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The multiplicity of eigenvalues of a vectorial diffusion equations with discontinuous function inside a finite interval

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Abstract. In this study , m-dimensional vectorial diffusion equation with discontinuous function inside a finite interval is considered. Considering the asymptotic representation of the solution of the problem, we have obtained some conclusions about the multiplicity of eigenvalues. We have proved that, under certain conditions on potential matrix, the problem can only have a finite number of eigenvalues with multiplicity m.

sectionIntroduction Consider the m-dimensional vectorial singular diffusion equations

$$-y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 \delta(x)\, y, \;\; x \in (0, \pi) \tag{1}$$

$$y'(0) = \theta \tag{2}$$

$$y'(\pi) = \theta \tag{3}$$

where λ is the spectral parameter ,$y = (y_1, y_2, ...y_m)^T$ is an m-dimensional vector function,

$$\delta(x) = \begin{cases} 1 \;\;, & x \in (0, a_1) \\ \alpha^2, & x \in (a_1, a_2) \\ \beta^2, & x \in (a_2, \pi) \end{cases}$$

and $\alpha > 0$, $\alpha \neq 1$,$\beta > 0$, $\beta \neq 1$, $q(x) \in L_2[0, \pi]$, $p(x) \in W_2^1[0, \pi]$, $a_1, a_2 \in (0, \pi)$, $a_1 < a_2$. The potential matrix $(2\lambda p(x) + q(x))$ is an $m \times m$ real symmetric matrix function. θ denotes the m-dimensional zero vector. Many studies on the theory of second-order differential operators have been studied in [7, 18]. One of the most important of these was made in 1946 by Titchmarsh [20]. In 1984, the studies on the spectral theory of singular differential operators were conducted by Levitan [21]. Many physical phenomena, such as fluid flow and heat dissipation [23], atomic mixing modelling [24] include a diffusion process. Singular differential operators with conditions of discontinuity are often used in mathematical physics, in geophysics and natural sciences. In general, these problems are associated with discontinuous material properties. For example; It is used to in determining the parameters of the electricity line in electronics [22]. Also, it is used to determine geophysical models for the release of the earth [9]. The discontinuity here is the reflection of

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the shear waves at the base of the earth's crust. In 1999, C. L. Shen and C.T. Shies [5] studied the multiplicity of eigenvalues of the m-dimensional the vectorial Sturm-Liouville problem

$$-y'' + Q(x)\,y = \lambda y\ ,\ \ y(0) = y(1) = \theta$$

where Q is continuous $m \times m$ Jacobi matrix-valued function defined on $0 \le x \le 1$. Q. Kong [4] generalized to the case when $Q$is real symmetric. However, there are no such result for the discontinuous problem (1) – (3).
In this study, firstly we define the characteristic function of the eigenvalues of vectorial problem (1) – (3). Following this, we prove the conclusion that the eigenvalues of the problem coincide with the zeros of characteristic function. Then, we show the asymptotic forms of the solutions and obtain some results about multiplicity of the eigenvalues.

1. Characteristic function and asymptotics of solutions

Denote $H = L^2(I, C^m)$ the Hilbert space of vector-valued functioons with the scalar product

$$(f,g) = \int_0^{a_1} g_1^* f_1 dx + \int_{a_1}^{a_2} g_2^* f_2 dx + \int_{a_2}^{\pi} g_r^* f_r dx = \int_0^{\pi} g^* f dx$$

where $f = (f_1, f_2, ..., f_m)^T$, $g = (g_1, g_2, ...g_m)^T$ and $f_i, g_i \in L^2(I)$, $f_1(x) = f(x)\big|_{(0,a_1)}$, $f_2(x) = f(x)\big|_{(a_1,a_2)}$ and $f_r(x) = f(x)\big|_{(a_2,\pi)}$. We can define an operator L associated with the problem (1) – (3) on H

$$L : -y'' + [2\lambda p(x) + q(x)]\,y = \lambda^2 \delta(x)\,y\ ,\ \ y \in D(L)$$
$$D(L) = \{y \in H;\ y, y' \in AC[I, C^m]\},\ Ly \in L^2[I, C^m]$$
$$y'(0) = y'(\pi) = \theta.$$

Lemma 1.1. *The operator L is self-adjoint.*

The proof is similar to the scaler case in [12].
We consider the problem on the three intervals $(0, a_1)$, (a_1, a_2) and (a_2, π) respectively, where θ_m denotes $m \times m$ zero matrix and $E_m$denotes $m \times m$ identify matrix. On $(0, a_1)$, the matrix initial value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2 \cdot 1 \cdot Y\ ,\ \ x \in (0, a_1) \\ \phi_1(0,\lambda) = E_m\ , \phi_1'(0,\lambda) = \theta_m \end{cases} \tag{4}$$

has a unique solution $\phi_1(x,\lambda)$. What's more, for any fixed $x \in (0, a_1)$, $\phi_1(x,\lambda)$ is an entire matrix function in λ [1],p17. By variation of constants, we have

$$\phi_1(x,\lambda) = \cos\lambda x E_m + \frac{1}{\lambda}\int_0^x \sin\lambda(x-t)\,(2\lambda p(t) + q(t))\,\phi_1(t,\lambda)\,dt. \tag{5}$$

on (a_1, a_2) the matrix value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))\,Y = \lambda^2 \alpha^2 Y\ ,\ \ x \in (a_1, a_2) \\ \phi_2(a_1 + 0, \lambda) = \phi_1(a_1 - 0, \lambda) \\ \phi_2'(a_1 + 0, \lambda) = \phi_1'(a_1 - 0, \lambda) \end{cases} \tag{6}$$

has a unique solution $\phi_2(x,\lambda)$. In addition to, for any fixed $x \in (a_1, a_2)$, $\phi_2(x,\lambda)$ is an entire matrix function in λ. By variation of constants, we have

$$\begin{aligned} \varphi(x,\lambda) = {} & \alpha^+ e^{i\lambda\mu^+(x)} + \alpha^- e^{i\lambda\mu^-(x)} + \alpha^+ \int_0^{a_1} \frac{\sin\lambda\left(\mu^+(x)-t\right)}{\lambda} Q(t)\,y(t,\lambda)\,dt \\ & + \alpha^- \int_0^{a_1} \frac{\sin\lambda\left(\mu^-(x)-t\right)}{\lambda} Q(t)\,y(t,\lambda)\,dt + \int_{a_1}^{x} \frac{\sin\lambda\alpha(x-t)}{\lambda\alpha} Q(t)\,y(t,\lambda)\,dt \end{aligned} \tag{7}$$

where $\mu^{\pm}(x) = \pm\alpha x \mp \alpha a_1 + a_1$, $Q(t) = 2\lambda p(t) + q(t)$,
or

$$\begin{array}{l}\phi_2(x,\lambda) = \cos\lambda\alpha(x-a_1)\phi_1(a_1-0)E_m + \frac{1}{\lambda\alpha}\sin\lambda\alpha(x-a_1)\phi_1'(a_1-0)E_m \\ +\int_{a_1}^{x}\frac{\sin\lambda\alpha(x-t)}{\lambda\alpha}(2\lambda p(t)+q(t))\phi_2(t,\lambda)\,dt.\end{array} \tag{8}$$

on (a_2,π) the matrix value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))Y = \lambda^2\beta^2 Y\ ,\ x \in (a_2,\pi) \\ \phi_3(a_2+0,\lambda) = \phi_2(a_2-0,\lambda) \\ \phi_3'(a_2+0,\lambda) = \phi_2'(a_2-0,\lambda) \end{cases} \tag{9}$$

has a unique solution $\phi_3(x,\lambda)$. In addition to, for any fixed $x \in (a_2,\pi)$, $\phi_3(x,\lambda)$ is an entire matrix function in λ. By variation of constants, we have

$$\begin{array}{l}\phi_3(x,\lambda) = \alpha^+\beta^+e^{i\lambda k^+(x)} + \alpha^-\beta^-e^{i\lambda k^-(x)} + \alpha^+\beta^-e^{i\lambda s^+(x)} + \alpha^-\beta^+e^{i\lambda s^-(x)} \\ +\alpha^+\beta^+\int_0^{a_1}\frac{\sin\lambda(k^+(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt + \alpha^+\beta^-\int_0^{a_1}\frac{\sin\lambda(s^+(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt \\ +\alpha^-\beta^-\int_0^{a_1}\frac{\sin\lambda(k^-(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt + \alpha^-\beta^+\int_{a_1}^{a_2}\frac{\sin\lambda(s^-(x)-t)}{\lambda}Q(t)\,y(t,\lambda)\,dt \\ +\frac{\beta^+}{\alpha}\int_0^{a_1}\frac{\sin\lambda\left(\beta x-\beta a_2+\alpha a_2-\alpha t\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt \\ -\frac{\beta^-}{\alpha}\int_0^{a_1}\frac{\sin\lambda\left(\beta x-\beta a_2-\alpha a_2+\alpha t\right)}{\lambda}Q(t)\,y(t,\lambda)\,dt + \int_{a_2}^{x}\frac{\sin\lambda\beta(x-t)}{\lambda\beta}Q(t)\,y(t,\lambda)\,dt\end{array} \tag{10}$$

where $Q(t) = 2\lambda p(t) + q(t)$, $\mu^{\pm}(x) = \pm\alpha x \mp \alpha a_1 + a_1$, $\alpha^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\alpha}\right)$, $\beta^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\beta}\right)$, $k^{\pm}(x) = \beta x - \beta a_2 + \mu^{\pm}(a_2)$, $s^{\pm}(x) = -\beta x + \beta a_2 + \mu^{\pm}(a_2)$,
or

$$\begin{array}{l}\phi_3(x,\lambda) = \cos\lambda\beta(x-a_2)\phi_2(a_2-0,\lambda)E_m + \frac{1}{\lambda\beta}\sin\lambda\beta(x-a_2)\phi_2'(a_2-0,\lambda)E_m \\ +\int_{a_2}^{x}\frac{\sin\lambda\beta(x-t)}{\lambda\beta}(2\lambda p(t)+q(t))\phi_3(t,\lambda)\,dt\,.\end{array} \tag{11}$$

Let

$$\phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda)\ ,\ x \in (0,a_1) \\ \phi_2(x,\lambda)\ ,x \in (a_1,a_2) \\ \phi_3(x,\lambda)\ ,x \in (a_2,\pi) \end{cases}.$$

Then, any solution of the equations (1) satisfying boundary condition (2) can be expressed as

$$y(x,\lambda) = \phi(x,\lambda)\,c_1 = \begin{cases} \phi_1(x,\lambda)\,c_0, & x \in (0,a_1) \\ \phi_2(x,\lambda)\,c_0, & x \in (a_1,a_2) \\ \phi_3(x,\lambda)\,c_0, & x \in (a_2,\pi) \end{cases} \tag{12}$$

where c_1 is an arbitrary m-dimensional constant vector. If λ is an eigenvalue of the problem (1) − (3), then $c_0 \neq \theta$ and $y(x,\lambda)$ satisfies the boundary condition at $x = \pi$, that is,

$$y'(\pi,\lambda) = \phi'(\pi,\lambda)\,c_0 = \phi_3'(\pi,\lambda)\,c_0 = \theta.$$

Thus, we get

$$\det\left(\phi_3'(\pi,\lambda)\right) = 0.$$

Similarly, on (a_2,π), consider the matrix initial value problem

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))Y = \lambda^2\beta^2 Y\ ,\ x \in (a_2,\pi) \\ \psi_3(\pi,\lambda) = E_m\ ,\psi_3'(\pi,\lambda) = \theta_m \end{cases}. \tag{13}$$

The problem (13) has a unique solution $\psi_3(x,\lambda)$. Furthermore, for any fixed $x \in (a_2,\pi)$, $\psi_3(x,\lambda)$ is an entire matrix function in λ.

Consider the matrix initial value problem on (a_1, a_2),

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))Y = \lambda^2\alpha^2 Y, \ x \in (a_1, a_2) \\ \psi_3(a_2+0,\lambda) = \psi_2(a_2-0,\lambda) \\ \psi_3'(a_2+0,\lambda) = \psi_2'(a_2-0,\lambda) \end{cases} . \tag{14}$$

The problem (14) has a unique solution $\psi_2(x,\lambda)$. Furthermore, for any fixed $x \in (a_1, a_2)$, $\psi_2(x,\lambda)$ is an entire matrix function in λ.
Consider the matrix initial value problem on $(0, a_1)$,

$$\begin{cases} -Y'' + (2\lambda p(x) + q(x))Y = \lambda^2 \cdot 1 \cdot Y, \ x \in (0, a_1) \\ \psi_2(a_1+0,\lambda) = \psi_1(a_1-0,\lambda) \\ \psi_2'(a_1+0,\lambda) = \psi_1'(a_1-0,\lambda) \end{cases} \tag{15}$$

The problem (15) has a unique solution $\psi_1(x,\lambda)$. Furthermore, for any fixed $x \in (0, a_1)$, $\psi_1(x,\lambda)$ is an entire matrix function in λ. Let

$$\psi(x,\lambda) = \begin{cases} \psi_1(x,\lambda), \ x \in (0, a_1) \\ \psi_2(x,\lambda), x \in (a_1, a_2) \\ \psi_3(x,\lambda), x \in (a_2, \pi) \end{cases} .$$

Then, any solution of the equations (1) satisfying boundary condition (3) can be expressed as

$$y(x,\lambda) = \psi(x,\lambda)c_2 = \begin{cases} \psi_1(x,\lambda)c_1, \ x \in (0, a_1) \\ \psi_2(x,\lambda)c_1, x \in (a_1, a_2) \\ \psi_3(x,\lambda)c_1, x \in (a_2, \pi) \end{cases} \tag{16}$$

where c_2 is an arbitrary m-dimensional constant vector. If λ is an eigenvalue of the problem (1) – (3), then $c_1 \neq \theta$and $y(x,\lambda)$ satisfies the boundary condition at $x = 0$, that is,

$$y'(0,\lambda) = \psi'(0,\lambda)c_1 = \psi_1'(0,\lambda)c_1 = \theta$$

Thus, we get

$$\det\left(\psi_1'(0,\lambda)\right) = 0.$$

Let $\Delta_j(\lambda) = W\left(\phi_j(x,\lambda), \psi_j(x,\lambda)\right)$ be the Wronskian of solution matrices $\phi_j(x,\lambda)$and $\psi_j(x,\lambda)$, $j = 1, 2, 3$, that is,

$$\Delta_1(\lambda) = \begin{vmatrix} \phi_1(x,\lambda) & \psi_1(x,\lambda) \\ \phi_1'(x,\lambda) & \psi_1'(x,\lambda) \end{vmatrix} . \ \Delta_2(\lambda) = \begin{vmatrix} \phi_2(x,\lambda) & \psi_2(x,\lambda) \\ \phi_2'(x,\lambda) & \psi_2'(x,\lambda) \end{vmatrix} . \\ \Delta_3(\lambda) = \begin{vmatrix} \phi_3(x,\lambda) & \psi_3(x,\lambda) \\ \phi_3'(x,\lambda) & \psi_3'(x,\lambda) \end{vmatrix} . \tag{17}$$

Lemma 1.2. $\Delta_1(\lambda) = \Delta_2(\lambda) = \Delta_3(\lambda)$ *for all* $\lambda \in C$.

Proof. Because the Wronskian of the solution matrices $\phi_j(x,\lambda)$ and $\psi_j(x,\lambda)$ is independent of x ,

$$\Delta_3(\lambda) = \Delta_3(\lambda)|_{x=a_2+0} = \begin{vmatrix} \phi_3(a_2+0,\lambda) & \psi_3(a_2+0,\lambda) \\ \phi_3'(a_2+0,\lambda) & \psi_3'(a_2+0,\lambda) \end{vmatrix} = \begin{vmatrix} \phi_2(a_2-0,\lambda) & \psi_2(a_2-0,\lambda) \\ \phi_2'(a_2-0,\lambda) & \psi_2'(a_2-0,\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \phi_2(x,\lambda) & \psi_2(x,\lambda) \\ \phi_2'(x,\lambda) & \psi_2'(x,\lambda) \end{vmatrix}_{x=a_2-0} = \Delta_2(\lambda) = \Delta_2(\lambda)|_{x=a_1+0} = \begin{vmatrix} \phi_2(a_1+0,\lambda) & \psi_2(a_1+0,\lambda) \\ \phi_2'(a_1+0,\lambda) & \psi_2'(a_1+0,\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \phi_1(a_1-0,\lambda) & \psi_1(a_1-0,\lambda) \\ \phi_1'(a_1-0,\lambda) & \psi_1'(a_1-0,\lambda) \end{vmatrix} = \begin{vmatrix} \phi_1(x,\lambda) & \psi_1(x,\lambda) \\ \phi_1'(x,\lambda) & \psi_1'(x,\lambda) \end{vmatrix}_{x=a_1-0} = \Delta_1(\lambda)$$

the proof is completed. □

Denote $\Delta(\lambda) = \Delta_1(\lambda) = \Delta_2(\lambda) = \Delta_3(\lambda)$, we have the following lemma.

Lemma 1.3. *λ is an eigenvalue of* $(1)-(3)$ *if any only if* $\Delta(\lambda) = 0$.

Proof. Necessity: Assume that $\lambda_0$is an eigenvalue of $(1)-(3)$. $y(x,\lambda_0)$ is the eigenfunctions corresponding to λ_0, then by (16) we have

$$y(x,\lambda_0) = \phi(x,\lambda_0)c_{30} = \begin{cases} \phi_1(x,\lambda_0)c_{30}, & x \in (0,a_1) \\ \phi_2(x,\lambda_0)c_{30}, & x \in (a_1,a_2) \\ \phi_3(x,\lambda_0)c_{30}, & x \in (a_2,\pi) \end{cases} \tag{18}$$

$$y(x,\lambda_0) = \psi(x,\lambda_0)c_{40} = \begin{cases} \psi_1(x,\lambda_0)c_{40}, & x \in (0,a_1) \\ \psi_2(x,\lambda_0)c_{40}, & x \in (a_1,a_2) \\ \psi_3(x,\lambda_0)c_{40}, & x \in (a_2,\pi) \end{cases} \tag{19}$$

c_{30}, c_{40} are m-dimensional nonzero constant vector. So from (18) and (19), we have

$$\left.\begin{aligned} \phi_1(x,\lambda_0)c_{30} &= \psi_1(x,\lambda_0)c_{40} \\ \phi_1'(x,\lambda_0)c_{30} &= \psi_1'(x,\lambda_0)c_{40} \end{aligned}\right\} x \in (0,a_1).$$

By direct simplification, we get

$$\begin{pmatrix} \phi_1(x,\lambda_0) & -\psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & -\psi_1'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_{30} \\ c_{40} \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}.$$

Because $c_{30}, c_{40} \neq 0$, the coefficient determinant of above linear system of equations

$$\begin{aligned} &\begin{vmatrix} \phi_1(x,\lambda_0) & -\psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & -\psi_1'(x,\lambda_0) \end{vmatrix} = (-1)^m \begin{vmatrix} \phi_1(x,\lambda_0) & \psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0) \end{vmatrix} \\ &= (-1)^m \Delta_1(\lambda_0) = \Delta_2(\lambda_0) = \Delta_3(\lambda_0) = \Delta(\lambda_0) = 0 \end{aligned}$$

Sufficiency:
If $\lambda_0 \in \mathrm{C}$, $\Delta(\lambda_0) = 0$. Then the linear systems of equations

$$\begin{pmatrix} \phi_1(x,\lambda_0) & \psi_1(x,\lambda_0) \\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}, \begin{pmatrix} \phi_2(x,\lambda_0) & \psi_2(x,\lambda_0) \\ \phi_2'(x,\lambda_0) & \psi_2'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}$$

$$\begin{pmatrix} \phi_3(x,\lambda_0) & \psi_3(x,\lambda_0) \\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}$$

have nonzero solutions. By a direct computation, we get

$$\left.\begin{aligned} \phi_1(x,\lambda_0)c_0 &= -\psi_1(x,\lambda_0)c_1 \\ \phi_1'(x,\lambda_0)c_0 &= -\psi_1'(x,\lambda_0)c_1 \end{aligned}\right\} x \in (0,a_1), \quad \left.\begin{aligned} \phi_2(x,\lambda_0)c_0 &= -\psi_2(x,\lambda_0)c_1 \\ \phi_2'(x,\lambda_0)c_0 &= -\psi_2'(x,\lambda_0)c_1 \end{aligned}\right\} x \in (a_1,a_2)$$

and

$$\left.\begin{aligned} \phi_3(x,\lambda_0)c_0 &= -\psi_3(x,\lambda_0)c_1 \\ \phi_3'(x,\lambda_0)c_0 &= -\psi_3'(x,\lambda_0)c_1 \end{aligned}\right\} x \in (a_2,\pi).$$

Denote

$$y(x,\lambda_0) = \begin{cases} \phi_1(x,\lambda_0)c_0 = -\psi_1(x,\lambda_0)c_1, & x \in (0,a_1) \\ \phi_2(x,\lambda_0)c_0 = -\psi_2(x,\lambda_0)c_1, & x \in (a_1,a_2) \\ \phi_3(x,\lambda_0)c_0 = -\psi_3(x,\lambda_0)c_1, & x \in (a_2,\pi) \end{cases}.$$

We note that $y(x,\lambda_0)$ satisfies the boundary condition $(2),(3)$. That is, $y(x,\lambda_0)$ is the eigenfunctions corresponding to λ_0. Thus λ_0 is an eigenvalue of the problem $(1)-(3)$.

□

Remark 1.4. *As two especial case*

$$\Delta(\lambda)=\begin{vmatrix}\phi_1(x,\lambda_0) & \psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0)\end{vmatrix}_{x=0}=\begin{vmatrix}E_m & \psi_1(0,\lambda_0)\\ \theta_m & \psi_1'(0,\lambda_0)\end{vmatrix}=\det\left(\psi_1'(0,\lambda)\right)$$

$$\Delta(\lambda)=\begin{vmatrix}\phi_3(x,\lambda_0) & \psi_3(x,\lambda_0)\\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0)\end{vmatrix}_{x=\pi}=\begin{vmatrix}\phi_3(\pi,\lambda_0) & E_m\\ \phi_3'(\pi,\lambda_0) & \theta_m\end{vmatrix}=(-1)^m\det\left(\phi_3'(\pi,\lambda)\right).$$

Definition 1.5. $\Delta(\lambda)$ *will be called the characteristic function of the eigenvalues of the problem* (1) − (3).

Definition 1.6. *If there is a* $\Delta_1(\lambda)$ *to be* $\Delta(\lambda)=(\lambda-\lambda_0)^m\Delta_1(\lambda)$, *algebraic multiplicity of eigenvalue* λ*is called* m. *The geometric multiplicity of* λ *as an eigenvalue of the problem* (1) − (3) *is defined to be the number of linearly independent solutions of the boundary value problem. If we denote* $2m\times 2m$*matrices*
$A(x,\lambda_0)=\begin{pmatrix}\phi_1(x,\lambda_0) & \psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0)\end{pmatrix}$,$B(x,\lambda_0)=\begin{pmatrix}\phi_2(x,\lambda_0) & \psi_2(x,\lambda_0)\\ \phi_2'(x,\lambda_0) & \psi_2'(x,\lambda_0)\end{pmatrix}$ *and*
$C(x,\lambda_0)=\begin{pmatrix}\phi_3(x,\lambda_0) & \psi_3(x,\lambda_0)\\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0)\end{pmatrix}$
the rank of matrix $A(x,\lambda_0)$ *as* $R(A(x,\lambda_0))$. *Similarly,* $B(x,\lambda_0)$ *as* $R(B(x,\lambda_0))$ *and* $C(x,\lambda_0)$*as* $R(C(x,\lambda_0))$.

Corollary 1.7. *The geometric multiplicity of* λ_0 *as an eigenvalue of the problem* (1)−(3) *is equal to* $2m-R(A(x,\lambda_0))$ *or* $2m-R(B(x,\lambda_0))$ *or* $2m-R(C(x,\lambda_0))$.

Corollary 1.8. $R(A(x,\lambda_0))$, $R(B(x,\lambda_0))$ *or* $R(C(x,\lambda_0))$ *is at least equal to* m, *so the geometric multiplicity of* λ_0 *varies from 1 to* m. *When the geometric multiplicity of an eigenvalue is* m, *we say the eigenvalue has maximal (full) multiplicity. In this study, we refer multiplicity as the geometric multiplicity.*
An entire function of non-integer order has an infinite set of zeros. The zeros of an analytic function which does not vanish identically are isolated [3]. $\psi_1'(0,\lambda)$ *and* $\phi_3'(\pi,\lambda)$ *are entire function of order* $\frac{1}{2}$ *matrices. The sums and products of such functions are entire of order not exceeding* $\frac{1}{2}$. *Hence, the determinants of* $\psi_1'(0,\lambda)$ *and* $\phi_3'(\pi,\lambda)$, *that is, the caracteristic functions are also non-integer.*

Eigenvalues for (1)−(3) are real. The boundary value problem (1)−(3)has a countable number of eigenvalues that grow unlimitedly, when those are ordered according to their absolute value.
The norm of a constant matrix as well as the norm of a matrix function A is denoted by $\|A\|$.
$A(x)=\left(a_{ij}\right)_{i,j=1}^m : I\to M_{mxm}^{\mathrm{R}}$, for any $x\in I$, the norm of $A(x)$ may be taken as

$$\|A(x)\|=\max_{1\le i\le m}\sum_{j=1}^m\left|a_{ij}\right| \tag{20}$$

Let $\lambda=s^2, s=\sigma+i\tau$, $\sigma,\tau\in\mathrm{R}$. We have the following three lemmas.

Lemma 1.9. *When* $|\lambda|\to\infty$, *the following asymptotic formulas hold on*
$0<x<a_1$,

$$\phi_1(x,\lambda)=\cos(\lambda x)E_m+O\left(|\lambda|^{-1}e^{|\sigma|x}\right) \tag{21}$$

$$\phi_1'(x,\lambda)=-\lambda\sin(\lambda x)E_m+O\left(e^{|\sigma|x}\right) \tag{22}$$

Proof. See [1]. □

Lemma 1.10. *When* $|\lambda|\to\infty$, $\phi_2(x,\lambda)$ *and* $\phi_2'(x,\lambda)$ *have the following asymptotic formulas on* $a_1<x<a_2$,

$$\phi_2(x,\lambda)=\frac{1}{2}\alpha^+\exp\left(-i\left(\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right)\right)E_m\left(1+O\left(\frac{1}{\lambda}\right)\right) \tag{23}$$

$$\phi_2'(x,\lambda)=\frac{1}{2}\alpha^+(p(x)-\lambda\alpha)\,i\exp\left(-i\left(\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right)\right)E_m+O(1) \tag{24}$$

where $\mu^{\mp}(x)=\mp\alpha x\pm\alpha a_1+a_1$, $\alpha^{\pm}=\frac{1}{2}\left(1\pm\frac{1}{\alpha}\right)$.

Proof. Since $\phi_2(x,\lambda)$ is the solution of initial value problem (6) , we have

$$\begin{array}{l}\phi_2(x,\lambda)=\alpha^+\cos\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]E_m\\ +\alpha^-\cos\left[\lambda\mu^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right]E_m+O\left(\frac{1}{\lambda}e^{\sigma\mu^+(x)}\right)\end{array}.$$

We get

$$\begin{array}{l}\phi_2(x,\lambda)=\frac{1}{2}\alpha^+e^{i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m+\frac{1}{2}\alpha^+e^{-i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m\\ +\frac{1}{2}\alpha^-e^{i\left[\lambda\mu^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m+\frac{1}{2}\alpha^-e^{-i\left[\lambda\mu^-(x)+\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m+O\left(\frac{1}{\lambda}e^{\sigma\mu^+(x)}\right)\end{array} \tag{25}$$

Let $f(x,\lambda):=O\left(\frac{1}{\lambda}e^{\sigma\mu^+(x)}\right)$ and note that

$$\phi_2(x,\lambda)=\frac{1}{2}\alpha^+e^{-i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m+(1+g(x,\lambda)).$$

From a simple computation at equations (25), we get

$$\begin{array}{l}g(x,\lambda)=e^{2i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m+\frac{\alpha-1}{\alpha+1}e^{2i\lambda a_1}E_m+\frac{\alpha-1}{\alpha+1}e^{2i\left[\lambda\alpha(x-a_1)-\frac{v(x)}{\alpha}\right]}E_m\\ +\frac{2e^{i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}}{\alpha^+}f(x,\lambda)E_m.\end{array}$$

Let's examine $g(x,\lambda)=O\left(\frac{1}{\lambda}\right)$ accuracy.

$$\begin{array}{l}\left|g(x,\lambda)\right|\le\left|e^{2i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}E_m\right|+\left|\frac{\alpha-1}{\alpha+1}e^{2i\lambda a_1}E_m\right|+\left|\frac{\alpha-1}{\alpha+1}e^{2i\left[\lambda\alpha(x-a_1)-\frac{v(x)}{\alpha}\right]}E_m\right|\\ +\left|\frac{e^{i\left[\lambda\mu^+(x)-\frac{v(x)}{\alpha}\right]}}{s^+}E_mf(x,\lambda)\right|+\left|\frac{2e^{i\left[\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)dt\right]}}{\alpha^+}f(x,\lambda)E_m\right|\\ \le e^{-2\sigma\mu^+(x)}E_m+\left|\frac{s^-}{s^+}\right|e^{-2\sigma a_1}E_m+\left|\frac{s^-}{s^+}\right|e^{-2\sigma\alpha x}E_m+\frac{c}{\lambda}e^{-\sigma\mu^+(x)}e^{\sigma\mu^+(x)}E_m\end{array}$$

Furthermore , $\sigma>\varepsilon|\lambda|$, $\varepsilon>0$ in D. Thus,$-\sigma<-\varepsilon|\lambda|$ and $e^{-2\sigma\mu^+(x)}<e^{-\varepsilon|\lambda|\mu^+(x)}$
Since $\frac{x}{e^x}\to 0$, $x<ce^{\mu^+(x)}$ $(c>0)$. Thus, $e^{-2\sigma\mu^+(x)}<\frac{c}{\varepsilon|\lambda|\mu^+(x)}$. We get
$g(x,\lambda)=O\left(\frac{1}{\lambda}\right)$ $\lambda\to\infty$. Hence,

$$\phi_2(x,\lambda)=\frac{1}{2}\alpha^+\exp\left(-i\left(\lambda\mu^+(x)-\frac{1}{\alpha}\int_{a_1}^x p(t)\,dt\right)\right)E_m\left(1+O\left(\frac{1}{\lambda}\right)\right),|\lambda|\to\infty.$$

Derivativing both sides of (23) and using the first formula (25), we could get the formula of (24) similarly. □

Lemma 1.11. *When $|\lambda|\to\infty$, $\phi_3(x,\lambda)$ and $\phi_3'(x,\lambda)$ have the following asymptotic formulas on $a_2<x<\pi$,*

$$\phi_3(x,\lambda)=\frac{1}{2}\beta^+\exp\left(-i\left(\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right)\right)E_m\left(1+O\left(\frac{1}{\lambda}\right)\right) \tag{26}$$

$$\phi_3'(x,\lambda)=\frac{1}{2}\beta^+(p(x)-\lambda\beta)\,i\exp\left(-i\left(\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right)\right)E_m+O(1) \tag{27}$$

where $k^\pm(x)=\pm\beta x\mp\beta a_2+\mu^+(a_2)$, $s^\pm(x)=\pm\beta x\mp\beta a_2+\mu^-(a_2)$,$\beta_2^\mp=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right)$.

Proof. Since $\phi_3(x,\lambda)$ is the solution of initial value problem (9) , we have

$$\begin{array}{l}\phi_3(x,\lambda)=\beta^+\cos\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right]+\beta^-\cos\left[\lambda k^-(x)-\frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right]\\ +\beta^-\cos\left[\lambda s^+(x)+\frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right]+\beta^+\cos\left[\lambda s^-(x)+\frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right]+O\left(\frac{1}{\lambda}e^{\sigma k^+(x)}\right)\end{array}$$

We get

$$\begin{aligned}\phi_3(x,\lambda) &= \tfrac{\beta^+}{2}e^{i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^+}{2}e^{-i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}\\ &+\tfrac{\beta^-}{2}e^{i\left[\lambda k^-(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^-}{2}e^{-i\left[\lambda k^-(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}\\ &+\tfrac{\beta^-}{2}e^{i\left[\lambda s^+(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^-}{2}e^{-i\left[\lambda s^+(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}\\ &+\tfrac{\beta^+}{2}e^{i\left[\lambda s^-(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+\tfrac{\beta^+}{2}e^{-i\left[\lambda s^-(x)+\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}+O\left(\tfrac{1}{\lambda}e^{\sigma k^+(x)}\right)\end{aligned} \tag{28}$$

Let $f(x,\lambda) := O\left(\frac{1}{\lambda}e^{\sigma k^+(x)}\right)$ and note that

$$\phi_3(x,\lambda) = \frac{\beta^+}{2}e^{-i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m + (1+g(x,\lambda))$$

From a simple calculation at equation (28), we get

$$\begin{aligned}g(x,\lambda) &= e^{2i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m + \tfrac{\beta^-}{\beta^+}e^{2i\left[(\beta\pi-\beta a_2+a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\\ &+\tfrac{\beta^-}{\beta^+}e^{2i\left[\alpha(a_2-a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m+\tfrac{\beta^-}{\beta^+}e^{2i\mu^+(a_2)}+\tfrac{\beta^-}{\beta^+}e^{2i\beta(\pi-a_2)}\\ &+e^{2ia_1}+e^{2i[\beta\pi-\beta a_2+\alpha a_2-\alpha a_1]}+\frac{e^{i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}}{\beta^+}f(x,\lambda)E_m\end{aligned}$$

Let's examine $g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ accuracy.

$$\begin{aligned}\left|g(x,\lambda)\right| &\le \left|e^{2i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\right| + \left|\tfrac{\beta^-}{\beta^+}e^{2i\left[(\beta\pi-\beta a_2+a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\right|\\ &+\left|\tfrac{\beta^-}{\beta^+}e^{2i\left[\alpha(a_2-a_1)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}E_m\right|+\left|\tfrac{\beta^-}{\beta^+}e^{2i\mu^+(a_2)}E_m\right|+\left|\tfrac{\beta^-}{\beta^+}e^{2i\beta(\pi-a_2)}E_m\right|\\ &+\left|e^{2ia_1}E_m\right|+\left|e^{2i[\beta\pi-\beta a_2+\alpha a_2-\alpha a_1]}E_m\right|+\left|\frac{e^{i\left[\lambda k^+(x)-\frac{1}{\beta}\int_{a_2}^x p(t)dt\right]}}{\beta^+}f(x,\lambda)E_m\right|\end{aligned}$$

$$\begin{aligned}&\le e^{-2\sigma k^+(x)}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma k^+(x)}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma a_2}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma a_2}+\left|\tfrac{\beta^-}{\beta^+}\right|e^{-2\sigma\beta x}\\ &+e^{-2\sigma a_1}+e^{-2\sigma k^+(x)}+\tfrac{c}{\lambda}e^{-2\sigma k^+(x)}e^{2\sigma k^+(x)}\end{aligned}$$

In addition to , $\sigma > \varepsilon|\lambda|$, $\varepsilon > 0$ in D . Thus,$-\sigma < -\varepsilon|\lambda|$ and $e^{-2\sigma k^+(x)} < e^{-\varepsilon|\lambda|k^+(x)}$
Since $\frac{x}{e^x} \to 0$, $x < ce^{k^+(x)}$ $(c > 0)$. Thus, $e^{-2\sigma k^+(x)} < \frac{c}{\varepsilon|\lambda|k^+(x)}$. We get
$g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ $\lambda \to \infty$. Hence,

$$\phi_3(x,\lambda) = \frac{1}{2}\beta^+\exp\left(-i\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^x p(t)\,dt\right)\right)E_m\left(1+O\left(\frac{1}{\lambda}\right)\right), |\lambda| \to \infty.$$

Derivativing both sides of (26) and using the first formula (28), we could get the formula of (27) similarly. □

2. Multiplicities of eigenvalues of the vectorial problem

In the section, we find the conditions on the potential matrix function $(2\lambda p(x) + q(x))$, under some conditions, the problem (1) – (3) can only have a finite number of eigenvalues with multiplicity m. Where $p(x) \in W_2^1[0,\pi]$ ve $p(x) = \{p_{ij}(x)\}_{i,j=1}^m$, $q(x) \in L_2[0,\pi]$ and $q(x) = \{q_{ij}(x)\}_{i,j=1}^m$.

Theorem 2.1. *Let $m \ge 2$. Assume that, for some $i, j \in \{1, 2, ..., m\}$ with $i \ne j$*

either

$$(i)\quad \begin{array}{l} \int_0^{a_1} p_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\,dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\,dx \neq 0 \\ \int_0^{a_1} q_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\,dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\,dx \neq 0 \end{array} \tag{29}$$

or

$$(ii)\quad \begin{array}{l} \int_0^{a_1}\left[p_{ii}(x)-p_{jj}(x)\right]dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[p_{ii}(x)-p_{jj}(x)\right]dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[p_{ii}(x)-p_{jj}(x)\right]dx \neq 0 \\ \int_0^{a_1}\left[q_{ii}(x)-q_{jj}(x)\right]dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[q_{ii}(x)-q_{jj}(x)\right]dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[q_{ii}(x)-q_{jj}(x)\right]dx \neq 0 \end{array} \tag{30}$$

where $\alpha^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\alpha}\right)$, $\beta^{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{\beta}\right)$.*Then, with finitely many exceptions. The multiplicities of the eigenvalues of the problem* (1) – (3) *are at most* $m-1$.

Proof. (i) We assume that (29) holds. Suppose, to the contrary, that there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ whole multiplicities are all m. Obviously, $\lambda_n \to \infty$ as $n \to \infty$. From the equations in (9). Denoting $\phi_3(x,\lambda) = \left\{y_{ij}^{+}(x)\right\}_{i,j=1}^{m}$, when $\lambda = \lambda_n$ for $n = 1, 2, ...$, we get

$$\left(y_{ii}^{+}\right)''(x) + \left(\lambda - \left(2\lambda p_{ii}(x) + q_{ii}(x)\right)\right) y_{ii}^{+}(x) - \sum_{k\neq i}\left(2\lambda p_{ii}(x) + q_{ii}(x)\right) y_{ki}^{+}(x) = 0 \tag{31}$$

and

$$\left(y_{ij}^{+}\right)''(x) + \left(\lambda - \left(2\lambda p_{ii}(x) + q_{ii}(x)\right)\right) y_{ij}^{+}(x) - \sum_{k\neq j}\left(2\lambda p_{ii}(x) + q_{ii}(x)\right) y_{kj}^{+}(x) = 0 \tag{32}$$

Multiplying (31) and (32) by $y_{ij}^{+}(x)$ and $y_{ii}^{+}(x)$ respectively, then subtructing one fom the other and using (26), nothing that the eigenvalues of the problem are all real, we have

$$\begin{aligned} \left(\left(y_{ii}^{+}\right)'(x)\, y_{ij}^{+}(x) - y_{ii}^{+}(x)\left(y_{ij}^{+}\right)'(x)\right)' &= \sum_{k\neq i}\left(2\lambda p_{ik}(x) + q_{ik}(x)\right)\left(y_{ki}^{+}(x)\, y_{ij}^{+}(x) - y_{ii}^{+}(x)\, y_{kj}^{+}(x)\right) \\ = \left(2\lambda p_{ij}(x) + q_{ij}(x)\right)&\left[y_{ij}^{+}(x)\, y_{ji}^{+}(x) - y_{ii}^{+}(x)\, y_{ij}^{+}(x)\right] \\ &+ \textstyle\sum_{k\neq i,j}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left(y_{ki}^{+}(x)\, y_{ij}^{+}(x) - y_{ii}^{+}(x)\, y_{kj}^{+}(x)\right) \\ = -\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)&\left[\frac{(\beta^+)^2}{4}\cos^2\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^{x} p(t)\,dt\right)\right] + O\left(1 + \frac{1}{\lambda}\right) \end{aligned} \tag{33}$$

similarly, from the equations in (6), denoting $\phi_2(x,\lambda) = \left\{y_{ij}^{-}(x)\right\}_{i,j=1}^{m}$, we get

$$\begin{aligned} &\left(\left(y_{ii}^{-}\right)'(x)\, y_{ij}^{-}(x) - y_{ii}^{-}(x)\left(y_{ij}^{-}\right)'(x)\right)' = \\ &\qquad -\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4} cos^2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right] O\left(1 + \frac{1}{\lambda}\right) \end{aligned} \tag{34}$$

similarly, from the equations in (4), denoting $\phi_1(x,\lambda) = \left\{y_{ij}^{0}(x)\right\}_{i,j=1}^{m}$, we get

$$\left(\left(y_{ii}^{0}\right)'(x)\, y_{ij}^{0}(x) - y_{ii}^{0}(x)\left(y_{ij}^{0}\right)'(x)\right)' = -\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\cos^2(\lambda x)\right] + O\left(\frac{1}{\lambda}\right) \tag{35}$$

When λ is an eigenvalue with multiplicity m, we have $\phi_3'(\pi,\lambda) = 0_m$. By integrating both sides of (33) from a_2 to π, for $\lambda_n \to \lambda$ and $n \to \infty$, we obtain

$$\begin{aligned} &-\left(\left(y_{ii}^{+}\right)'(x)\, y_{ij}^{+}(x) - y_{ii}^{+}(x)\left(y_{ij}^{+}\right)'(x)\right) = \\ &\quad = \int_{a_2}^{\pi}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\beta^+)^2}{4}\cos^2\left(\lambda k^+(x) - \frac{1}{\beta}\int_{a_2}^{x} p(t)\,dt\right)\right] + O\left(\frac{1}{\lambda}\right)\right]dx \end{aligned} \tag{36}$$

By integrating both sides of (34) from a_1 to a_2 and appliying the boundary condition

$$\begin{aligned} &-\left(\left(y_{ii}^{-}\right)'(x)\, y_{ij}^{-}(x) - y_{ii}^{-}(x)\left(y_{ij}^{-}\right)'(x)\right) = \\ &= \int_{a_1}^{a_2}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4}\cos^2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right] + O\left(\frac{1}{\lambda}\right)\right]dx \end{aligned} \tag{37}$$

By integrating both sides of (35) from 0 to a_1 and applying the boundary condition $\phi_1'(0,\lambda) = 0_m$, we obtain, for $\lambda_n \to \lambda$ and $n \to \infty$,

$$\left(\left(y_{ii}^{0}\right)'(x)\, y_{ij}^{0}(x) - y_{ii}^{0}(x)\left(y_{ij}^{0}\right)'(x)\right) = -\int_0^{a_1}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\cos^2(\lambda x)\right] + O\left(\frac{1}{\lambda}\right)\right]dx \tag{38}$$

Sum the above (36), (37) and (38), then use the initial conditions at point $x = a_1$ and $x = a_2$, we get

$$\begin{aligned} 0 = &-\int_0^{a_1}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\cos^2(\lambda x)\right] + O\left(\frac{1}{\lambda}\right)\right]dx \\ &+ \int_{a_1}^{a_2}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4}\cos^2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx \\ &+ \int_{a_2}^{\pi}\left[-\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\beta^+)^2}{4}\cos^2\left(\lambda k^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx + O\left(\frac{1}{\lambda}\right) \end{aligned}$$

By a simple computation, one can see that

$$\begin{aligned} &\int_0^{a_1}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)dx \\ &= -\int_0^{a_1}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\cos 2\lambda x\right]dx \\ &- \int_{a_1}^{a_2}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\alpha^+)^2}{4}\cos 2\left(\lambda\mu^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx \\ &- \int_{a_2}^{\pi}\left[\left(2\lambda p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\beta^+)^2}{4}\cos 2\left(\lambda k^+(x) - \frac{1}{\alpha}\int_{a_1}^{x} p(t)\,dt\right)\right]\right]dx + O\left(\frac{1}{\lambda}\right) \end{aligned} \tag{39}$$

$$\begin{aligned} &= -2\lambda\int_0^{a_1} p_{ij}(x)\cos(2\lambda x)\,dx - \int_0^{a_1} q_{ij}(x)\cos(2\lambda x)\,dx - 2\lambda\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\cos 2\lambda\mu^+(x)\cos\frac{2v(x)}{\alpha}dx \\ &-\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\cos 2\lambda\mu^+(x)\cos\frac{2v(x)}{\alpha}dx - 2\lambda\frac{(\alpha^+)^2}{4}in\int_{a_1}^{a_2} p_{ij}(x)\,2\lambda\mu^+(x)\sin\frac{2v(x)}{\alpha}dx \\ &-\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\sin 2\lambda\mu^+(x)\sin\frac{2v(x)}{\alpha}dx - 2\lambda\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\cos 2\lambda k^+(x)\cos\frac{2t(x)}{\beta}dx \\ &-\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\cos 2\lambda k^+(x)\cos\frac{2t(x)}{\beta}dx - 2\lambda\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\sin 2\lambda k^+(x)\sin\frac{2t(x)}{\beta}dx \\ &-\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\sin 2\lambda k^+(x)\sin\frac{2t(x)}{\beta}dx \end{aligned}$$

where $v(x) = \int_{a_1}^{x} p(t)\,dt$, $t(x) = \int_{a_2}^{x} p(t)\,dt$. Then, we obtain, for $\lambda_n \to \infty$ and $n \to \infty$,

$$\begin{aligned} &= -2\int_0^{a_1} p_{ij}(x)\cos(2\lambda x)\,dx - 2\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\cos 2\lambda\mu^+(x)\cos\frac{2v(x)}{\alpha}dx \\ &-2\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\,2\lambda\mu^+(x)\sin\frac{2v(x)}{\alpha}dx - 2\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\cos 2\lambda k^+(x)\cos\frac{2t(x)}{\beta}dx \\ &-2\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\sin 2\lambda k^+(x)\sin\frac{2t(x)}{\beta}dx \end{aligned}$$

By Riemann-Lebesgue Lemma, the right side of (39) approaches 0 as $\lambda_n = \lambda$ and $n \to \infty$. This implies that

$$\begin{aligned} &\int_0^{a_1} p_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} p_{ij}(x)\,dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} p_{ij}(x)\,dx = 0 \\ &\int_0^{a_1} q_{ij}(x)\,dx + \frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} q_{ij}(x)\,dx + \frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} q_{ij}(x)\,dx = 0 \end{aligned}$$

We have reached a contradiction. The conclusion for this case is proved.

(ii) Next, we assume that

$$\int_0^{a_1}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx=0$$

or
$\int_0^{a_1} s_{ij}(x)\,dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} s_{ij}(x)\,dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} s_{ij}(x)\,dx=0, \forall i\neq j,$
where $s_{ij}(x)=\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)$.
and
$\int_0^{a_1}\left[s_{ii}(x)-s_{jj}(x)\right]dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}\left[s_{ii}(x)-s_{jj}(x)\right]dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}\left[s_{ii}(x)-s_{jj}(x)\right]dx\neq 0$
without loss of generality, we assume that for $i=1, j=2$

$$\int_0^{a_1}[s_{11}(x)-s_{22}(x)]\,dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}[s_{11}(x)-s_{22}(x)]\,dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}[s_{11}(x)-s_{22}(x)]\,dx\neq 0$$

$$K=\begin{bmatrix}\frac12 & -\frac12 & & & \\ \frac12 & \frac12 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{bmatrix} K=\begin{bmatrix}\frac12 & -\frac12 & & & \\ \frac12 & \frac12 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{bmatrix}$$

and $y=K\cdot t$. Then, the problem $(1)-(3)$ becomes

$$\left.\begin{array}{l} t''+\left(\lambda^2\delta(x)-R(x)\right)t=0 \\ t'(0)=t'(\pi)=0\end{array}\right\} \tag{40}$$

where $R(x)=K^{-1}S(x)K$. By making a simple computation, we get

$$R(x)=\begin{bmatrix}\frac14(s_{11}+s_{22})+s_{12} & \frac14(s_{22}-s_{11}) & * & * & * \\ \frac14(s_{22}-s_{11}) & \frac14(s_{11}+s_{22})+s_{12} & * & * & * \\ * & * & q_{33} & \cdots & \\ * & * & \vdots & \ddots & \\ * & * & & \cdots & q_{mm}\end{bmatrix}(x)$$

We note that the two poblems $(1)-(3)$and (40)have exactly the same spectral structure. Denote $R(x)=\left\{r_{ij}(x)\right\}_{i,j=1}^{m}$. Since

$$\int_0^{a_1} r_{12}(x)\,dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2} r_{12}(x)\,dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi} r_{12}(x)\,dx=$$
$$\int_0^{a_1}[s_{11}(x)-s_{22}(x)]\,dx+\frac{(\alpha^+)^2}{4}\int_{a_1}^{a_2}[s_{11}(x)-s_{22}(x)]\,dx+\frac{(\beta^+)^2}{4}\int_{a_2}^{\pi}[s_{11}(x)-s_{22}(x)]\,dx\neq 0$$

By part (i) , the conclusion of the theorem holds for the problem (40), and hence holds for the problem $(1)-(3)$. □

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