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Half inverse problems for the impulsive singular diffusion operator

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Abstract. In this paper, we consider the inverse spectral problem for the impulsive Sturm-Liouville differential pencils on $[0,\pi]$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. We prove that two potentials functious on the whole interval and the parameters in the boundary and jump conditions can be determined from a set of eigenvalues for two cases: (i) The potentials is given on $\left(0, \frac{\pi}{4}(\alpha+\beta)\right)$. (ii) The potentials is given on $\left(\alpha+\beta, \frac{\alpha+\beta}{2}\right)$, where $0 < \alpha+\beta < 1$, $\alpha+\beta > 1$ respectively. Finally, was given interior inverse problem for same boundary problem.

1. Introduction

We consider the impulsive quadratic pencils of Sturm-Liouville operator of the form

$$ly := -y'' + [q(x) + 2\lambda p(x)]\, y = \lambda^2 \rho(x)\, y\ ,\ \ x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0 \tag{2}$$

$$V(y) := y'(\pi) + Hy(\pi) = 0 \tag{3}$$

and the jump conditions

$$\begin{aligned} y\left(\frac{\pi}{2}+0\right) &= ay\left(\frac{\pi}{2}-0\right) \\ y'\left(\frac{\pi}{2}+0\right) &= a^{-1}y'\left(\frac{\pi}{2}-0\right) + \gamma y\left(\frac{\pi}{2}\right) \end{aligned} \tag{4}$$

Where λ is the spectral parameter, $p(x) \in W_2^1[0,\pi]$, $q(x) \in L_2[0,\pi]$ are real valued functions, $h, H \in$ R, a, γ, α, β are real numbers, $0 < \alpha < \beta < 1$, $\alpha+\beta > 1$, $a > 0$, $|a-1|^2 + \gamma^2 \neq 0$ and

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$$\rho(x) = \begin{cases} \alpha^2, & 0 < x < \dfrac{\pi}{2} \\ \beta^2, & \dfrac{\pi}{2} < x < \pi, \end{cases}$$

Here we denote by $W_2^m[0,\pi]$ the space of functions $f(x), x \in [0,\pi]$ such that the derivatives $f^{(m)}(x)$ $\left(m = \overline{0, n-1}\right)$ are absolute continuous and $f^{(n)}(x) \in L_2[0,\pi]$.
We can get $p(0) = 0$without general exposure, otherwise, if $c_0 = p(0) \neq 0$by direct calculation we note that equations (1) is equivalent to

$$ly := -y'' + \left[q(x) + 2p(x)c_0 - c_0^2 + 2(\lambda - c_0)(p(x) - c_0)\right]y = (\lambda - c_0)^2 \rho(x) y \tag{5}$$

Let

$$\hat{q}(x) = q(x) + 2p(x)c_0 - c_0^2, \hat{p}(x) = p(x) - c_0, \hat{\lambda} = \lambda - c_0$$

then for the problem with the form (5) we have $\hat{p}(0) = 0$.

Inverse spectral problems consist in recovering the coefficients of an operator from their spectral characteristics. The first results on inverse problems theory of classical Sturm-Liouville operator where given by Ambarzumyan and Borg (*see*[13, 24]). Inverse Sturm-Liouville problems which appear in mathematical physics, mechanics, electronics, geophysics an other branches of natural sciences have been studied for about ninety years (*see*[8, 9, 12]).

The half inverse Sturm-Liouville problem which is one of the important subjects of the inverse spectral theory has been studied firstly by Hochstadt and Lieberman in 1978 [*see*[20]]. They proved that spectrum of the problem

$$-y'' + q(x)y = \lambda y, \quad x \in (0,1)$$

$$y'(0) - hy(0) = 0 = y'(1) + Hy(1)$$

and potential $q(x)$ on the $\left(\frac{1}{2}, 1\right)$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$ almost everywhere. Since then, this result has been generalized to various versions. In 1984, Hald [15] proved similar results in the case when there exist a impulse conditions inside the interval. He also gave some applications of this kinds of problem to geophysics. Recently, some new uniqueness results in inverse spectral analysis with partial information on the potential for some classes of differential equations have been given (see for example [18, 25, 32]). These kinds of results are known as Hochstadt and Lieberman type theorems. In particular, in the work [6] studied the inverse spectral problem for the impulsive Sturm-Liouville problem on $(0,\pi)$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. They proved that the potential $q(x)$ on the whole interval and the paremeters in the boundary conditions and jump conditions can be determined from a set of eigenvalues for two cases:

i) The potential $q(x)$ is given on $\left(0, \frac{1+\alpha}{4}\pi\right)$,

ii) The potential $q(x)$ is given on $\left(\frac{1+\alpha}{4}\pi, \pi\right)$, where $0 < \alpha < 1$,

and also shown that the potential and all the parameters can be uniquely recovered by one spectrum and some information on the eigenfunctions at some interior point. Similary problem studied in [25]. In particulary, they discuss Gesztesy-Simon theorem and show that if the potential function $q(x)$ is preseribed on the interval $\left[\frac{\pi}{2(1-\alpha)}, \pi\right]$ for some $\alpha \in (0,1)$, then parts of a finite number of spectra suffice to determine $q(x)$ on $[0,\pi]$.

2. Preliminaries

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of the equation (1), satisfying the initial conditions $\varphi(0,\lambda)=1, \varphi'(0,\lambda)=h, \psi(\pi,\lambda)=1, \psi'(\pi,\lambda)=-H$ and the jump condition (4). Denote

$\sigma(x)=\int_0^x \sqrt{\rho(t)}dt, \tau=Im\lambda$, for every $\lambda\in C$

It is shown in [2] if $q(x)\in L_2[0,\pi]$ and $p(x)\in W_2^1[0,1]$ for every $\lambda\in C$,that there exist funvtions $A(x,t)$and $B(x,t)$ whose first order partial derivatives are summable on $[0,\pi]$ for each $x\in[0,\pi]$ such that

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^{\sigma(x)} A(x,t)\cos\lambda t dt+\int_0^{\sigma(x)} B(x,t)\sin\lambda t dt \tag{6}$$

Where

$$\varphi_0(x,\lambda)=\begin{cases}\cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+\frac{h}{\lambda\alpha}\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right], & 0\le x<\frac{\pi}{2}\\ a^+\cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\cos\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right] & \\ +\frac{h}{\lambda\alpha}\left\{a^+\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\sin\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\right\}, & \frac{\pi}{2}<x\le\pi\end{cases} \tag{7}$$

and $a^{\pm}=\frac{1}{2}\left(a\pm\frac{\alpha}{a\beta}\right), w^+(x)=\int_0^x p(t)dt, w^-(x)=\int_{\frac{\pi}{2}}^x p(t)dt$
It easy to verify from the integral representation (6) above that the solution $\varphi(x,\lambda)$ following asimptotic relation is valid as $|\lambda|\to\infty$. For $\frac{\pi}{2}<x\le\pi$

$$\begin{aligned}\varphi(x,\lambda)&=a^+\cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\cos\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\\&+\frac{h}{\lambda\alpha}\left\{a^+\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]+a^-\sin\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\right\}\\&+O\left(\lambda^{-2}\exp(|\tau|\sigma(x))\right)\end{aligned} \tag{8}$$

$$\begin{aligned}\varphi'(x,\lambda)&=-a^+\left(\lambda\beta-\tfrac{1}{\beta}p(x)\right)\sin\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\&+a^-\left(\lambda\beta-\tfrac{1}{\beta}p(x)\right)\sin\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\\&+\frac{h}{\lambda\alpha}a^+\left(\lambda\beta-\tfrac{1}{\beta}p(x)\right)\cos\left[\lambda\sigma(x)-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\&-\frac{h}{\lambda\alpha}a^-\left(\lambda\beta-\tfrac{1}{\beta}p(x)\right)\cos\left[\lambda(\alpha\pi-\sigma(x))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]+O\left(\lambda^{-1}\exp(|\tau|\sigma(x))\right)\end{aligned} \tag{9}$$

Similarly, for the solution $\psi(x,\lambda)$ following asiymptotic relation hold as $|\lambda|\to\infty$. For $0\le x<\frac{\pi}{2}$,

$$\begin{aligned}\psi(x,\lambda)&=R^+\cos\left[\lambda(\sigma(\pi)-\sigma(x))-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\&+R^-\cos\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x)))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]\\&+\frac{1}{\lambda}\left(\frac{H}{\beta}R^++\frac{\gamma}{\alpha}\right)\sin\left[\lambda(\sigma(\pi)-\sigma(x))-\frac{w^+(x)}{\sqrt{\rho(x)}}\right]\\&+\frac{1}{\lambda}\left(\frac{H}{\beta}R^-+\frac{\gamma}{\alpha}\right)\sin\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x)))+\frac{w^-(x)}{\sqrt{\rho(x)}}\right]+O\left(\lambda^{-2}\exp(|\tau|(\sigma(\pi)-\sigma(x)))\right)\end{aligned} \tag{10}$$

$$\begin{aligned}\psi'(x,\lambda) &= R^{+}\left(\lambda\alpha - \tfrac{1}{\alpha}p(x)\right)\sin\left[\lambda(\sigma(\pi)-\sigma(x)) - \tfrac{w^{+}(x)}{\sqrt{\rho(x)}}\right]\\ &-R^{-}\left(\lambda\alpha - \tfrac{1}{\alpha}p(x)\right)\sin\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x))) + \tfrac{w^{-}(x)}{\sqrt{\rho(x)}}\right]\\ &+\tfrac{1}{\lambda}\left(\tfrac{H}{\beta}R^{+}+\tfrac{\gamma}{\alpha}\right)\left(\lambda\alpha - \tfrac{1}{\alpha}p(x)\right)\cos\left[\lambda(\sigma(\pi)-\sigma(x)) - \tfrac{w^{+}(x)}{\sqrt{\rho(x)}}\right]\\ &+\tfrac{1}{\lambda}\left(\tfrac{H}{\beta}R^{-}+\tfrac{\gamma}{\alpha}\right)\left(\lambda\alpha - \tfrac{1}{\alpha}p(x)\right)\cos\left[\lambda(\beta\pi-(\sigma(\pi)-\sigma(x))) + \tfrac{w^{-}(x)}{\sqrt{\rho(x)}}\right]\\ &+O\left(\lambda^{-1}\exp\left(|\tau|(\sigma(\pi)-\sigma(x))\right)\right)\end{aligned} \tag{11}$$

where $R^{\pm} = \dfrac{1}{2}\left(\dfrac{1}{a} \pm \dfrac{\beta a}{\alpha}\right)$.

Define

$$\langle \varphi(x,\lambda), \psi(x,\lambda)\rangle := \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda)$$

It is easy to verify that if $y(x)$ and $z(x)$ satisfy equations (1) and jump conditions (4), then $\langle y,z\rangle$ is independent of x, and

$$\langle y,z\rangle\big|_{x=\frac{\pi}{2}-0} = \langle y,z\rangle\big|_{x=\frac{\pi}{2}+0}$$

Denote

$$\Delta(\lambda) = \langle\varphi,\psi\rangle = V(\varphi) = -U(\psi) \tag{12}$$

The function $\Delta(\lambda)$ is called the characteristic function of L,which is entire in λ and it has an at most countable set of zeros $\{\lambda_n\}$,$n \in Z$. It follows from (3) and (4) that the characteristic function of the pencil L can be reduced

$$\Delta(\lambda) = \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda) \tag{13}$$

or

$$\Delta(\lambda) = \Delta_0(\lambda) + \int_0^{\sigma(\pi)} A(\pi,t)\cos\lambda t dt + \int_0^{\sigma(\pi)} B(\pi,t)\sin\lambda t dt \tag{14}$$

Where $\Delta_0(\lambda) = \varphi_0'(\pi,\lambda) + H\varphi_0'(\pi,\lambda)$. Denote by $G_\delta = \{\lambda : |\lambda-\lambda_n| \geq \delta, n \in Z\}$ with fixed $\delta > 0$. Then exist a constant $C_\delta > 0$ such that

$$|\Delta(\lambda)| \geq C_\delta(C+\beta(\lambda))\exp(|\tau|\sigma(\pi))\, for \lambda \in G_\delta \tag{15}$$

On here supposes that the function $q(x)$satisfies the additional condition

$$\int_0^{\pi}\left\{\left|y'(x)\right|^2 + q(x)\left|y(x)\right|^2\right\}dx > 0 \tag{16}$$

For all $y(x) \in W_2^2\left(\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]\right)$ such that $y(x) \neq 0$and

$$y'(0)\,\overline{y(0)} - y'(\pi)\,\overline{y(\pi)} = 0. \tag{17}$$

Lemma 2.1. *The following statements hold:*

i) The zeros $\{\lambda_n\}_{n\geq 0}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem L.

ii) The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are corresponding eigenfunctions and exists a sequence $\{\beta_n\}$, $\beta_n \neq 0$, $n = 0,1,2,...,$ such that

$$\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n). \tag{18}$$

Next, we denote by $L_2((0,\pi);\rho(x))$ a space which has the inner product

$$(\varphi,\psi) = \int_0^{\pi}\rho(x)\varphi(x,\lambda)\psi(x,\lambda)dx$$

Then it is shown in [2] that the eigenvalues of the boundary values problem L are real, nonzero, simple and does not have associated functions. Additionaly, eigenfunctions correspondings to different eigenvalues of the problem L are orthogonal in the sense of the equality

$$(\lambda_1 + \lambda_2)(\rho(x) y_1, y_2) - 2(\rho(x) y_1, y_2) = 0$$

Lemma 2.2. *The eigenvalues $\{k_n\}_{n\geq 0}$ of the problem L are real and simple. The eigenfunctions corresponding to the different eigenvalues are orthogonal in the weighted space $L_2((0,\pi);\rho(x))$ and for sufficiently large values of n, the eigenvalue k_n has the following behavior*

$$k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{k_n}{k_n^0} \tag{19}$$

where, λ_n^0 are zeros of $\Delta_0(\lambda) = \varphi_0'(\pi, \lambda) + H\varphi_0(\pi, \lambda)$, d_n is bounded and $k_n \in \ell_2$,

$$k_n^0 = \frac{n\pi}{\sigma(\pi)} + \theta_n, \ \sup_n |\theta_n| < +\infty$$

Proof of lemmas similarly to the proof of [7], so we omit the proof. Let α_n $(n \geq 0)$ be the normalized constants, which are defined as $\alpha_n := \int_0^\pi \rho(x)\varphi^2(x, \lambda_n)\,dx$ for all $n \geq 0$.

Lemma 2.3. *The following relation holds:*

$$\overset{\bullet}{\Delta}(k_n) = -2\alpha_n\beta_n k_n \tag{20}$$

where $\overset{\bullet}{\Delta}(k_n) = \left(\frac{d}{d\lambda}\Delta(\lambda)\right)_{k=k_n}$, $\beta_n = -[\varphi(\pi, k_n)]^{-1}$.

In particular, it follows from (19) that all eigenvalues k_n are simple.

Let be $\delta > 0$ and fixed. Define $G_\delta := \left\{k \in \mathbb{C} : \left|k - k_n^0\right| \geq \delta, n = 1, 2, ...\right\}$. The following inequality can be deduced using the asymptotic formula for $\Delta(\lambda)$,

$$\Delta_0(k) \geq c\,|k| \exp(|\tau|\,\sigma(\pi)), \ \ k \in G_\delta \tag{21}$$

for some pozitive constant c.

3. Main Results

Now we state the main result of this work. It is assumed in what follows that if a certain symbol s denotes an object related to L, then the corresponding symbol $\widetilde{s}$ with tilde denote the analogous object related to $\tilde{L}$.

Lemma 3.1. *If $\lambda_n = \widetilde{\lambda}_n$, $n = 0, 1, 2, ...$ then $\sigma(\pi) = \widetilde{\sigma}(\pi)$.*

Proof of Lemma is easily obtained from the asymptotic expression of λ_n.

Lemma 3.2. *If $k_n = \widetilde{k}_n$, $n = 0, 1, 2, ...$ then $a = \widetilde{a}$, $\alpha = \widetilde{\alpha}$, $\beta = \widetilde{\beta}$, $\rho(x) = \widetilde{\rho}(x)$, $h = \widetilde{h}$ and $H = \widetilde{H}$.*

Proof. Since, $k_n = \widetilde{k}_n$, $n = 0, 1, 2, ...$, Lemma 2.2 requires $\sigma(\pi) = \widetilde{\sigma}(\pi)$ or $\alpha + \beta = \widetilde{\alpha} + \widetilde{\beta}$. $\Delta(k), \widetilde{\Delta}(k)$ are entire functions of order one by Hadamard factorization theorem, for $\lambda \in \mathbb{C}$

$$\Delta(k) \equiv C\widetilde{\Delta}(k). \tag{22}$$

Then from Lemma 2.3 and $\sigma(\pi) = \widetilde{\sigma}(\pi)$ we obtain $C = 1$.

On the other hand, (22) can be written as

$$\Delta_0(k) - C\widetilde{\Delta}_0(k) = \left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right] \tag{23}$$

Hence

$$\begin{aligned}
&\left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right] = \\
&= -r^+ k\beta \sin k\sigma(\pi) + r^- k\beta \sin k(\alpha\pi - \sigma(\pi)) \\
&+ h\frac{\beta}{\alpha}\left[r^+ \cos k\sigma(\pi) - r^- \cos k(\alpha\pi - \sigma(\pi))\right] \\
&+ H\{r^+ \cos k\sigma(\pi) + r^- \cos k(\alpha\pi - \sigma(\pi)) \\
&+ \frac{h}{k\alpha}\left[r^+ \sin k\sigma(\pi) + r^- \sin k(\alpha\pi - \sigma(\pi))\right]\} \\
&- \{\widetilde{r}^+ k\beta \sin k\sigma(\pi) + \widetilde{r}^- k\beta \sin k(\alpha\pi - \sigma(\pi)) \\
&+ \widetilde{h}\frac{\beta}{\alpha}\left[\widetilde{r}^+ \cos k\sigma(\pi) - \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi))\right]\} \\
&- \widetilde{H}\{\widetilde{r}^+ \cos k\sigma(\pi) + \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi)) \\
&+ \frac{\widetilde{h}}{k\alpha}\left[\widetilde{r}^+ \sin k\sigma(\pi) + \widetilde{r}^- \sin k(\alpha\pi - \sigma(\pi))\right]\}
\end{aligned} \tag{24}$$

if we multiply both sides of (24) with $\sin k\sigma(\pi)$ and integrate with respect to k in (ε, T) (ε is sufficiently small pozitive number) for any pozitive real number T, then we get

$$\begin{aligned}
&\int_{\varepsilon}^{T} \left(\left[\widetilde{\Delta}(k) - \widetilde{\Delta}_0(k)\right] - \left[\Delta(k) - \Delta_0(k)\right]\right) \sin k\sigma dk = \\
&\int_{\varepsilon}^{T} \{-r^+ k\beta \sin k\sigma(\pi) + r^- k\beta \sin k(\alpha\pi - \sigma(\pi)) + h\frac{\beta}{\alpha}\left[r^+ \cos k\sigma(\pi) - r^- \cos k(\alpha\pi - \sigma(\pi))\right] \\
&+H\left[r^+ \cos k\sigma(\pi) - r^- \cos k(\alpha\pi - \sigma(\pi)) + \frac{h}{k\alpha}\left(r^+ \sin k\sigma(\pi) + r^- \sin k(\alpha\pi - \sigma(\pi))\right)\right] \\
&- \left[\widetilde{r}^+ k\beta \sin k\sigma(\pi) + \widetilde{r}^- k\beta \sin k(\alpha\pi - \sigma(\pi)) + \widetilde{h}\frac{\beta}{\alpha}\left(\widetilde{r}^+ \cos k\sigma(\pi) - \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi))\right)\right] \\
&-\widetilde{H}\left[\widetilde{r}^+ \cos k\sigma(\pi) + \widetilde{r}^- \cos k(\alpha\pi - \sigma(\pi)) + \frac{\widetilde{h}}{k\alpha}\left(\widetilde{r}^+ \sin k\sigma(\pi) + \widetilde{r}^- \sin k(\alpha\pi - \sigma(\pi))\right)\right]\Bigg\} \sin k\sigma dk
\end{aligned}$$

Since

$$\Delta(k) - \Delta_0(k) = O\left(k^{-2}\exp(|\tau|\sigma(\pi))\right), \widetilde{\Delta}(k) - \widetilde{\Delta}_0(k) = O\left(k^{-2}\exp(|\tau|\sigma(\pi))\right)$$

for all k in (ε, T)

$$\frac{\beta}{4}\widetilde{r}^+ - \frac{\beta}{4}r^+ = O(\frac{1}{T^2})$$

By letting T tend to infinity we see that

$$r^+ = \widetilde{r}^+ \tag{25}$$

Similarly, if we multiply both sides of (24) with $\sin k(\alpha\pi - \sigma(\pi))$ and integrate again with respect to k in (ε, T), and by letting T tend to infinity, then we get

$$r^- = \widetilde{r}^- \tag{26}$$

Taking $a > 0$ into account, (25) and (26) implies that $a = \widetilde{a}, \alpha = \widetilde{\alpha}, \beta = \widetilde{\beta}$.

Considering that Lemma 3.2, and $a = \widetilde{a}$, if both sides of the last expression are multiplied by the $\cos k\sigma(\pi)$ and integrate with respect to k in (ε, T) , then we get

$$h\frac{\beta}{\alpha}r^{+} + Hr^{+} = \widetilde{h}\frac{\beta}{\alpha}r^{+} + \widetilde{H}r^{+} \tag{27}$$

Similary, if we multiply both sides of the last expression are with $\cos k(\alpha\pi - \sigma(\pi))$ and integrate again with respect to k in (ε, T) , and by letting T tend to infinity, then we get

$$h\frac{\beta}{\alpha}r^{-} - Hr^{-} = \widetilde{h}\frac{\beta}{\alpha}r^{-} - \widetilde{H}r^{-} \tag{28}$$

Finaly, from (27) and (28) implies that $h = \widetilde{h}$ and $H = \widetilde{H}$. □

Theorem 3.3. *If for any* $n \in Z$, $\lambda_n = \tilde{\lambda}_n$,

$$\frac{y'(c_1, \lambda_n)}{y(c_2, \lambda_n)} = \frac{\tilde{y}'(c_1, \lambda_n)}{\tilde{y}(c_2, \lambda_n)} \tag{29}$$

Then $p(x) = \tilde{p}(x)$ *on* $[0, \pi]$, $q(x) = \tilde{q}(x)$ *a. e. on* $[0, \pi]$, *and* $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

Proof. Let $\varphi(x, \lambda)$ be the solution of the equations (1) satisfying the initial conditions $\varphi(0, \lambda) = 1, \varphi'(0, \lambda) = h$ and the jump conditions (4). Let $\tilde{\varphi}(x, \lambda)$be the solution of the equations

$$-\tilde{\varphi}''(x, \lambda) + [\tilde{q}(x) + 2\lambda\tilde{p}(x)]\tilde{\varphi}(x, \lambda) = \lambda^2\tilde{\rho}(x)\tilde{\varphi}(x, \lambda) \tag{30}$$

With the initial conditions

$$\tilde{\varphi}(0, \lambda) = 1,\ \tilde{\varphi}'(0, \lambda) = \tilde{h} \tag{31}$$

and the jump conditions (4). Multiplying (1) by $\tilde{\varphi}(x, \lambda)$ and (30) by $\varphi(x, \lambda)$, respectively, and subtracting, we get

$$\frac{d}{dx}[\tilde{\varphi}(x, \lambda)\varphi'(x, \lambda) - \tilde{\varphi}'(x, \lambda)\varphi(x, \lambda)] = [(q(x) - \tilde{q}(x)) + 2\lambda(p(x) - \tilde{p}(x))]\varphi(x, \lambda)\tilde{\varphi}(x, \lambda) \tag{32}$$

Integrating the above equality from 0 to c_1 with respect to x, using the initial conditions at $x = 0$ and Lemma 3.1, we have

$$\begin{aligned} H(\lambda) &= \int_0^{c_1}[(q(x) - \tilde{q}(x)) + 2\lambda(p(x) - \tilde{p}(x))]\varphi(x, \lambda)\tilde{\varphi}(x, \lambda)\,dx \\ &= \tilde{\varphi}(c_1, \lambda)\varphi'(c_1, \lambda) - \tilde{\varphi}'(c_1, \lambda)\varphi(c_1, \lambda) \end{aligned} \tag{33}$$

It follows from (6)-(7) that $H(\lambda)$ is an entire function of exponential type and there are some pozitive constant A and B such that

$$|H(\lambda)| \leq (A + B|\lambda|)\exp(|\tau|\sigma(\pi)) \quad for\, all\, \lambda \in C \tag{34}$$

From the assumption (29) we have

$$H(\lambda_n) = 0\ ,\ \ n \in Z \tag{35}$$

Define

$$F(\lambda) = \frac{H(\lambda)}{\Delta(\lambda)} \tag{36}$$

Which is entire function from the above arguments and it follows from (14) and (35) that

$$F(\lambda) = O(1)$$

For sufficiently large $|\lambda|, \lambda \in G_\delta$, thus, by liouville's theorem [4], we obtain for all λ that $F(\lambda) = C$.

Where c is a constant. Let us show that the constant $C = 0$. Based on (24) and (14), we can rewrite the equations $H(\lambda) = C\Delta(\lambda)$ in the form

$$2\lambda \int_0^{c_1} (p(x) - \tilde{p}(x))\, \varphi(x,\lambda)\, \tilde{\varphi}(x,\lambda)\, dx + \int_0^{c_1} (q(x) - \tilde{q}(x))\, \varphi(x,\lambda)\, \tilde{\varphi}(x,\lambda)\, dx$$
$$= C\left\{\Delta_0(\lambda) + \int_0^{\sigma(\pi)} A(\pi,t)\cos\lambda t dt + \int_0^{\sigma(\pi)} B(\pi,t)\sin\lambda t dt\right\}$$

By use of Riemann-Lebesgue lemma [4], we see that the limit of the left-hand side of the above equality exists as $\lambda \to \infty$, $\lambda \in R$ thus we obtain that $C = 0$. So we have $H(\lambda) = 0$ for all $\lambda \in C$.
As already mentioned, if $H(\lambda) = 0$ for all $\lambda \in C$, then from (33) we have
$\tilde{\varphi}(c_1,\lambda)\, \varphi'(c_1,\lambda) - \tilde{\varphi}'(c_1,\lambda)\, \varphi(c_1,\lambda) = 0$ for all $\lambda \in C$
so
$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)}$ for all $\lambda \in C$.
The function $M(\lambda) := \frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}$is the Weyl function of the boundary value problem for equation (1) on $(0,c_1)$with boundary conditoons $V(y) = 0$, $y'(c_1) = 0$ and without jump conditions.
By [2], the Weyl function uniquely species $p(x)$ and $q(x)$a.e. on $(0,c_1)$and the coefficients in boundary and jump conditions and $\rho(x)$. □

Theorem 3.4. *If for any* $n \in Z$, $\lambda_n = \tilde{\lambda}_n$, $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$, $p(x) = \tilde{p}(x)$ *and* $q(x) = \tilde{q}(x)$ *on* $\left(0, \frac{\alpha+\beta}{4}\pi\right)$, *then* $p(x) = \tilde{p}(x)$ *and* $q(x) = \tilde{q}(x)$ *a.e. on* $\left(\frac{\alpha+\beta}{4}\pi, \frac{\alpha+\beta}{2}\pi\right)$ *and* $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$

.

Proof. Let the boundary value problems $L$and $\tilde{L}$ satisfy the conditions of Teorem 3.4, then by virtue of Lemma 2.4 and Lemma 3.2 $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$ and $\rho(x) = \tilde{\rho}(x)$. For brevity, denote $c_1 = \frac{\alpha+\beta}{4}\pi$, $c_2 = \frac{\alpha+\beta}{2}\pi$.
Let $\psi(x,\lambda)$, $\tilde{\psi}(x,\lambda)$ be the solutions of the equations

$$-\psi''(x,\lambda) + [q(x) + 2\lambda p(x)]\, \psi(x,\lambda) = \lambda^2 \rho(x)\, \psi(x,\lambda) \tag{37}$$

$$-\tilde{\psi}''(x,\lambda) + [\tilde{q}(x) + 2\lambda \tilde{p}(x)]\, \tilde{\psi}(x,\lambda) = \lambda^2 \tilde{\rho}(x)\, \tilde{\psi}(x,\lambda) \tag{38}$$

With the initial conditions, respectively

$$\psi(\pi,\lambda) = 1\, ,\ \psi'(\pi,\lambda) = -H \tag{39}$$

$$\tilde{\psi}(\pi,\lambda) = 1\, ,\ \tilde{\psi}'(\pi,\lambda) = -\tilde{H} \tag{40}$$

and the jump conditions (4). After multipliying (37) by $\tilde{\psi}(x,\lambda)$ and (38) by $\psi(x,\lambda)$, we subtract these equations from each other. Then by integrating on$[c_1,\pi]$ with respect to x, using the initial conditions (39) and (40)and jump conditions (4), we have

$$\int_{c_1}^{\pi} [(q(x) - \tilde{q}(x)) + 2\lambda (p(x) - \tilde{p}(x))]\, \psi(x,\lambda)\, \tilde{\psi}(x,\lambda)\, dx = \tilde{\psi}(c_1,\lambda)\, \psi'(c_1,\lambda) - \tilde{\psi}'(c_1,\lambda)\, \psi(c_1,\lambda) \tag{41}$$

From the hypothesis $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $(0,c_1)$.
Denote $Q(x) = q(x) - \tilde{q}(x)$, $P(x) = p(x) - \tilde{p}(x)$ and

$$F_0(\lambda) = 2\lambda \int_{c_1}^{\pi} P(x)\, \psi(x,\lambda)\, \tilde{\psi}(x,\lambda)\, dx + \int_{c_1}^{\pi} Q(x)\, \psi(x,\lambda)\, \tilde{\psi}(x,\lambda)\, dx \tag{42}$$

It follows from (10) and (41) that $F_0(\lambda)$ is an entire function of exponential type and there are some pozitive constants A_1 and B_1 such that

$$|F_0(\lambda)| \le (A_1 + B_1 |\lambda|) \exp(|\tau|\, \sigma(\pi)) \quad \textit{for all}\ \lambda \in C \tag{43}$$

It is clear from the properties $\psi(x,\lambda)$, $\tilde{\psi}(x,\lambda)$ and the boundary conditions (2)

$$F_0(\lambda_n) = 0\ ,\ n \in Z \tag{44}$$

Define

$$F(\lambda) := \frac{F_0(\lambda)}{\Delta(\lambda)}$$

Which is an entire function from the above arguments and it follows from (15) and (43) that

$$F(\lambda) = O(1)$$

For sufficiently large $|\lambda|, \lambda \in G_\delta$. Using Liouville's theorem [4], we obtain for all λ that $F(\lambda) = C$.
Where C is a constant. Let us Show that the constant $C = 0$. We can rewrite the equations $F_0(\lambda) = C\Delta(\lambda)$ as

$$\begin{aligned}
&2\lambda \int_{c_1}^{\pi} P(x)\,\psi(x,\lambda)\,\tilde{\psi}(x,\lambda)\,dx + \int_{c_1}^{\pi} Q(x)\,\psi(x,\lambda)\,\tilde{\psi}(x,\lambda)\,dx \\
&= -a^+C\left(\lambda\beta - \tfrac{1}{\beta}p(\pi)\right)\sin\left[\lambda\sigma(\pi) - \tfrac{w^+(\pi)}{\beta}\right] \\
&+a^-C\left(\lambda\beta - \tfrac{1}{\beta}p(\pi)\right)\sin\left[\lambda(\alpha\pi - \sigma(\pi)) + \tfrac{w^-(\pi)}{\beta}\right] \\
&+Ha^+C\cos\left[\lambda\sigma(\pi) - \tfrac{w^+(\pi)}{\beta}\right] + Ha^-C\cos\left[\lambda(\alpha\pi - \sigma(\pi)) + \tfrac{w^-(\pi)}{\beta}\right] \\
&+O(\exp(|\tau|\,\sigma(\pi)))
\end{aligned}$$

By use of Riemann-Lebesgue lemma [4], we see that the limit of the left-hand side of the above equality exists as $\lambda \to \infty$, $\lambda \in R$. Therefore, we get that $C = 0$. So, we have $F_0(\lambda) = 0$ for all $\lambda \in C$.
Then, from teh equality (41) we obtain
$\tilde{\psi}(c_1,\lambda)\,\psi'(c_1,\lambda) - \tilde{\psi}'(c_1,\lambda)\,\psi(c_1,\lambda) = 0$ for all $\lambda \in C$. Hence,

$$\frac{\psi(c_1,\lambda)}{\psi'(c_1,\lambda)} = \frac{\tilde{\psi}(c_1,\lambda)}{\tilde{\psi}'(c_1,\lambda)}. \tag{45}$$

Note that $M(\lambda) := -\frac{\psi(c_1,\lambda)}{\psi'(c_1,\lambda)}$is the Weyl function, defined [2], of the boundary value problem for equation (1) on the interval $(c_1,\pi)$with the boundary conditoons $V(y) = 0$, $y'(c_1) = 0$ and jump conditions (4). It has been show in [2] that the Weyl function species the function $p(x)$ and $q(x)$on (c_1,π), consequently on (c_1,c_2). Theorem is proved. □

Corollary.
If for any $n \in Z$, $\lambda_n = \tilde{\lambda}_n$, $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$, $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $(0,c_1)$, then $p(x) = \tilde{p}(x)$ on $(0,\pi)$ and $q(x) = \tilde{q}(x)$ a.e. on $(0,\pi)$ and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

Theorem 3.5. *If $\lambda_n = \tilde{\lambda}_n$ for all $n \in Z$,, $\frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}}$, $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on $\left(\frac{\alpha+\beta}{4}\pi, \frac{\alpha+\beta}{2}\pi\right)$, then $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ a.e. on $\left(0, \frac{\alpha+\beta}{4}\pi\right)$ and $\left(\frac{\alpha+\beta}{2}\pi, \pi\right)$ and$\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.*

Proof. By the Lemma 3.1 and the condition of Teorem 3.5, we have $h = \tilde{h}$, $H = \tilde{H}$, $a = \tilde{a}$, $\rho(x) = \tilde{\rho}(x)$ and $p(x) = \tilde{p}(x)$ and $q(x) = \tilde{q}(x)$ on (c_1,c_2).
Let

$$-\varphi''(x,\lambda) + [q(x) + 2\lambda p(x)]\,\varphi(x,\lambda) = \lambda^2\rho(x)\,\varphi(x,\lambda) \tag{46}$$

$$-\tilde{\varphi}''(x,\lambda) + [\tilde{q}(x) + 2\lambda\tilde{p}(x)]\,\tilde{\varphi}(x,\lambda) = \lambda^2\tilde{\rho}(x)\,\tilde{\varphi}(x,\lambda) \tag{47}$$

With the initial conditions, respectively

$$\varphi(0,\lambda) = 1\,,\ \varphi'(0,\lambda) = h \tag{48}$$

$$\tilde{\varphi}(0,\lambda) = 1\,,\ \tilde{\varphi}'(0,\lambda) = \tilde{h} \tag{49}$$

and the jump conditions (4). Multipliying (46) by $\tilde{\varphi}(x,\lambda)$ and (47) by $\varphi(x,\lambda)$, we subtract these equations from each other. Then by integrating on$[0,c_2]$ with respect to x, using the initial conditions (48) and (49) and jump conditions (4), we have

$$\begin{aligned} H(\lambda) &= 2\lambda\int_0^{c_1} P(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx + \int_0^{c_1} Q(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)\,dx \\ &= \varphi'(c_1,\lambda)\tilde{\varphi}(c_1,\lambda) - \tilde{\varphi}'(c_1,\lambda)\varphi(c_1,\lambda) \end{aligned} \tag{50}$$

From the hypothesis $p(x)=\tilde{p}(x)$ and $q(x)=\tilde{q}(x)$ on (c_1,c_2). Similarly to proof of Theorem 3.5, we have that $H(\lambda)=0$ for all $\lambda\in C$. Then, from equality
$\varphi'(c_1,\lambda)\tilde{\varphi}(c_1,\lambda)-\tilde{\varphi}'(c_1,\lambda)\varphi(c_1,\lambda)=0$ for all $\lambda\in C$.
so

$$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)}.$$

The function $M(\lambda) := -\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}$is the Weyl function of the boundary value problem for the equation (1) on $(0,c_1)$with boundary conditoons $V(y)=0$, $y'(c_1)=0$ and without jump conditions (4) (*see*[2]). By [2], the Weyl function uniquely species $p(x)$ and $q(x)$a.e. on $(0,c_1)$. Next, now using Theorem 3.6 we obtain $p(x)=\tilde{p}(x)$and $q(x)=\tilde{q}(x)$ a.e. on (c_2,π). Theorem is proved.
□

4. An interior inverse problems.

We cconsider the interior inverse problem for the same boundary problem L and obtain the corresponding result.

Theorem 4.1. *If* $\lambda_n=\tilde{\lambda}_n$ *for all* $n\in Z$, $\frac{\alpha}{\beta}=\frac{\tilde{\alpha}}{\tilde{\beta}}$, *and*

$$\frac{y(c_1,\lambda_n)}{y'(c_1,\lambda_n)} = \frac{\tilde{y}(c_1,\lambda_n)}{\tilde{y}'(c_1,\lambda_n)} \tag{51}$$

, then $p(x)=\tilde{p}(x)$ *on* $[0,\pi]$, $q(x)=\tilde{q}(x)$ *a.e. on* $[0,\pi]$*and* $\rho(x)=\tilde{\rho}(x)$, $a=\tilde{a}$, $h=\tilde{h}$, $H=\tilde{H}$.

Proof. Let $\varphi(x,\lambda)$ be the solution of the equations (1) satisfying the initial conditions $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=h$ and jump conditions (4). Firstly, the assumption that $\lambda_n=\tilde{\lambda}_n$ and $\frac{\alpha}{\beta}=\frac{\tilde{\alpha}}{\tilde{\beta}}$ can determine $\rho(x)=\tilde{\rho}(x)$, $a=\tilde{a}$, $h=\tilde{h}$, $H=\tilde{H}$ by Lemma 3.1 the other hand from (50), we see that

$$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)},$$

Then from (50), the entire function $H(\lambda)$ has zeros $\{\lambda_n\}$, $n\in Z$, i.e. $H(\lambda_n)=0$. Similarly to the proof of Theorem4, we have that $p(x)=\tilde{p}(x)$ and $q(x)=\tilde{q}(x)$ on $(0,c_1)$. Once we get that $p(x)=\tilde{p}(x)$ and $q(x)=\tilde{q}(x)$, by Corollary of Theorem 3.4 we have that $p(x)=\tilde{p}(x)$ on $[0,\pi]$, $q(x)=\tilde{q}(x)$ a.e. on $[0,\pi]$. Theorem is proved. □

Theorem 4.2. *Let* $m(n)$*be a sequence of integers such that* $\inf\limits_{n\in Z}\frac{m(n)}{\lambda_n}\le 1$
(i) If for any $n\in Z$,

$$\lambda_{m(n)}=\tilde{\lambda}_{m(n)},\ \frac{y\left(c_1,\lambda_{m(n)}\right)}{y'\left(c_1,\lambda_{m(n)}\right)} = \frac{\tilde{y}\left(c_1,\lambda_{m(n)}\right)}{\tilde{y}'\left(c_1,\lambda_{m(n)}\right)}\ \text{and}\ \frac{\alpha}{\beta}=\frac{\tilde{\alpha}}{\tilde{\beta}} \tag{52}$$

Then $p(x)=\tilde{p}(x)$ *on* $(0,c_1)$ *and* $q(x)=\tilde{q}(x)$ *a.e. on* $(0,c_1)$ *and* $\rho(x)=\tilde{\rho}(x)$, $a=\tilde{a}$, $h=\tilde{h}$, $H=\tilde{H}$.

(*ii*) *If for any* $n \in Z$,

$$\lambda_{m(n)} = \tilde{\lambda}_{m(n)}, \frac{y\left(c_2, \lambda_{m(n)}\right)}{y'\left(c_2, \lambda_{m(n)}\right)} = \frac{\tilde{y}\left(c_2, \lambda_{m(n)}\right)}{\tilde{y}'\left(c_2, \lambda_{m(n)}\right)} \textit{ and } \frac{\alpha}{\beta} = \frac{\tilde{\alpha}}{\tilde{\beta}} \tag{53}$$

Then $p(x) = \tilde{p}(x)$ *on* (c_2, π)*and* $q(x) = \tilde{q}(x)$ *a.e. on* (c_2, π) *and* $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$.

Proof. (*i*) from the assumption (52) and (50) we have

$$\varphi'\left(c_1, \lambda_{m(n)}\right)\tilde{\varphi}\left(c_1, \lambda_{m(n)}\right) - \tilde{\varphi}'\left(c_1, \lambda_{m(n)}\right)\varphi\left(c_1, \lambda_{m(n)}\right) = 0$$

Which means

$$H\left(\lambda_{m(n)}\right) = 0\,,\ n \in Z \tag{54}$$

Next, we shall show that $H(\lambda) = 0$ on the whole λ-plane. From (50) and (6) on has

$$|H(\lambda)| \leq (A + Br)\, e^{2c_1 r|\sin\theta|} \tag{55}$$

For some pozitive costants A and B, where $\lambda = re^{i\theta}$. Moreover, we see that the entire function $H_1(\lambda)$ is a function of exponential type less than $2c_1$.
Define the indicator of function $H_1(\lambda)$ by

$$h(\theta) = \lim_{r\to\infty}\sup \frac{\ln\left|H_1\left(re^{i\theta}\right)\right|}{r} \tag{56}$$

One obtain the following estimate from (55)and (56) that $h(\theta) \leq 2c_1|\sin\theta|$.
Let us denote by $n(r)$the number of zeros of $H_1(\lambda)$ in the disk $|\lambda| \leq r$. From the equations (4.4), the assimption of (52) and known asymtotic expreession of the eigenvalues λ_n, we have the following estimate for the number of zeros of $H_1(\lambda)$ in the disk $|\lambda| \leq r$.

$$n(r) = 1 + 2\left[\sigma r(1 + \varepsilon(r))\right] = 2\sigma r(1 + \varepsilon(r))\,.$$

Here $\varepsilon(r) \to 0\, for\, r \to \infty$, $\sigma$is number such that $\sigma > \frac{\alpha+\beta}{2} = \frac{2c_1}{\pi}$and $[x]$is the integer part of x. It follows that in the case under consideration

$$\lim_{r\to\infty}\frac{n(r)}{r} = 2\sigma > \frac{4c_1}{\pi} = \frac{c_1}{\pi}\int_0^{2\pi}|\sin\theta|\,d\theta \geq \frac{1}{2\pi}\int_0^{2\pi}h(\theta)\,d\theta \tag{57}$$

To complate the proof we have to recall the following theorem [4]: the set of zeros of every entire function of the exponential type, not identically zero, satisfy the inequality

$$\lim_{r\to\infty}\inf\frac{n(r)}{r} \leq \frac{1}{2\pi}\int_0^{2\pi}h(\theta)\,d\theta \tag{58}$$

Inequalities (57) and (58) implay that $H_1(\lambda) \equiv 0$ on the whole λ-plane. As already mentioned, if $H_1(\lambda) \equiv 0$, then from (52) we have

$$\tilde{\varphi}(c_1, \lambda)\,\varphi'(c_1, \lambda) - \tilde{\varphi}'(c_1, \lambda)\,\varphi(c_1, \lambda) = 0$$

so
$\frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)} = \frac{\tilde{\varphi}(c_1,\lambda)}{\tilde{\varphi}'(c_1,\lambda)}$ on the whole λ-plane.
The function $M(\lambda) := \frac{\varphi(c_1,\lambda)}{\varphi'(c_1,\lambda)}$is the Weyl function of the boundary value problem for the equation (1) on $(0, c_1)$with boundary conditoons $U(y) = 0$, $y'(c_1) = 0$ and without jump conditions (4) (*see*[2]). By [2], the Weyl function uniquely species $p(x)$ and $q(x)$a.e. on $(0, c_1)$ and coefficient h.
(*ii*)

To prove that $p(x) = \tilde{p}(x)$ on $(c_2, \pi)$and $q(x) = \tilde{q}(x)$ a.e. on (c_2, π) and $\rho(x) = \tilde{\rho}(x)$, $a = \tilde{a}$, $h = \tilde{h}$, $H = \tilde{H}$. We will consider the supplementary problem L

$$\begin{cases} -y'' + [q_1(x) + 2\lambda p_1(x)]\, y = \lambda^2 \rho(x)\, y\ ,\ x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \\ y(0) - Hy(0) = 0 \\ y(\pi) - hy(\pi) = 0 \\ y\left(\frac{\pi}{2} + 0\right) = a^{-1} y\left(\frac{\pi}{2} - 0\right) \\ y'\left(\frac{\pi}{2} + 0\right) = ay'\left(\frac{\pi}{2} - 0\right) + \gamma\left(\frac{\pi}{2} - 0\right) \end{cases}$$

Where $q_1(x) = q(\pi - x)$ and $p_1(x) = p(\pi - x)$. A direct calculation implies that $\hat{y}_n := y_n(\pi - x)$ is the solution to the supplementary problem $\hat{L}$ and $\hat{y}_n(\pi - x) = y_n(c_2)$. Note that $\pi - c_2 \in \left(0, \frac{\pi}{2}\right)$. Thus the assmption conditions for $\hat{L}$ in the case (i) are still satisfied. Repeting the above arguments we can obtain the proof of this Theorem 4.2. □

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