_0(s)}. \tag{30}$$

where $A_0^{(i)}(s,t)$ $(i = 1,2,3)$ *denote the coordinate functions of the surface,and* $B_0(s)$ *represents the common normalization term arising from the Frenetframe of the T-pedal curve.Gauss curvature belonged to the surface M of* $T - Pedal$ *curve is calculated as follows:*

$$K_M = -\frac{\begin{array}{c}\big(2\cos(\sqrt{2}s)^2\sqrt{5}\sqrt{2}s^2 + 2\sin(\sqrt{2}s)^2\sqrt{5}\sqrt{2}s^2 \\ +7\cos(\sqrt{2}s)^2\sqrt{5}\sqrt{2} + 7\sin(\sqrt{2}s)^2\sqrt{5}\sqrt{2} \\ +16\cos(\sqrt{2}s)^2 + 16\sin(\sqrt{2}s)^2 + 4\big)^2\end{array}}{\left(2\sqrt{5}\sqrt{2}s^2 + 6\sqrt{5}\sqrt{2} + 11s^2 + 25\right)^2 s^2}$$

where x_2,y_2,z_2 *are follows:*

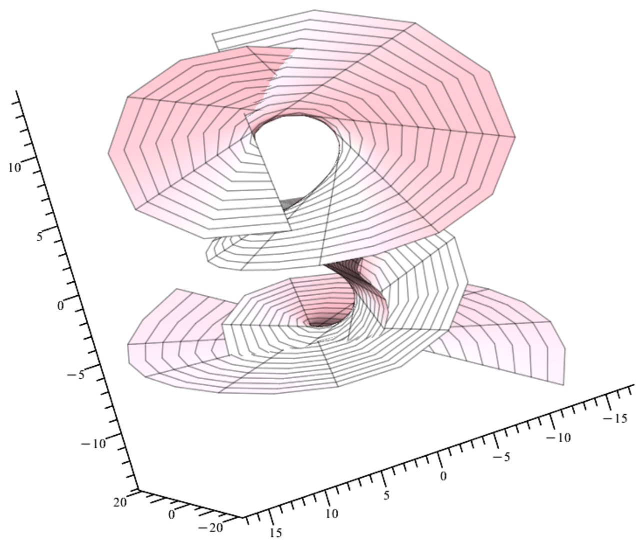

Figure 1: The surface M of T-pedal curve of the curve on α the intervals $-2\pi \leq s \leq 2\pi$ and $-3 \leq t \leq 3$.

$$x_2 = (x_1)_s - \mu\kappa_1 y_1 \left(1 + \left(\frac{\sqrt{1+s^2}}{2} - \frac{st\sqrt{1+2s^2}}{1+s^2}\right)\right)_s - \frac{s^2\sqrt{1+2s^2}}{1+s^2}\left(\frac{s^3\sqrt{1+2s^2}}{1+s^2} + \frac{2t\sqrt{1+s^2}}{1+2s^2}\right),$$

$$\begin{aligned} y_2 &= \mu\, x_1\kappa_1\,(y_1)_s - \mu\, z_1\tau_1 \\ &= \frac{1}{2}\sqrt{1+s^2}\left(\frac{\sqrt{1+s^2}}{2} - \frac{st\sqrt{1+2s^2}}{1+s^2}\right)\frac{2s^2\sqrt{1+2s^2}}{(1+s^2)\sqrt{1+s^2}}\left(\frac{s^3\sqrt{1+2s^2}}{1+s^2} + \frac{2t\sqrt{1+s^2}}{1+2s^2}\right)_s \\ &\quad + \frac{2\sqrt{1+s^2}}{s(1+2s^2)}\left(\frac{2t\sqrt{1+s^2}}{s(1+2s^2)} + t\right), \end{aligned}$$

$$z_2 = \mu\, y_1\tau_1 + (z_1)_s - \frac{2\sqrt{1+s^2}}{s(1+2s^2)}\left(\frac{s^3\sqrt{1+2s^2}}{1+s^2} + \frac{2t\sqrt{1+s^2}}{1+2s^2}\right) + \left(\frac{2t\sqrt{1+s^2}}{s(1+2s^2)}\right)_s .$$

Mean curvature belonged to the surface of T-pedal curve is given by following equality:

$$H_M = \frac{\begin{array}{l} 2\sqrt{s^4+25s^2+160}\,A\cos(s)\,s^{15}t + 2\sqrt{s^4+25s^2+160}\,B\sin(s)\,s^{15}t \\ + 4\sqrt{s^4+25s^2+160}\,A\sin(s)\,s^{14}t - 4\sqrt{s^4+25s^2+160}\,B\cos(s)\,s^{14}t \\ + 188\sqrt{s^4+25s^2+160}\,A\cos(s)\,s^{13}t + 188\sqrt{s^4+25s^2+160}\,B\sin(s)\,s^{13}t \\ + 139200000\sqrt{s^4+25s^2+160}\,B\sin(s)\,s\,t + 7580\sqrt{s^4+25s^2+160}\,A\cos(s)\,s^{11}t \\ + 168920\sqrt{s^4+25s^2+160}\,A\cos(s)\,s^{9}t + 15320\sqrt{s^4+25s^2+160}\,A\sin(s)\,s^{10}t \\ \vdots \end{array}}{\begin{array}{l} -4s^{16}t^2 + 582400\sqrt{s^4+25s^2+160}\sqrt{s^2+10}\,s^9t \\ + 7344800\sqrt{s^4+25s^2+160}\sqrt{s^2+10}\,s^7t + 55312000\sqrt{s^4+25s^2+160}\sqrt{s^2+10}\,s^5t \\ - 27935s^{16} - 820045s^{14} - 15616800s^{12} - 200814200s^{10} \\ - 256000000t^2 - 1755628000s^8 - 19620\sqrt{s^2+10}\,s^{14} \end{array}}$$

Example 3.8. *By recalling Example 3.7., different selections of marching-scale functions of surface M obtained from T-Pedal curve are calculated and illustrated as follows:*

a) If marching scale functions are selected as $f(s,t) = s,\ g(s,t) = 0,\ h(s,t) = t$

$$M(s,t) = A_1(s) + B_1(s)\,\frac{C_1(s)}{D_1(s)},$$

where

$$A_1(s) = -2\sin(\sqrt{2}s)\,2^{3/4},$$

$$B_1(s) = \frac{\begin{array}{c} \sqrt{2}\cos(\sqrt{2}s)\sqrt{2} - 7\sqrt{2} \\ + 9\sin(\sqrt{2}s) \end{array}}{\begin{array}{l} 448\sin(\sqrt{2}s)\cos(\sqrt{2}s)^4\sqrt{2} + 64\cos(\sqrt{2}s)^6 \\ - 5576\sin(\sqrt{2}s)\cos(\sqrt{2}s)^2\sqrt{2} - 2928\cos(\sqrt{2}s)^4 \\ + 10378\sin(\sqrt{2}s)\sqrt{2} + 15089\cos(\sqrt{2}s)^2 - 14725 \end{array}},$$

$$C_1(s) = 20\sin(\sqrt{2}s)\sqrt{2} - 33 + 8\cos(\sqrt{2}s)^2,$$

$$\begin{array}{rl} D_1(s) = & 3840\sin(\sqrt{2}s)\cos(\sqrt{2}s)^5\sqrt{2} + 512\cos(\sqrt{2}s)^7 \\ & - 47680\sin(\sqrt{2}s)\cos(\sqrt{2}s)^3\sqrt{2} - 25536\cos(\sqrt{2}s)^5 \\ & + 81340\sin(\sqrt{2}s)\sqrt{2} + 124536\cos(\sqrt{2}s)^3 - 115137. \end{array}$$

By taking the derivatives the expressions (12) and (13), following equalities are obtained:

$$f_s = 1, f_t = 1, g_s = 0, g_t = 0, h_s = 0, h_t = 1.$$

$$x_1 = \frac{\sqrt{1+s^2}}{2} + 1, y_1 = \frac{s^2\sqrt{1+2s^2}}{1+s^2}, z_1 = 0, \mu = \frac{1}{2}\sqrt{1+s^2}$$

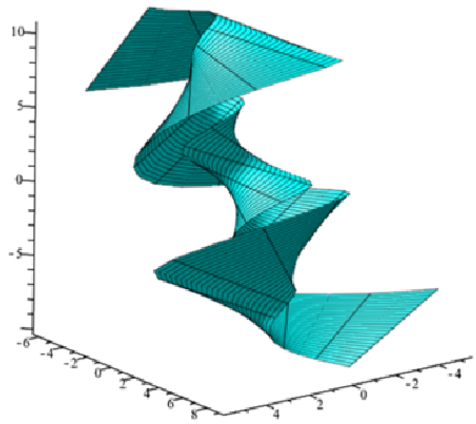

Figure 2: The surface M of T-pedal curve of the curve on α the intervals $-2\pi \leq s \leq 2\pi$ and $-3 \leq t \leq 3$.

$$x_2 = \frac{s}{2\sqrt{1+s^2}} - \frac{s^4\left(1+2s^2\right)}{(1+s^2)^2}, y_2 = \frac{s^4\left(1+2s^2\right)}{2(1+s^2)^2}\left(2+\sqrt{1+s^2}\right), z_2 = \frac{-2s}{\sqrt{1+s^2}\left(\sqrt{1+2s^2}\right)}.$$

If these expressions are substituted into relations (10) and (11) and some necessary algebraic operations are done, then Gauss and Mean curvatures of surface is computed as follows, respectively.

$K_M = 0$ *and* $H_M = 0$.

b) If marching-scale functions are selected as $f(s,t) = s$, $g(s,t) = 0$, $h(s,t) = st$ *then the parametric equation of surface is calculated like as:*

$$M(s,t) = \frac{A_2(s,t)}{B_2(s)},$$

where

$$A_2(s,t) = -\frac{1}{2}\cos(\sqrt{2}s) - \frac{1}{2}s + \frac{s\left(\sin(\sqrt{2}s)\sqrt{2}-1\right) + s\,t\left(5\sin(\sqrt{2}s)\sqrt{2} + 4\cos(\sqrt{2}s)^2 - 4\right)}{\sqrt{5-2\sin(\sqrt{2}s)\sqrt{2}}},$$

$$B_2(s) = \frac{\begin{array}{l}448\sin(\sqrt{2}s)\cos(\sqrt{2}s)^4\sqrt{2} + 64\cos(\sqrt{2}s)^6 \\ \quad - 5576\sin(\sqrt{2}s)\cos(\sqrt{2}s)^2\sqrt{2} - 2928\cos(\sqrt{2}s)^4 \\ \quad + 10378\sin(\sqrt{2}s)\sqrt{2} + 15089\cos(\sqrt{2}s)^2 - 14725\end{array}}{\begin{array}{l}3840\sin(\sqrt{2}s)\cos(\sqrt{2}s)^5\sqrt{2} + 512\cos(\sqrt{2}s)^7 \\ \quad - 47680\sin(\sqrt{2}s)\cos(\sqrt{2}s)^3\sqrt{2} - 25536\cos(\sqrt{2}s)^5 \\ \quad + 81340\sin(\sqrt{2}s)\sqrt{2} + 124536\cos(\sqrt{2}s)^3 - 115137\end{array}} \cdot \left(20\sin(\sqrt{2}s)\sqrt{2} - 33 + 8\cos(\sqrt{2}s)^2\right).$$

By taking the derivatives the expressions (12) and (13), following equalities are obtained:

$$f_s = 1, f_t = 0, g_s = 0, g_t = 0, h_s = t, h_t = s.$$

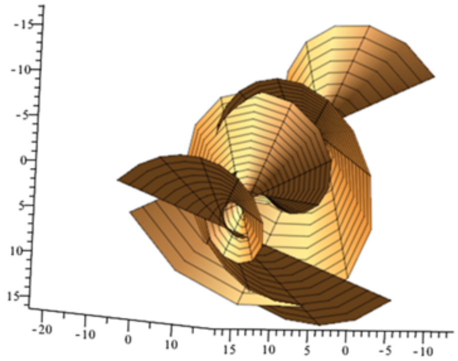

Figure 3: The surface M of T-pedal curve of the curve on α the intervals $-2\pi \leq s \leq 2\pi$ and $-3 \leq t \leq 3$

$$x_1 = \frac{\sqrt{1+s^2}}{2} + 1,$$

$$y_1 = \frac{s^3\sqrt{1+2s^2}}{1+s^2},$$

$$z_1 = t, \mu = \frac{1}{2}\sqrt{1+s^2}.$$

$$(x_1)_s = \frac{s}{2\sqrt{1+s^2}},$$

$$(y_1)_s = \left(\frac{s^3\sqrt{1+2s^2}}{1+s^2}\right)_s',$$

$$(x_1)_t = (y_1)_t = (z_1)_s = 0, (z_1)_t = 1.$$

$$x_2 = \frac{s}{2\sqrt{1+s^2}} - \frac{1}{2}\sqrt{1+s^2}\frac{2s^2\sqrt{1+2s^2}}{(1+s^2)\sqrt{1+s^2}}\frac{s^3\sqrt{1+2s^2}}{1+s^2},$$

$$y_2 = \frac{s^5\left(1+2s^2\right)}{2(1+s^2)^2}\left(2+\sqrt{1+s^2}\right),$$

$$z_2 = \left(\frac{-2s^2}{\sqrt{1+s^2}\sqrt{1+2s^2}}\right) + 1.$$

If these expressions are substituted into relations (10) and (11) and some necessary algebraic operations are done, then Gauss and Mean curvatures of surface is computed as follows, respectively.

$K_M = 0$ *and* $H_M = \frac{s\left(x_2 y_1 - x_1 y_2\right)}{\left(x_1^2+y_1^2+t^2\right)-st^2 y_1}$.

c) If marching-scale functions are selected as $f(s,t) = s$, $g(s,t) = t$, $h(s,t) = t^2$ *, then the parametric equation of surface is calculated like as:*

$$M(s,t) = \left(A_3(s,t), B_3(s,t), C_3(s,t)\right),$$

where

$$D(s)=\frac{\begin{array}{l}448\sin(\sqrt{2}s)\cos(\sqrt{2}s)^4\sqrt{2}+64\cos(\sqrt{2}s)^6\\ \quad -5576\sin(\sqrt{2}s)\cos(\sqrt{2}s)^2\sqrt{2}-2928\cos(\sqrt{2}s)^4\\ \quad +10378\sin(\sqrt{2}s)\sqrt{2}+15089\cos(\sqrt{2}s)^2-14725\end{array}}{\begin{array}{l}3840\sin(\sqrt{2}s)\cos(\sqrt{2}s)^5\sqrt{2}+512\cos(\sqrt{2}s)^7\\ \quad -47680\sin(\sqrt{2}s)\cos(\sqrt{2}s)^3\sqrt{2}-25536\cos(\sqrt{2}s)^5\\ \quad +81340\sin(\sqrt{2}s)\sqrt{2}+124536\cos(\sqrt{2}s)^3-115137\end{array}}.$$

and

$$A_3(s,t)=\frac{D(s)\Big(-2\sin(\sqrt{2}s)\sqrt{2}\Big)^{3/2}-t^2\cos(\sqrt{2}s)\sqrt{2}}{(2\sin(\sqrt{2}s)\sqrt{2}-5)\sqrt{D(s)}},$$

$$B_3(s,t)=\frac{\frac{1}{2}\sqrt{2}s+\frac{s\sqrt{2}}{\sqrt{5-2\sin(\sqrt{2}s)\sqrt{2}}}+s\,t\sqrt{2}\cos(\sqrt{2}s)}{D(s)},$$

$$C_3(s,t)=\frac{D(s)\Big(-2\sin(\sqrt{2}s)\sqrt{2}\Big)^{3/2}+t^2\Big(2\cos(\sqrt{2}s)^2\sqrt{2}-7\sqrt{2}+9\sin(\sqrt{2}s)\Big)}{D(s)\Big(20\sin(\sqrt{2}s)\sqrt{2}-33+8\cos(\sqrt{2}s)^2\Big)}.$$

By taking the derivatives the expressions (12) and (13), following equalities are obtained:

$$f_s=1, f_t=0, g_s=0, g_t=1, h_s=0, h_t=2t.$$

$$x_1=\frac{\sqrt{1+s^2}}{2}+1-\frac{ts^2\sqrt{1+2s^2}}{(1+s^2)},$$

$$y_1=\frac{s^3\sqrt{1+2s^2}}{(1+s^2)}+\frac{2t^2\sqrt{1+s^2}}{s(1+2s^2)},$$

$$z_1=-\frac{2t\sqrt{1+s^2}}{s(1+2s^2)}, \mu=\frac{1}{2}\sqrt{1+s^2}.$$

$$(x_1)_s=\left(\frac{\sqrt{1+s^2}}{2}+1-\frac{ts^2\sqrt{1+2s^2}}{(1+s^2)}\right)'_s,$$

$$(y_1)_s=\left(\frac{s^3\sqrt{1+2s^2}}{(1+s^2)}+\frac{2t^2\sqrt{1+s^2}}{s(1+2s^2)}\right)'_s,$$

$$(z_1)_s=\left(-\frac{2t\sqrt{1+s^2}}{s(1+2s^2)}\right)'_s,$$

$$(x_1)_t=-\frac{s^2\sqrt{1+2s^2}}{(1+s^2)}.$$

$$(y_1)_t=\frac{4t\sqrt{1+s^2}}{s(1+2s^2)},$$

$$(z_1)_t=-\frac{2\sqrt{1+s^2}}{s(1+2s^2)},$$

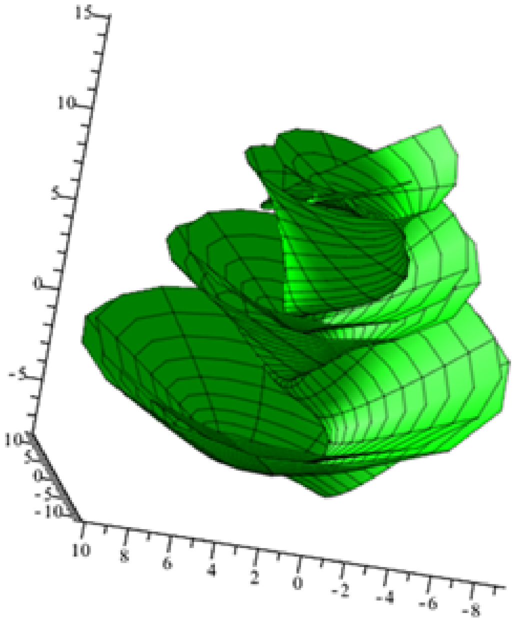

Figure 4: The surface M of T-pedal curve of the curve on α the intervals $-2\pi \leq s \leq 2\pi$ and $-3 \leq t \leq 3$.

$$x_2 = \left(\frac{\sqrt{1+s^2}}{2} + 1 - \frac{ts^2\sqrt{1+2s^2}}{(1+s^2)}\right)_s - \left(\frac{s^5\left(1+2s^2\right)}{(1+s^2)^2} + \frac{2st^2}{\sqrt{1+s^2}(\sqrt{1+2s^2})}\right),$$

$$y_2 = \left(\frac{\sqrt{1+s^2}}{2} + 1\right)\frac{s^2\sqrt{1+2s^2}}{(1+s^2)}\left(\frac{s^3\sqrt{1+2s^2}}{(1+s^2)} + \frac{2t^2\sqrt{1+s^2}}{s(1+2s^2)}\right)_s - \frac{4t\left(1+s^2\right)}{s^2(1+2s^2)^2}$$

$$z_2 = -\frac{2\sqrt{1+s^2}}{s(1+2s^2)}\left(\frac{s^3\sqrt{1+2s^2}}{(1+s^2)} + \frac{2t^2\sqrt{1+s^2}}{s(1+2s^2)}\right) + \left(-\frac{2t\sqrt{1+s^2}}{s(1+2s^2)}\right)_s$$

If these expressions are substituted into relations (10) and (11) and some necessary algebraic operations are done, then Gauss and Mean curvatures of surface is computed as follows, respectively

$$K_M = \frac{x_1(z_1)_t - z_1(x_1)_t + 2t\left(y_1(x_1)_t - x_1(y_1)_t\right)}{\left(x_1^2+y_1^2+t^2\right)(1+4t^2) - (y_1+2tz_1)^2},$$

$$H_M = \frac{\left(1+4t^2\right)x_2(2ty_1 - z_1) - 2tx_1y_2 + x_1z_2 - (y_1+2tz_1)\left[(x_1)_t(2ty_1 - z_1) - 2t(y_1)_t x_1 + (z_1)_t x_1\right]}{\left(x_1^2+y_1^2+t^2\right)(1+4t^2) - (y_1+2tz_1)\sqrt{(2ty_1 - z_1)^2 + x_1^2(1+4t^2)}}.$$

d) If marching-scale functions are selected as $f(s,t) = s$, $g(s,t) = st$, $h(s,t) = t$, then the parametric equation *of surface is calculated like as:*

$$D(s)=\frac{\begin{array}{l}448\sin(\sqrt{2}s)\cos(\sqrt{2}s)^4\sqrt{2}+64\cos(\sqrt{2}s)^6\\ \quad-5576\sin(\sqrt{2}s)\cos(\sqrt{2}s)^2\sqrt{2}-2928\cos(\sqrt{2}s)^4\\ \quad+10378\sin(\sqrt{2}s)\sqrt{2}+15089\cos(\sqrt{2}s)^2-14725\end{array}}{\begin{array}{l}3840\sin(\sqrt{2}s)\cos(\sqrt{2}s)^5\sqrt{2}+512\cos(\sqrt{2}s)^7\\ \quad-47680\sin(\sqrt{2}s)\cos(\sqrt{2}s)^3\sqrt{2}-25536\cos(\sqrt{2}s)^5\\ \quad+81340\sin(\sqrt{2}s)\sqrt{2}+124536\cos(\sqrt{2}s)^3-115137\end{array}}.$$

$$M(s,t)=\frac{A_4(s,t)}{\sqrt{B_4(s,t)}},$$

where

$$A_4(s,t)=-2\sin(\sqrt{2}s)\sqrt{2}^{3/2}+t^2\left(2\cos(\sqrt{2}s)^2\sqrt{2}-7\sqrt{2}+9\sin(\sqrt{2}s)\right),$$

$$B_4(s,t)=D(s)\left(20\sin(\sqrt{2}s)\sqrt{2}-33+8\cos(\sqrt{2}s)^2\right).$$

By taking the derivatives the expressions (12) and (13), following equalities are obtained:

$$f_s=1, f_t=0, g_s=t, g_t=s, h_s=0, h_t=1.$$

$$\begin{aligned} x_1 &= \frac{\sqrt{1+s^2}}{2}+1-\frac{s^3t\sqrt{1+2s^2}}{(1+s^2)},\\ y_1 &= \frac{s^3\sqrt{1+2s^2}}{(1+s^2)}-t+\frac{2t\sqrt{1+s^2}}{s(1+2s^2)},\\ z_1 &= -\frac{2t\sqrt{1+s^2}}{(1+2s^2)},\\ \mu &= \frac{1}{2}\sqrt{1+s^2}.\end{aligned}$$

$$\begin{aligned}(x_1)_s &= \left(\frac{\sqrt{1+s^2}}{2}+1-\frac{s^3t\sqrt{1+2s^2}}{(1+s^2)}\right)_s,\\ (y_1)_s &= \frac{s^3\sqrt{1+2s^2}}{(1+s^2)}-t+\frac{2t\sqrt{1+s^2}}{s(1+2s^2)}_s,\\ (z_1)_s &= \left(-\frac{2t\sqrt{1+s^2}}{(1+2s^2)}\right)_s.\end{aligned}$$

$$\begin{aligned}(x_1)_t &= -\frac{s^3\sqrt{1+2s^2}}{(1+s^2)},\\ (y_1)_t &= \frac{2\sqrt{1+s^2}}{s(1+2s^2)}-1,\\ (z_1)_t &= -\frac{2\sqrt{1+s^2}}{(1+2s^2)},\end{aligned}$$

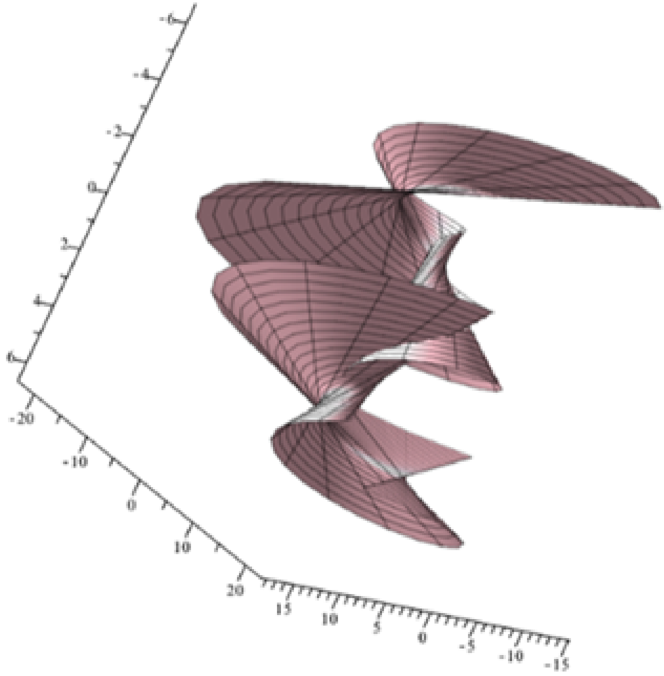

Figure 5: he surface M of T-pedal curve of the curve on α the intervals $-2\pi \leq s \leq 2\pi$ and $-3 \leq t \leq 3$

$$x_2 = \left(\frac{\sqrt{1+s^2}}{2} + 1 - \frac{s^3 t \sqrt{1+2s^2}}{(1+s^2)} \right)_s - \frac{s^2 \sqrt{1+2s^2}}{(1+s^2)} \left(\frac{s^3 \sqrt{1+2s^2}}{(1+s^2)} - t + \frac{2t \sqrt{1+s^2}}{s(1+2s^2)} \right),$$

$$y_2 = \left(\frac{\sqrt{1+s^2}}{2} + 1 - \frac{s^3 t \sqrt{1+2s^2}}{(1+s^2)} \right) \frac{s^2 \sqrt{1+2s^2}}{(1+s^2)} \left(\frac{s^3 \sqrt{1+2s^2}}{(1+s^2)} - t + \frac{2t \sqrt{1+s^2}}{s(1+2s^2)} \right)_s + \frac{4t\left(1+s^2\right)}{s(1+2s^2)^2},$$

$$z_2 = -\left(\frac{s^3 \sqrt{1+2s^2}}{(1+s^2)} - t + \frac{2t \sqrt{1+s^2}}{s(1+2s^2)} \right) \frac{2\sqrt{1+s^2}}{s(1+2s^2)} + \left(\frac{2t \sqrt{1+s^2}}{(1+2s^2)} \right)_s.$$

If these expressions are substituted into relations (10) and (11) and some necessary algebraic operations are done, then Gauss and Mean curvatures of surface is computed as follows, respectively.

$$K_M = \frac{s\left(x_1 (z_1)_t - z_1 (x_1)_t\right) + \left(y_1 (x_1)_t - x_1 (y_1)_t\right)}{\left(x_1^2 + y_1^2 + t^2\right)(1+s^2) - (sy_1 + z_1)^2},$$

$$H_M = \frac{\left(1+s^2\right) x_2 (y_1 - sz_1) - t x_1 y_2 + s x_1 z_2 - (sy_1 + z_1)\left[(x_1)_t (y_1 - sz_1) - (y_1)_t x_1 + s (z_1)_t x_1\right]}{\left(x_1^2 + y_1^2 + t^2\right)(1+s^2) - (sy_1 + z_1)\sqrt{(y_1 - sz_1)^2 + x_1^2 (1+s^2)}}.$$

Structural comparison. Although the marching-scale functions are deliberately varied in cases (a)–(d), this variation should not be interpreted as producing essentially different surface constructions. On the contrary, the four cases represent controlled modifications of the same underlying geometric mechanism. In each case, the surface $M(s,t)$ is generated by perturbing the T-pedal curve along its Frenet frame, while preserving the dominant dependence on the curve parameter s.

The purpose of considering different marching-scale functions is therefore not to alter the intrinsic nature of the surface, but to examine how distinct choices of tangential and normal contributions influence the resulting parametric structure and curvature behavior.

This observation allows all four surfaces to be interpreted within a unified framework, where the differences arise from the distribution of the marching-scale functions rather than from fundamentally different geometric constructions. Consequently, the comparable behavior observed in the Gauss and mean curvatures across cases (a)–(d) is not coincidental, but rather a direct consequence of this shared structural backbone.

From this perspective, the decomposition adopted in the examples serves a dual purpose: it clarifies the role of marching-scale functions in the surface generation process and highlights the robustness of the proposed construction under different functional selections, thereby providing a coherent basis for further analytical and geometric investigations.

4. Conclusion

In this study, first, the T-pedal curve, which is the geometric locus of the perpendicular projection points drawn from any point not on the curve onto the tangent vector of any curve, is defined. Subsequently, surfaces containing this T-pedal curve are created using Frenet vectors along the curve, with the help of marching-scale functions. Finally, the geometric properties, Gaussian and mean curvatures of these obtained surfaces are calculated.

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