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Integral Inequalities Involving $k-$Atangana-Baleanu Fractional Integral Operators

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Abstract. As an alternative to the classical Riemann–Liouville and Caputo definitions, newly developed fractional derivative and integral operators allow for a more accurate representation of complex processes such as memory and uncertainty. Introduced in 2017 by Atangana and Baleanu, Atangana–Baleanu fractional integral and derivative are distinguished by their Mittag–Leffler type kernels, which provide a non-singular structure and enable more realistic models for numerous physical and engineering problems. The advantages offered by Atangana–Baleanu fractional integral, when combined with inequality theory, yield significant contributions by offering new perspectives in theoretical mathematics and effective methods for solving applied problems. In 2023, Kermausuor and Nwaeze introduced the $k-$Atangana–Baleanu fractional integral, a generalized version of the Atangana–Baleanu fractional integral. Based on the newly introduced $k-$Atangana–Baleanu fractional integral operator, novel Bullen-type inequalities involving convex functions have been established.

1. Introduction

Many processes encountered in nature and engineering systems develop not only based on current states but also influenced by past events. Classical derivative and integral definitions often fall short in accounting for these past effects. At this point, fractional derivative and integral approaches, which carry traces of the past and contain system memory, come to the forefront.

Although the foundations of fractional calculus date back to the 17th century, concrete mathematical developments in this field gained momentum only toward the end of the 19th century. The subject attracted the attention of many mathematicians, who proposed various definitions of fractional derivatives and integrals. The Riemann–Liouville and Caputo definitions stand out as the most well-known forms of fractional derivatives [1]-[5]. Both definitions are generalized forms of integral operators and have been widely used in modeling many physical processes. However, these approaches have certain limitations in some applications, particularly regarding initial conditions and the properties of kernel functions. To overcome these limitations, new definitions have been proposed in recent years. One of these, Atangana–Baleanu fractional derivatives, particularly stands out due to the non-singularity of its kernel function [6], [1]. Proposed in 2017, this approach enables the construction of more stable and physically meaningful

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models with its kernel structure based on Mittag-Leffler function. Additionally, as it reflects memory effects more accurately, it yields remarkable results in many application areas. With the help of Atangana-Baleanu fractional integrals, numerous articles and thesis studies related to existing inequalities can be found in the literature [7]-[13]. The possibility of obtaining more general forms of these inequalities proven through demonstrations using $k-$Atangana-Baleanu fractional integrals has motivated us to work on this topic [14].

Definition 1.1. *(See [1], [6]) The left-sided and the right-sided Atangana–Baleanu fractional integral of a real-valued function $\mathcal{F}$ of order $\varphi \in (0,1)$ is given by:*

$$^{AB}\mathfrak{I}^{\varphi}_{u^+}\mathcal{F}(\kappa) = \frac{1-\varphi}{B(\varphi)}\mathcal{F}(\kappa) + \frac{\varphi}{B(\varphi)\Gamma(\varphi)}\int_u^{\kappa}(\kappa - \varkappa)^{\varphi-1}\mathcal{F}(x)\,d\varkappa \qquad \kappa > u,$$

and

$$^{AB}\mathfrak{I}^{\varphi}_{v^-}\mathcal{F}(\kappa) = \frac{1-\varphi}{B(\varphi)}\mathcal{F}(\kappa) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_{\kappa}^{v}(x - \kappa)^{\varphi-1}\mathcal{F}(x)\,d\varkappa \qquad \kappa < v$$

where $B(\varphi) > 0$ and satisfies the property $B(0) = B(1) = 1$.

In 2023, Kermausuor and Nwaeze developed $k-$parameterized generalized version, namely $k-$Atangana-Baleanu fractional integrals, to further expand the advantages offered by Atangana-Baleanu fractional integrals as follows.

Definition 1.2. *([14]) The left- and right-sided $k-$Atangana-Baleanu fractional integral operators of a real-valued function $\mathcal{F}$ of order $\lambda > 0$, are defined as:*

$$^{AB}_{k}\mathfrak{I}^{\varphi}_{u^+}\mathcal{F}(\kappa) = \frac{1-\varphi}{B(\varphi)}\mathcal{F}(\kappa) + \frac{\varphi}{kB(\varphi)\Gamma_k(\varphi)}\int_u^{\kappa}(\kappa - \varkappa)^{\frac{\varphi}{k}-1}\mathcal{F}(x)\,d\varkappa \qquad \kappa > u,$$

and

$$^{AB}_{k}\mathfrak{I}^{\varphi}_{v^-}\mathcal{F}(\kappa) = \frac{1-\varphi}{B(\varphi)}\mathcal{F}(\kappa) + \frac{\varphi}{kB(\varphi)\Gamma_k(\varphi)}\int_{\kappa}^{v}(x - \kappa)^{\frac{\varphi}{k}-1}\mathcal{F}(x)\,d\varkappa \qquad \kappa < v$$

where $k > 0$ and $B(\varphi) > 0$ satisfies the property $B(0) = B(1) = 1$. Γ_k is the $k-$gamma function presented by Diaz et al. [15] as:

$$\Gamma_k(\varphi) = \int_0^{\infty}\varsigma^{\varphi-1}e^{-\frac{\varsigma^k}{k}}\,d\varsigma, \qquad \mathcal{R}\rceil(\varphi) > 0.$$

Remark 1.3. *If $k = 1$ in Definition 1.2, Atangana-Baleanu fractional integral is derived.*

Convex functions, a foundation of mathematical analysis and optimization theory, have a broad impact in both theoretical and applied sciences. The convexity of a function is defined such that the line segment connecting any two points on its graph lies above the graph itself, and this property plays a critical role in numerous fields, ranging from optimization problems to machine learning, from modeling physical systems to economic theories [16]. In particular, convex functions offer significant advantages in inequality theory and optimization algorithms, such as the coincidence of local and global minimum, making them a powerful tool in mathematical analysis.

Definition 1.4. *([17]) The function $\mathcal{F} : I \subset R \to R$ is said to be convex if we have*

$$\mathcal{F}(u\varsigma + v(1-\varsigma)) \leq \varsigma\mathcal{F}(u) + (1-\varsigma)\mathcal{F}(v)$$

for all $u, v \in I$ and $\varsigma \in [0,1]$.

Inequalities based on convex functions hold a fundamental place in mathematical analysis, with Hermite-Hadamard inequalities being among the most notable results in this field. The Hermite-Hadamard inequality expresses the relationship between the integral average of a convex function and the arithmetic mean of its values at the endpoints, and it is regarded as a fundamental outcome of the concept of convexity [18]. This inequality is formulated in the literature as follows:

Let $\mathcal{F}:[u,v]\to R$, be a convex function. Then, we have the following double inequality:

$$\mathcal{F}\left(\frac{u+v}{2}\right)\leq\frac{1}{v-u}\int_u^v\mathcal{F}(x)\,dx\leq\frac{\mathcal{F}(u)+\mathcal{F}(v)}{2}.$$

On the other hand, the Bullen inequality emerges as a generalization of the Hermite-Hadamard inequality, offering a powerful tool for examining more complex averages of convex functions, particularly in cases involving symmetric structures [19]. By extending the fundamental structure of the Hermite-Hadamard inequality, the Bullen inequality addresses the relationships between the function's endpoints and midpoints within a more general framework, thereby establishing a natural connection between the two inequalities.

$$\frac{1}{v-u}\int_u^v\mathcal{F}(x)\,dx\leq\frac{1}{2}\left[\mathcal{F}\left(\frac{u+v}{2}\right)+\frac{\mathcal{F}(u)+\mathcal{F}(v)}{2}\right]$$

This relationship forms a significant foundation for both theoretical and applied studies in the analysis of convex functions. Furthermore, there are numerous studies in the literature related to this topic [20]-[29].

Lemma 1.5. *([30]) Suppose that $u,v\in I$, $u<v$, $0<\lambda\leq 1$, $0<\varsigma<1$, the function $\mathcal{F}:I\to[0,\infty)$ is positive and $\mathcal{F}^{\lambda}$ is convex on $[u,v]$*

i) If $0<\lambda\leq\frac{1}{2}$, then

$$\begin{aligned}&\mathcal{F}[\varsigma u+(1-\varsigma)v]\\ \leq\;&\lambda 2^{\frac{1}{\lambda}-1}\left[\varsigma^{\frac{1}{\lambda}}\mathcal{F}(u)+(1-\varsigma)^{\frac{1}{\lambda}}\mathcal{F}(v)+\left(\frac{2}{\lambda}-2\right)\varsigma^{\frac{1}{2\lambda}}(1-\varsigma)^{\frac{1}{2\lambda}}(\mathcal{F}(u)\mathcal{F}(v))^{\frac{1}{2}}\right].\end{aligned}$$

ii) If $\frac{1}{2}<\lambda<1$, then

$$\begin{aligned}&\mathcal{F}[\varsigma u+(1-\varsigma)v]\\ \leq\;&\left[\varsigma^{\frac{1}{\lambda}}\mathcal{F}(u)+(1-\varsigma)^{\frac{1}{\lambda}}\mathcal{F}(v)+\left(2^{\frac{1}{\lambda}}-2\right)\varsigma^{\frac{1}{2\lambda}}(1-\varsigma)^{\frac{1}{2\lambda}}(\mathcal{F}(u)\mathcal{F}(v))^{\frac{1}{2}}\right].\end{aligned}$$

The subsequent section focuses on establishing a new identity for Bullen-type inequalities involving k-Atangana-Baleanu fractional integrals. Then, by considering this identity and with the help of some fundamental integral inequalities such as Hölder inequality, power-mean inequality, new Bullen-type inequalities are presented.

2. Bounds for Bullen type inequalities

The lemma presented below plays a vital role in proving our remaining primary results.

Lemma 2.1. *Suppose that $\mathcal{F}:[u,v]\to R$ be a differentiable function on (u,v), $u<v$ and $\mathcal{F}'\in L_1[u,v]$, $\rho_1,\rho_2\in[0,1]$. Then, the following identity holds:*

$$\begin{aligned}&\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\\ &-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\\ =\;&\frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left[\int_0^1\left(\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right)\mathcal{F}'\left(\varsigma\frac{u+v}{2}+(1-\varsigma)v\right)d\varsigma\right.\\ &\left.+\int_0^1\left((1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right)\mathcal{F}'\left(\varsigma u+(1-\varsigma)\frac{u+v}{2}\right)d\varsigma\right],\end{aligned}\tag{1}$$

where $\varphi \in (0,1)$, $k>0$, Γ_k is $k-$gamma function and $B(\varphi)$ is normalization function.

Proof. By applying integration by parts to equations

$$\Upsilon_1 = \int_0^1 \left(\rho_1 - \varsigma^{\frac{\varphi}{\lambda}}\right)\mathcal{F}'\left(\varsigma\frac{u+v}{2} + (1-\varsigma)v\right)d\varsigma$$

and

$$\Upsilon_2 = \int_0^1 \left((1-\varsigma)^{\frac{\varphi}{\lambda}} - \rho_2\right)\mathcal{F}'\left(\varsigma u + (1-\varsigma)\frac{u+v}{2}\right)d\varsigma$$

and summing the resulting identities side by side, the following equation

$$\begin{aligned}
&\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u) + (2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right) + \rho_1\mathcal{F}(v)\right)\\
&-\left[{}^{AB}_{k;\frac{u+v}{2}}\Im^{\varphi}_{u}\mathcal{F}(u) + {}^{AB}_{k;v}\Im^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right] + \frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\\
=\ & \frac{2(1-\rho_1)}{(v-u)}\mathcal{F}\left(\frac{u+v}{2}\right) + \frac{2\rho_1}{(v-u)}\mathcal{F}(v)\\
&-\frac{2\varphi}{k(v-u)}\int_0^1 \varsigma^{\frac{\varphi}{\lambda}-1}\mathcal{F}\left(\varsigma\frac{u+v}{2} + (1-\varsigma)v\right)d\varsigma\\
&+\frac{2\rho_2}{(v-u)}\mathcal{F}(u) + \frac{2(1-\rho_2)}{(v-u)}\mathcal{F}\left(\frac{u+v}{2}\right)\\
&-\frac{2\varphi}{k(v-u)}\int_0^1 (1-\varsigma)^{\frac{\varphi}{\lambda}-1}\mathcal{F}\left(\varsigma u + (1-\varsigma)\frac{u+v}{2}\right)d\varsigma
\end{aligned}$$

is obtained. After applying the change of variable, the result is obtained by subtracting the value of $\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))$ and the proof is completed. □

Theorem 2.2. *Suppose that $\mathcal{F}:[u,v]\to R$ be a differentiable function on (u,v), $u<v$ and $\mathcal{F}'\in L_1[u,v]$, $\rho_1,\rho_2\in[0,1]$. If $\|\mathcal{F}'\|_\infty = \sup_{x\in(u,v)}|\mathcal{F}'| \le \infty$ then, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u) + (2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right) + \rho_1\mathcal{F}(v)\right)\right. \qquad (2)\\
&\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\Im^{\varphi}_{u}\mathcal{F}(u) + {}^{AB}_{k;v}\Im^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right] + \frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|\\
\le\ & \frac{(v-u)^{\frac{\varphi}{k}+2}\|\mathcal{F}'\|_\infty}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)},
\end{aligned}$$

where $\varphi \in (0,1)$, $k>0$, Γ_k is $k-$gamma function and $B(\varphi)$ is normalization function.

Proof. By Lemma 2.1, employing the properties of the modulus and the boundedness of $\mathcal{F}'$, we obtain

$$\begin{aligned}
&\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u) + (2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right) + \rho_1\mathcal{F}(v)\right)\right.\\
&\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\Im^{\varphi}_{u}\mathcal{F}(u) + {}^{AB}_{k;v}\Im^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right] + \frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|\\
\le\ & \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left[\int_0^1 \left|\left(\rho_1 - \varsigma^{\frac{\varphi}{\lambda}}\right)\right|\left|\mathcal{F}'\left(\varsigma\frac{u+v}{2} + (1-\varsigma)v\right)\right|d\varsigma\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \left|\left((1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right)\right|\left|\mathcal{F}'\left(\varsigma u+(1-\varsigma)\frac{u+v}{2}\right)\right|d\varsigma\Bigg]\\
\leq\ & \frac{(v-u)^{\frac{\varphi}{k}+1}\|\mathcal{F}'\|_\infty}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\int_0^1\left|\left(\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right)\right|d\varsigma+\int_0^1\left|\left((1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right)\right|d\varsigma\right)\\
=\ & \frac{(v-u)^{\frac{\varphi}{k}+2}\|\mathcal{F}'\|_\infty}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}.
\end{aligned}
$$

The proof is completed. □

Corollary 2.3. *Under the same assumptions of Theorem 2.2 with $k=1$, then we get*

$$
\begin{aligned}
&\Bigg|\frac{(v-u)^{\varphi}}{2^{\varphi}B(\varphi)\Gamma(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right) \qquad (3)\\
&-\left[{}^{AB}_{\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\Bigg|\\
\leq\ & \frac{(v-u)^{\varphi+2}\|\mathcal{F}'\|_\infty}{2^{\varphi+1}B(\varphi)\Gamma(\varphi)}.
\end{aligned}
$$

Theorem 2.4. *Suppose that $\mathcal{F}:[u,v]\to R$ be a differentiable function on (u,v), $u<v$ and $\mathcal{F}'\in L_1[u,v]$, $\rho_1,\rho_2\in[0,1]$. If $|\mathcal{F}'|$ is convex on $[u,v]$ then, the following inequality holds:*

$$
\begin{aligned}
&\Bigg|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right) \qquad (4)\\
&-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\Bigg|\\
\leq\ & \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\Bigg[|\mathcal{F}'(u)|\left(\frac{k^2}{(u+k)(v+2k)}-\frac{\rho_2}{2}\right)\\
&+|\mathcal{F}'(v)|\left(-\frac{k^2}{(u+k)(v+2k)}+\frac{\rho_1}{2}\right)\\
&+\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\left(\frac{\rho_1-\rho_2}{2}\right)\right|\Bigg]
\end{aligned}
$$

where $\varphi\in(0,1)$, $k>0$, Γ_k is $k-$gamma function and $B(\varphi)$ is normalization function.

Proof. From Lemma 2.1 and properties of modulus, we have

$$
\begin{aligned}
&\Bigg|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\\
&-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\Bigg|\\
\leq\ & \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\Bigg[\int_0^1\left|\left(\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right)\right|\left|\mathcal{F}'\left(\varsigma\frac{u+v}{2}+(1-\varsigma)v\right)\right|d\varsigma\\
&+\int_0^1\left|\left((1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right)\right|\left|\mathcal{F}'\left(\varsigma u+(1-\varsigma)\frac{u+v}{2}\right)\right|d\varsigma\Bigg]
\end{aligned}
$$

Using the convexity of $|\mathcal{F}'|$, we get

$$
\begin{aligned}
&\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right.\\
&\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|\\
\le\ &\frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left[\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|\left(\varsigma\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|+(1-\varsigma)|\mathcal{F}'(v)|\right)d\varsigma\right.\\
&\left.+\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|\left(\varsigma|\mathcal{F}'(u)|+(1-\varsigma)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\right)d\varsigma\right].
\end{aligned}
$$

This led us to the desired inequality. □

Corollary 2.5. *Under the same assumptions of Theorem 2.4 with $k=1$, then we get*

$$
\begin{aligned}
&\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \qquad (5)\\
&\left.-\left[{}^{AB}_{\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|\\
\le\ &\frac{(v-u)^{\varphi+1}}{2^{\varphi+1}B(\varphi)\Gamma_k(\varphi)}\left[|\mathcal{F}'(u)|\left(\frac{1}{(u+k)(v+2)}-\frac{\rho_2}{2}\right)\right.\\
&+|\mathcal{F}'(v)|\left(-\frac{1}{(u+k)(v+2)}+\frac{\rho_1}{2}\right)\\
&\left.+\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\left(\frac{\rho_1-\rho_2}{2}\right)\right|\right].
\end{aligned}
$$

Theorem 2.6. *Suppose that $\mathcal{F}:[u,v]\to R$ be a differentiable function on (u,v), $u<v$ and $\mathcal{F}'\in L_1[u,v]$, $\rho_1,\rho_2\in[0,1]$. If $|\mathcal{F}'|^{\lambda}$ is convex, then the following inequality holds:*

i) if $0<\lambda<\frac{1}{2}$

$$
\begin{aligned}
&\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \qquad (6)\\
&\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|\\
\le\ &\frac{(v-u)^{\frac{\varphi}{k}+1}\lambda 2^{\frac{1}{\lambda}-1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{k\lambda}{k\lambda+u\lambda+k}-\frac{\lambda\rho_1}{1+\lambda}\right)\right.\right.\\
&+|\mathcal{F}'(v)|\left(\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma\left(\frac{\varphi+k}{k}\right)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\frac{\varphi}{k}+\frac{1}{\lambda}\right)}\right)\\
&+\left(\frac{2}{\lambda}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right||\mathcal{F}'(v)|\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\times\left(\Gamma\left(1+\frac{1}{\lambda}\right)\left(\frac{\rho_1\Gamma\left(1+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\frac{1}{\lambda}\right)}-\frac{\Gamma\left(1+\frac{\varphi}{k}+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\frac{\varphi}{k}+\frac{1}{2\lambda}\right)}\right)\right)\Bigg)\Bigg)$$
$$+\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{k\lambda}{k\lambda+\varphi\lambda+k}-\frac{\lambda\rho_2}{1+\lambda}\right)\right.$$
$$+\left|\mathcal{F}'(u)\right|\left(-\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma\left(\frac{\varphi+k}{k}\right)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\frac{\varphi}{k}+\frac{1}{\lambda}\right)}\right)$$
$$+\left(\frac{2}{\lambda}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left|\mathcal{F}'(u)\right|\right)^{\frac{1}{2}}$$
$$\left.\times\frac{\left(2^{-\left(\frac{1+\lambda}{\lambda}\right)}\sqrt{\pi}\,\Gamma\left(1+\frac{1}{2\lambda}\right)\right)\left(-\rho_2+{}_2F_1\left(-\frac{\varphi}{k},1+\frac{1}{2\lambda},2+\frac{1}{\lambda},1\right)\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2\lambda}\right)}\right)\Bigg).$$

ii) *if* $\frac{1}{2}<\lambda<1$

$$\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \tag{7}$$
$$\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}\left(\mathcal{F}(u)+\mathcal{F}(v)\right)\right|$$
$$\leq\frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{k\lambda}{k\lambda+u\lambda+k}-\frac{\lambda\rho_1}{1+\lambda}\right)\right.\right.$$
$$+\left|\mathcal{F}'(v)\right|\left(\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma\left(\frac{\varphi+k}{k}\right)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\frac{\varphi}{k}+\frac{1}{\lambda}\right)}\right)$$
$$+\left(2^{\frac{1}{\lambda}}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left|\mathcal{F}'(v)\right|\right)^{\frac{1}{2}}$$

$$\times\left(\Gamma\left(1+\frac{1}{\lambda}\right)\left(\frac{\rho_1\Gamma\left(1+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\frac{1}{\lambda}\right)}-\frac{\Gamma\left(1+\frac{\varphi}{k}+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\frac{\varphi}{k}+\frac{1}{2\lambda}\right)}\right)\right)\Bigg)\Bigg)$$
$$+\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{k\lambda}{k\lambda+\varphi\lambda+k}-\frac{\lambda\rho_2}{1+\lambda}\right)\right.$$
$$+\left|\mathcal{F}'(u)\right|\left(-\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma\left(\frac{\varphi+k}{k}\right)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\frac{\varphi}{k}+\frac{1}{\lambda}\right)}\right)$$
$$+\left(2^{\frac{1}{\lambda}}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left|\mathcal{F}'(u)\right|\right)^{\frac{1}{2}}$$
$$\left.\times\frac{\left(2^{-\left(\frac{1+\lambda}{\lambda}\right)}\sqrt{\pi}\,\Gamma\left(1+\frac{1}{2\lambda}\right)\right)\left(-\rho_2+{}_2F_1\left(-\frac{\varphi}{k},1+\frac{1}{2\lambda},2+\frac{1}{\lambda},1\right)\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2\lambda}\right)}\right)\Bigg).$$

where $\varphi\in(0,1)$, $k>0$, Γ_k is $k-$gamma function and $B(\varphi)$ is normalization function.

Proof. Employing Lemma 1.5, Lemma 2.1 and the convexity of $|\mathcal{F}'|^{\lambda}$

i) if $0 < \lambda \leq \frac{1}{2}$

$$\begin{aligned}
&\left| \frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}} B(\varphi)\Gamma_k(\varphi)} \left(\rho_2 \mathcal{F}(u) + (2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right) + \rho_1 \mathcal{F}(v) \right) \right. \\
&\left. - \left[{}^{AB}_{k;\frac{u+v}{2}} \mathfrak{I}^{\varphi}_{u} \mathcal{F}(u) + {}^{AB}_{k;v} \mathfrak{I}^{\varphi}_{\frac{u+v}{2}} \mathcal{F}(v) \right] + \frac{(1-\varphi)}{B(\varphi)} \left(\mathcal{F}(u) + \mathcal{F}(v) \right) \right| \\
\leq\ & \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1} B(\varphi)\Gamma_k(\varphi)} \left[\int_0^1 \left| \rho_1 - \varsigma^{\frac{\varphi}{\lambda}} \right| \left| \mathcal{F}'\left(\varsigma \frac{u+v}{2} + (1-\varsigma)v \right) \right| d\varsigma \right. \\
&\left. + \int_0^1 \left| (1-\varsigma)^{\frac{\varphi}{\lambda}} - \rho_2 \right| \left| \mathcal{F}'\left(\varsigma u + (1-\varsigma)\frac{u+v}{2} \right) \right| d\varsigma \right] \\
\leq\ & \frac{(v-u)^{\frac{\varphi}{k}+1} \lambda 2^{\frac{1}{\lambda}-1}}{2^{\frac{\varphi}{k}+1} B(\varphi)\Gamma_k(\varphi)} \left(\int_0^1 \left| \rho_1 - \varsigma^{\frac{\varphi}{\lambda}} \right| \times \left(\varsigma^{\frac{1}{\lambda}} \mathcal{F}'\left(\frac{u+v}{2}\right) + (1-\varsigma)^{\frac{1}{\lambda}} \mathcal{F}'(v) \right. \right. \\
&\left. + \left(\frac{2}{\lambda} - 2 \right) \varsigma^{\frac{1}{2\lambda}} (1-\varsigma)^{\frac{1}{2\lambda}} \left(\mathcal{F}'\left(\frac{u+v}{2}\right) \mathcal{F}'(v) \right)^{\frac{1}{2}} \right) d\varsigma \\
&+ \int_0^1 \left| (1-\varsigma)^{\frac{\varphi}{\lambda}} - \rho_2 \right| \times \left(\varsigma^{\frac{1}{\lambda}} \mathcal{F}'(u) + (1-\varsigma)^{\frac{1}{\lambda}} \mathcal{F}'\left(\frac{u+v}{2}\right) \right. \\
&\left. \left. + \left(\frac{2}{\lambda} - 2 \right) \varsigma^{\frac{1}{2\lambda}} (1-\varsigma)^{\frac{1}{2\lambda}} \left(\mathcal{F}'\left(\frac{u+v}{2}\right) \mathcal{F}'(u) \right)^{\frac{1}{2}} \right) d\varsigma \right),
\end{aligned}$$

ii) if $\frac{1}{2} < \lambda < 1$

$$\begin{aligned}
&\left| \frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}} B(\varphi)\Gamma_k(\varphi)} \left(\rho_2 \mathcal{F}(u) + (2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right) + \rho_1 \mathcal{F}(v) \right) \right. \\
&\left. - \left[{}^{AB}_{k;\frac{u+v}{2}} \mathfrak{I}^{\varphi}_{u} \mathcal{F}(u) + {}^{AB}_{k;v} \mathfrak{I}^{\varphi}_{\frac{u+v}{2}} \mathcal{F}(v) \right] + \frac{(1-\varphi)}{B(\varphi)} \left(\mathcal{F}(u) + \mathcal{F}(v) \right) \right| \\
\leq\ & \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1} B(\varphi)\Gamma_k(\varphi)} \left[\int_0^1 \left| \rho_1 - \varsigma^{\frac{\varphi}{\lambda}} \right| \left| \mathcal{F}'\left(\varsigma \frac{u+v}{2} + (1-\varsigma)v \right) \right| d\varsigma \right. \\
&\left. + \int_0^1 \left| (1-\varsigma)^{\frac{\varphi}{\lambda}} - \rho_2 \right| \left| \mathcal{F}'\left(\varsigma u + (1-\varsigma)\frac{u+v}{2} \right) \right| d\varsigma \right] \\
\leq\ & \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1} B(\varphi)\Gamma_k(\varphi)} \left(\int_0^1 \left| \rho_1 - \varsigma^{\frac{\varphi}{\lambda}} \right| \times \left(\varsigma^{\frac{1}{\lambda}} \mathcal{F}'\left(\frac{u+v}{2}\right) + (1-\varsigma)^{\frac{1}{\lambda}} \mathcal{F}'(v) \right. \right. \\
&\left. + \left(2^{\frac{1}{\lambda}} - 2 \right) \varsigma^{\frac{1}{2\lambda}} (1-\varsigma)^{\frac{1}{2\lambda}} \left(\mathcal{F}'\left(\frac{u+v}{2}\right) \mathcal{F}'(v) \right)^{\frac{1}{2}} \right) d\varsigma \\
&+ \int_0^1 \left| (1-\varsigma)^{\frac{\varphi}{\lambda}} - \rho_2 \right| \times \left(\varsigma^{\frac{1}{\lambda}} \mathcal{F}'(u) + (1-\varsigma)^{\frac{1}{\lambda}} \mathcal{F}'\left(\frac{u+v}{2}\right) \right. \\
&\left. \left. + \left(2^{\frac{1}{\lambda}} - 2 \right) \varsigma^{\frac{1}{2\lambda}} (1-\varsigma)^{\frac{1}{2\lambda}} \left(\mathcal{F}'\left(\frac{u+v}{2}\right) \mathcal{F}'(u) \right)^{\frac{1}{2}} \right) d\varsigma \right)
\end{aligned}$$

The proof is completed upon evaluating the integrals in inequalities (i) and (ii). □

Corollary 2.7. *Under the same assumptions of Theorem 2.6 with $k = 1$, then we get*

i) $0 < \lambda \leq \frac{1}{2}$

$$\left|\frac{(v-u)^{\varphi}}{2^{\varphi}B(\varphi)\Gamma(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \tag{8}$$

$$\left.-\left[{}^{AB}_{\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|$$

$$\leq \frac{(v-u)^{\varphi+1}\lambda 2^{\frac{1}{\lambda}-1}}{2^{\varphi+1}B(\varphi)\Gamma(\varphi)}\left(\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{\lambda}{\lambda+u\lambda+1}-\frac{\lambda\rho_1}{1+\lambda}\right)\right.\right.$$

$$+|\mathcal{F}'(v)|\left(\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma(\varphi+1)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\varphi+\frac{1}{\lambda}\right)}\right)$$

$$+\left(\frac{2}{\lambda}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right||\mathcal{F}'(v)|\right)^{\frac{1}{2}}$$

$$\left.\times\left(\Gamma\left(1+\frac{1}{\lambda}\right)\left(\frac{\rho_1\Gamma\left(1+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\frac{1}{\lambda}\right)}-\frac{\Gamma\left(1+\varphi+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\varphi+\frac{1}{2\lambda}\right)}\right)\right)\right)$$

$$+\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{\lambda}{\lambda+\varphi\lambda+1}-\frac{\lambda\rho_2}{1+\lambda}\right)\right.$$

$$+|\mathcal{F}'(u)|\left(-\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma(\varphi+1)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\varphi+\frac{1}{\lambda}\right)}\right)$$

$$+\left(\frac{2}{\lambda}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right||\mathcal{F}'(u)|\right)^{\frac{1}{2}}$$

$$\left.\left.\times\frac{\left(2^{-\left(\frac{1+\lambda}{\lambda}\right)}\sqrt{\pi}\Gamma\left(1+\frac{1}{2\lambda}\right)\right)\left(-\rho_2+{}_2F_1\left(-\varphi,1+\frac{1}{2\lambda},2+\frac{1}{\lambda},1\right)\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2\lambda}\right)}\right)\right).$$

ii) if $\frac{1}{2} < \lambda < 1$

$$\left|\frac{(v-u)^{\varphi}}{2^{\varphi}B(\varphi)\Gamma(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \tag{8}$$

$$\left.-\left[{}^{AB}_{\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|$$

$$\leq \frac{(v-u)^{\varphi+1}}{2^{\varphi+1}B(\varphi)\Gamma(\varphi)}\left(\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{\lambda}{\lambda+u\lambda+1}-\frac{\lambda\rho_1}{1+\lambda}\right)\right.\right.$$

$$+|\mathcal{F}'(v)|\left(\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma(\varphi+1)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\varphi+\frac{1}{\lambda}\right)}\right)$$

$$+\left(2^{\frac{1}{\lambda}}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right||\mathcal{F}'(v)|\right)^{\frac{1}{2}}$$

$$\times\left(\Gamma\left(1+\frac{1}{\lambda}\right)\left(\frac{\rho_1\Gamma\left(1+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\frac{1}{\lambda}\right)}-\frac{\Gamma\left(1+\varphi+\frac{1}{2\lambda}\right)}{\Gamma\left(2+\varphi+\frac{1}{2\lambda}\right)}\right)\right)\Bigg)\Bigg)$$

$$+\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left(-\frac{\lambda}{\lambda+\varphi\lambda+1}-\frac{\lambda\rho_2}{1+\lambda}\right)\right.$$

$$+\left|\mathcal{F}'(u)\right|\left(-\frac{\lambda\rho_2}{1+\lambda}-\frac{\Gamma(\varphi+1)\Gamma\left(1+\frac{1}{\lambda}\right)}{\Gamma\left(2+\varphi+\frac{1}{\lambda}\right)}\right)$$

$$+\left(2^{\frac{1}{\lambda}}-2\right)\left(\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|\left|\mathcal{F}'(u)\right|\right)^{\frac{1}{2}}$$

$$\left.\times\frac{\left(2^{-\left(\frac{1+\lambda}{\lambda}\right)}\sqrt{\pi}\Gamma\left(1+\frac{1}{2\lambda}\right)\right)\left(-\rho_2+{}_2F_1\left(-\varphi,1+\frac{1}{2\lambda},2+\frac{1}{\lambda},1\right)\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2\lambda}\right)}\right)\Bigg).$$

Theorem 2.8. *Suppose that $\mathcal{F}:[u,v]\to R$ be a differentiable function on (u,v), $u<v$ and $\mathcal{F}'\in L_1[u,v]$, $\rho_1,\rho_2\in[0,1]$. If $|\mathcal{F}'|^{\lambda}$ is convex, then the following inequality holds:*

$$\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \tag{9}$$

$$\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|$$

$$\leq\frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left(\rho_1^{\ell}\times{}_2F_1\left(\frac{k}{\varphi},-\ell,\frac{\varphi+k}{\varphi},\frac{1}{\rho_1}\right)\right)^{\frac{1}{\ell}}\right.$$

$$\times\left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+|\mathcal{F}'(v)|^{\lambda}}{2}\right)^{\frac{1}{\lambda}}$$

$$\left.+\left(\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|^{\ell}d\varsigma\right)^{\frac{1}{\ell}}\left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+|\mathcal{F}'(u)|^{\lambda}}{2}\right)^{\frac{1}{\lambda}}\right),$$

where $\varphi\in(0,1)$, $k>0$, $\lambda>1$, $\frac{1}{\ell}+\frac{1}{\lambda}=1$, Γ_k is $k-$gamma function and $B(\varphi)$ is normalization function.

Proof. By the convexity of $|\mathcal{F}'|^{\lambda}$ and Hölder inequality, from Lemma 2.1 it follows that

$$\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right.$$

$$\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|$$

$$\leq\frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left[\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|\left|\mathcal{F}'\left(\varsigma\frac{u+v}{2}+(1-\varsigma)v\right)\right|d\varsigma\right.$$

$$\left.+\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|\left|\mathcal{F}'\left(\varsigma u+(1-\varsigma)\frac{u+v}{2}\right)\right|d\varsigma\right]$$

$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left(\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|^{\ell}d\varsigma\right)^{\frac{1}{\ell}}\right.$$
$$\times\left(\int_0^1\left(\varsigma\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+(1-\varsigma)\left|\mathcal{F}'(v)\right|^{\lambda}\right)d\varsigma\right)^{\frac{1}{\lambda}}$$
$$+\left(\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|^{\ell}d\varsigma\right)^{\frac{1}{\ell}}$$
$$\left.\times\left(\int_0^1\left(\varsigma\left|\mathcal{F}'(u)\right|^{\lambda}+(1-\varsigma)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}\right)d\varsigma\right)^{\frac{1}{\lambda}}\right).$$

By calculating the above integrals and simplifying, then the proof is completed. □

Corollary 2.9. *Let all the assumptions of Theorem 2.8 hold and $k=1$, then we get*

$$\left|\frac{(v-u)^{\varphi}}{2^{\varphi}B(\varphi)\Gamma(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \tag{10}$$
$$\left.-\left[{}^{AB}_{\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}\left(\mathcal{F}(u)+\mathcal{F}(v)\right)\right|$$

$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left(\rho_1^{\ell}\times{}_2F_1\left(\frac{k}{\varphi},-\ell,\frac{\varphi+k}{\varphi},\frac{1}{\rho_1}\right)\right)^{\frac{1}{\ell}}\right.$$
$$\times\left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+\left|\mathcal{F}'(v)\right|^{\lambda}}{2}\right)^{\frac{1}{\lambda}}$$

$$\left.+\left(\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|^{\ell}d\varsigma\right)^{\frac{1}{\ell}}\left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+\left|\mathcal{F}'(u)\right|^{\lambda}}{2}\right)^{\frac{1}{\lambda}}\right).$$

Theorem 2.10. *Suppose that $\mathcal{F}:[u,v]\to R$ be a differentiable function on (u,v), $u<v$ and $\mathcal{F}'\in L_1[u,v]$, $\rho_1,\rho_2\in[0,1]$. If $|\mathcal{F}'|^{\lambda}$ is convex, then the following inequality holds:*

$$\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \tag{11}$$
$$\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}\left(\mathcal{F}(u)+\mathcal{F}(v)\right)\right|$$

$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left|\rho_1-\frac{k}{\varphi+k}\right|^{1-\frac{1}{\lambda}}\right.$$
$$\times\left(\left(\frac{\rho_1}{2}-\frac{k}{\varphi+2k}\right)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+\left(\frac{\rho_1}{2}-\frac{k^2}{(\varphi+k)(\varphi+2k)}\right)\left|\mathcal{F}'(v)\right|^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$+\left|\frac{k}{\varphi+k}-\rho_2\right|^{1-\frac{1}{\lambda}}$$
$$\times\left(\left(\frac{k}{\varphi+2k}-\frac{\rho_2}{2}\right)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+\left(\frac{k^2}{(\varphi+k)(\varphi+2k)}-\frac{\rho_2}{2}\right)\left|\mathcal{F}'(u)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\Bigg),$$

where $\varphi \in (0,1)$, $k>0$, $\lambda \geq 1$, Γ_k *is* k*–gamma function and* $B(\varphi)$ *is normalization function.*

Proof. Taking into account Lemma 2.1 and power mean inequality, we can write

$$\Bigg|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)$$
$$-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}\left(\mathcal{F}(u)+\mathcal{F}(v)\right)\Bigg|$$

$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left[\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|\left|\mathcal{F}'\left(\varsigma\frac{u+v}{2}+(1-\varsigma)v\right)\right|d\varsigma\right.$$
$$\left.+\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|\left|\mathcal{F}'\left(\varsigma u+(1-\varsigma)\frac{u+v}{2}\right)\right|d\varsigma\right]$$

$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left(\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|d\varsigma\right)^{1-\frac{1}{\lambda}}\right.$$
$$\times\left(\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|\left(\varsigma\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+(1-\varsigma)\left|\mathcal{F}'(v)\right|^{\lambda}\right)d\varsigma\right)^{\frac{1}{\lambda}}$$

$$+\left(\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|d\varsigma\right)^{1-\frac{1}{\lambda}}$$
$$\left.\times\left(\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|\left(\varsigma\left|\mathcal{F}'(u)\right|^{\lambda}+(1-\varsigma)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}\right)d\varsigma\right)^{\frac{1}{\lambda}}\right).$$

By computing the above integrals, the statement is obtained. □

Corollary 2.11. *If we choose* $k=1$ *in Theorem 2.10, then we have*

$$\Bigg|\frac{(v-u)^{\varphi}}{2^{\varphi}B(\varphi)\Gamma(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right) \tag{12}$$
$$-\left[{}^{AB}_{\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}\left(\mathcal{F}(u)+\mathcal{F}(v)\right)\Bigg|$$

$$\leq \frac{(v-u)^{\varphi+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\left|\rho_1-\frac{1}{\varphi+1}\right|^{1-\frac{1}{\lambda}}\right.$$
$$\times\left(\left(\frac{\rho_1}{2}-\frac{1}{\varphi+2}\right)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+\left(\frac{\rho_1}{2}-\frac{1}{(\varphi+1)(\varphi+2)}\right)\left|\mathcal{F}'(v)\right|^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$+\left|\frac{1}{\varphi+1}-\rho_2\right|^{1-\frac{1}{\lambda}}$$
$$\times\left(\left(\frac{1}{\varphi+2}-\frac{\rho_2}{2}\right)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+\left(\frac{1}{(\varphi+1)(\varphi+2)}-\frac{\rho_2}{2}\right)\left|\mathcal{F}'(u)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\Bigg].$$

Theorem 2.12. *Suppose that $\mathcal{F}:[u,v]\to R$ be a differentiable function on (u,v), $u<v$ and $\mathcal{F}'\in L_1[u,v]$, $\rho_1,\rho_2\in[0,1]$. If $|\mathcal{F}'|^{\lambda}$ is convex, then the following inequality holds:*

$$\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right. \tag{13}$$
$$\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|$$
$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\frac{1}{\ell}\left(\rho_1^{\ell}\times{}_2F_1\left(\frac{k}{\varphi},-\ell,\frac{\varphi+k}{\varphi},\frac{1}{\rho_1}\right)\right)\right.$$
$$+\left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+|\mathcal{F}'(v)|^{\lambda}}{2\lambda}\right)$$
$$\left.+\frac{1}{\lambda}\left(\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|^{\lambda}d\varsigma\right)+\left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+|\mathcal{F}'(u)|^{\lambda}}{2\lambda}\right)\right),$$

where $\varphi\in(0,1)$, $k>0$, $\lambda>1$, $\frac{1}{\ell}+\frac{1}{\lambda}=1$, Γ_k is $k-$gamma function and $B(\varphi)$ is normalization function.

Proof. By applying Young inequality and Lemma 2.1

$$\left|\frac{(v-u)^{\frac{\varphi}{k}}}{2^{\frac{\varphi}{k}}B(\varphi)\Gamma_k(\varphi)}\left(\rho_2\mathcal{F}(u)+(2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right)+\rho_1\mathcal{F}(v)\right)\right.$$
$$\left.-\left[{}^{AB}_{k;\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u}\mathcal{F}(u)+{}^{AB}_{k;v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}}\mathcal{F}(v)\right]+\frac{(1-\varphi)}{B(\varphi)}(\mathcal{F}(u)+\mathcal{F}(v))\right|$$
$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left[\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|\left|\mathcal{F}'\left(\varsigma\frac{u+v}{2}+(1-\varsigma)v\right)\right|d\varsigma\right.$$
$$\left.+\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|\left|\mathcal{F}'\left(\varsigma u+(1-\varsigma)\frac{u+v}{2}\right)\right|d\varsigma\right]$$
$$\leq \frac{(v-u)^{\frac{\varphi}{k}+1}}{2^{\frac{\varphi}{k}+1}B(\varphi)\Gamma_k(\varphi)}\left(\frac{1}{\ell}\int_0^1\left|\rho_1-\varsigma^{\frac{\varphi}{\lambda}}\right|^{\ell}d\varsigma\right.$$
$$+\frac{1}{\lambda}\int_0^1\left(\varsigma\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}+(1-\varsigma)|\mathcal{F}'(v)|^{\lambda}\right)d\varsigma$$
$$+\frac{1}{\ell}\int_0^1\left|(1-\varsigma)^{\frac{\varphi}{\lambda}}-\rho_2\right|^{\ell}d\varsigma$$
$$\left.+\frac{1}{\lambda}\int_0^1\left(\varsigma|\mathcal{F}'(u)|^{\lambda}+(1-\varsigma)\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda}\right)d\varsigma\right).$$

By a simple computation, we get the desired result. □

Corollary 2.13. *Let all the assumptions of Theorem 2.12 hold and $k = 1$, then we get*

$$\left| \frac{(v-u)^{\varphi}}{2^{\varphi} B(\varphi)\Gamma(\varphi)} \left(\rho_2 \mathcal{F}(u) + (2-\rho_1-\rho_2)\mathcal{F}\left(\frac{u+v}{2}\right) + \rho_1 \mathcal{F}(v) \right) - \left[{}^{AB}_{\frac{u+v}{2}}\mathfrak{I}^{\varphi}_{u} \mathcal{F}(u) + {}^{AB}_{v}\mathfrak{I}^{\varphi}_{\frac{u+v}{2}} \mathcal{F}(v) \right] + \frac{(1-\varphi)}{B(\varphi)} (\mathcal{F}(u) + \mathcal{F}(v)) \right| \tag{14}$$

$$\leq \frac{(v-u)^{\varphi+1}}{2^{\varphi+1} B(\varphi)\Gamma(\varphi)} \left(\frac{1}{\ell} \left(\rho_1^{\ell} \times {}_2F_1\left(\frac{1}{\varphi}, -\ell, \frac{\varphi+1}{\varphi}, \frac{1}{\rho_1} \right) \right) + \left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda} + |\mathcal{F}'(v)|^{\lambda}}{2\lambda} \right) + \frac{1}{\lambda} \left(\int_0^1 \left| (1-\varsigma)^{\frac{\varphi}{\lambda}} - \rho_2 \right|^{\lambda} d\varsigma \right) + \left(\frac{\left|\mathcal{F}'\left(\frac{u+v}{2}\right)\right|^{\lambda} + |\mathcal{F}'(u)|^{\lambda}}{2\lambda} \right) \right).$$

Example 2.14. *The following Figure 1 describe the validity of the inequality of Theorem 2.4 for $\mathcal{F}(x) = x^3$, $u = 0$, $v = 1$, $\varphi = 0.3$, $k = 1/3$, $\rho_1, \rho_2 \in [0,1]$. Additionally, Figure 2 illustrates the validity of the inequality in Corollary 2.5 for the case $k = 1$.*

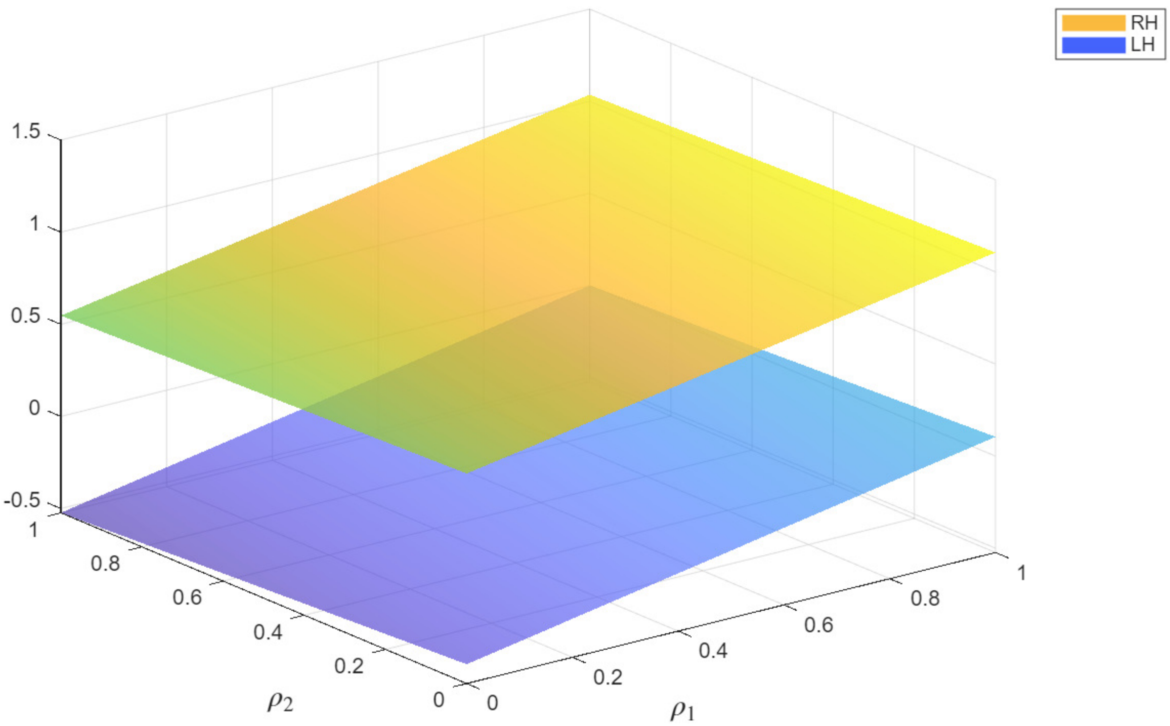

Figure 1:

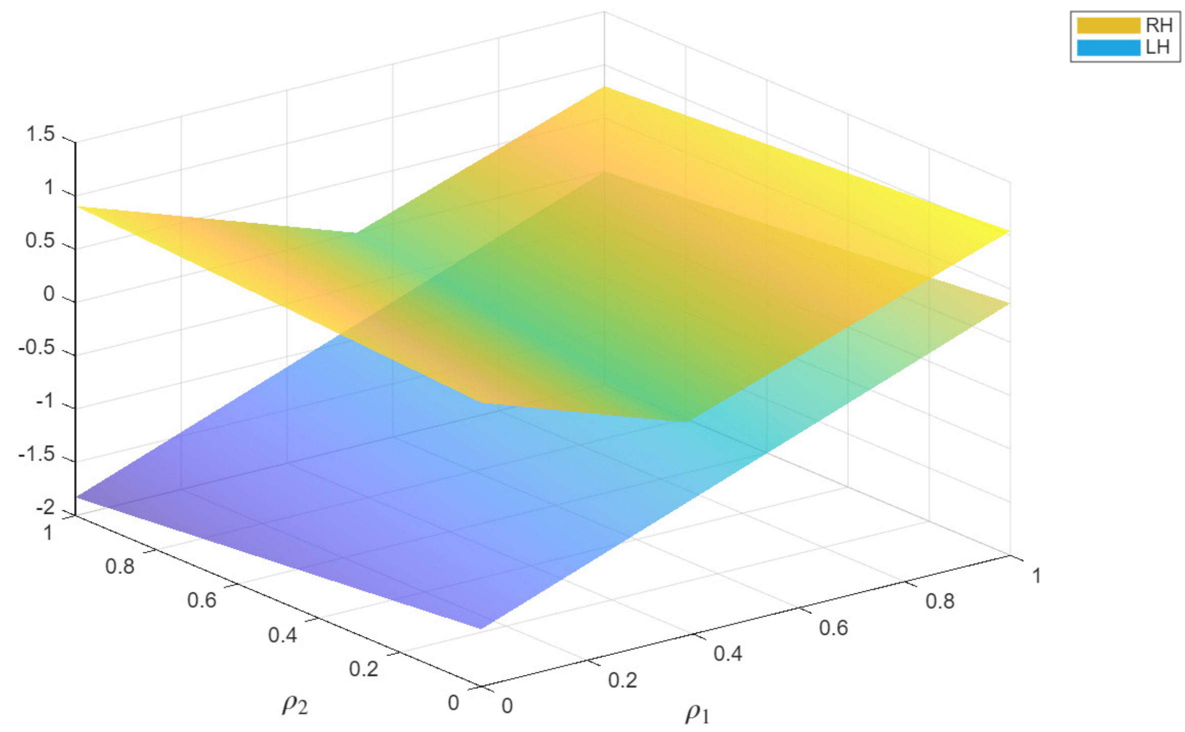

Figure 2:

3. Conclusion

In this study, new upper bounds have been derived for Bullen-type inequalities involving convex functions, based on the newly introduced k-Atangana–Baleanu fractional integral operator. This novel operator represents a generalized version of the classical Atangana–Baleanu fractional integral, enhancing the flexibility of the integral operator while enabling the examination of more general structures in fractional analysis. The obtained results constitute a significant step forward in expanding the inequality theory within the field of fractional calculus. In this context, researchers interested in the subject can generate new types of inequalities by building upon the identity we have defined and attain alternative bound values for these inequalities. Furthermore, the findings demonstrate that when the special case $k = 1$ is considered, the proposed operator reduces directly to the classical Atangana–Baleanu fractional integral operator. This clearly indicates that the suggested framework provides a more comprehensive structure encompassing the Atangana–Baleanu fractional integral operator. In this regard, the study is believed to offer novel methodological approaches that will contribute to inequality theory grounded in fractional analysis.

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