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Inequalities for strongly convex functions via Atangana-Baleanu Integral Operators

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Abstract. In this study, new results are generated for strongly convex functions with the help of Atangana-Baleanu integral operators.

1. Introduction

We will start by remembering the definitions of convex and strongly convex functions respectively. Let I be an interval in $\mathbb{R}$. Then $\rho : I \to \mathbb{R}$ is said to be convex if for all $n_1, n_2 \in I$ and all $t \in [0,1]$,

$$\rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) \tag{1}$$

holds. If the inequality in (1) is reversed, then ρ is said to be concave (See [36]).

Recall that a function $\rho : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus $c > 0$ if

$$\rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) - ct(1-t)(n_1 - n_2)^2, \tag{2}$$

for all $n_1, n_2 \in I$ and $t \in [0,1]$ (See [37]).

The following inequality is the Hermite-Hadamard inequality that has an important place for convex functions.

Theorem 1.1. *Suppose that $\rho : I \subset \mathbb{R} \to \mathbb{R}$ is convex function on $I \subseteq \mathbb{R}$ where $n_1, n_2 \in I$, with $n_1 < n_2$. The following double inequality is called Hermite-Hadamard's inequality for convex functions (See [36]):*

$$\rho\left(\frac{n_1 + n_2}{2}\right) \leq \frac{1}{n_2 - n_1} \int_{n_1}^{n_2} \rho(n)\, dn \leq \frac{\rho(n_1) + \rho(n_2)}{2}. \tag{3}$$

In [27], Merentes and Nikodem obtained the following inequality. This inequality is important as being a counterpart of the Hermite-Hadamard inequality for strongly convex functions.

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Theorem 1.2. *If a function $\rho : I \to \mathbb{R}$ is strongly convex with modulus c then*

$$\rho\left(\frac{n_1+n_2}{2}\right)+\frac{c}{12}(n_1-n_2)^2 \leq \frac{1}{n_2-n_1}\int_{n_1}^{n_2}\rho(n)\,dn \leq \frac{\rho(n_1)+\rho(n_2)}{2}-\frac{c}{6}(n_1-n_2)^2, \tag{4}$$

for all $n_1, n_2 \in I$, $n_1 < n_2$.

In [30], Ostrowski proved Ostrowski's inequality which is another important inequality in the theory of inequalities as the following:

Theorem 1.3. *Let ρ be a differentiable function on (n_1, n_2) and let, on (n_1, n_2), $|\rho'(n)| \leq K$. Then, for every $n \in (n_1, n_2)$*

$$\left|\rho(n)-\frac{1}{n_2-n_1}\int_{n_1}^{n_2}\rho(t)dt\right| \leq \left[\frac{1}{4}+\frac{\left(n-\frac{n_1+n_2}{2}\right)^2}{(n_2-n_1)^2}\right](n_2-n_1)K. \tag{5}$$

We recommend to see the studies [11], [12], [14], [16], [17], [21]-[29], [32]-[36] and [41]-[43] for results that include convex functions, strongly convex functions, Hermite-Hadamard and Ostrowski inequalities.

Now, we will give some of the derivative and integral operators.

Definition 1.4. *(See [15]) Let $\rho \in H^1(0, n_2)$, $n_2 > n_1$, $\xi \in [0, 1]$ then, the definition of the new Caputo fractional derivative is:*

$$^{CF}D^{\xi}\rho(t)=\frac{M(\xi)}{1-\xi}\int_{n_1}^{t}\rho'(s)exp\left[-\frac{\xi}{(1-\xi)}(t-s)\right]ds \tag{6}$$

where $M(\xi)$ is normalization function.

The integral operator associated to this fractional derivative has been given with a non-singular kernel structure as follows.

Definition 1.5. *(See [2]) Let $\rho \in H^1(0, n_2)$, $n_2 > n_1$, $\xi \in [0, 1]$ then, the definition of the left and right side of Caputo-Fabrizio fractional integral is:*

$$\left({}^{CF}_{n_1}I^{\xi}\rho\right)(t)=\frac{1-\xi}{B(\xi)}\rho(t)+\frac{\xi}{B(\xi)}\int_{n_1}^{t}\rho(y)dy,$$

and

$$\left({}^{CF}I^{\xi}_{n_2}\rho\right)(t)=\frac{1-\xi}{B(\xi)}\rho(t)+\frac{\xi}{B(\xi)}\int_{t}^{n_2}\rho(y)dy$$

where $B(\xi)$ is normalization function.

Atangana and Baleanu have defined the following fractional derivative and integral operators.

Definition 1.6. *(See [7]) Let $\rho \in H^1(n_1, n_2)$, $n_2 > n_1$, $\xi \in [0, 1]$ then, the definition of the new fractional derivative is given:*

$${}^{ABC}_{n_1}D^{\xi}_{t}[\rho(t)]=\frac{B(\xi)}{1-\xi}\int_{n_1}^{t}\rho'(x)E_{\xi}\left[-\xi\frac{(t-x)^{\xi}}{(1-\xi)}\right]dx. \tag{7}$$

Definition 1.7. *(See [7]) Let $\rho \in H^1(n_1, n_2)$, $n_2 > n_1$, $\xi \in [0, 1]$ then, the definition of the new fractional derivative is given:*

$${}^{ABR}_{n_1}D^{\xi}_{t}[\rho(t)]=\frac{B(\xi)}{1-\xi}\frac{d}{dt}\int_{n_1}^{t}\rho(x)E_{\xi}\left[-\xi\frac{(t-x)^{\xi}}{(1-\xi)}\right]dx. \tag{8}$$

The associated integral operator is presented as follows.

Definition 1.8. *(See [7]) The fractional integral associate to the new fractional derivative with non-local kernel of a function $\rho \in H^1(n_1, n_2)$ as defined:*

$$ {}^{AB}_{n_1}I^{\xi}_{t}\{\rho(t)\} = \frac{1-\xi}{B(\xi)}\rho(t) + \frac{\xi}{B(\xi)\Gamma(\xi)}\int_{n_1}^{t}\rho(u)(t-u)^{\xi-1}du $$

where $n_2 > n_1, \xi \in [0,1]$.

In [1], the authors have given the right hand side of integral operator as following;

$$ \left({}^{AB}I^{\xi}_{n_2}\right)\{\rho(t)\} = \frac{1-\xi}{B(\xi)}\rho(t) + \frac{\xi}{B(\xi)\Gamma(\xi)}\int_{t}^{n_2}\rho(u)(u-t)^{\xi-1}du. $$

Here, $\Gamma(\xi)$ is the Gamma function. Since the normalization function $B(\xi) > 0$ is positive, it immediately follows that the fractional Atangana-Baleanu integral of a positive function is positive. It should be noted that, when the order $\xi \longrightarrow 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\xi \longrightarrow 0$.

We recommend to see the studies [2]-[6], [8], [9], [13], [18]-[20], [31] and [38]-[40] for results that include fractional operators.

In this study, we obtained Hermite-Hadamard and Ostrowski type inequalities for strongly convex functions with the help of Atangana-Baleanu integral operators. In addition, we obtained inequalities on the product of convex and strongly convex functions and the product of two strongly convex functions via Atangana-Baleanu integral operators.

2. New results for strongly convex functions

Theorem 2.1. *Let $\rho : I \to \mathbb{R}$ be a strongly convex function with modulus c $(c > 0)$. If $\rho \in L[n_1, n_2]$, for all $n_1, n_2 \in I$, $n_1 < n_2$ following inequality which involves Atangana-Baleanu integral operators holds*

$$ \begin{aligned} & \frac{1}{(n_2-n_1)^{\xi}}\left({}^{AB}_{n_1}I^{\xi}_{n_2}\{\rho(n_2)\}\right) \\ \leq\ & \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\rho(n_1)}{\xi+1} + \frac{\rho(n_2)}{\xi(\xi+1)} - \frac{c(n_2-n_1)^2}{(\xi+1)(\xi+2)}\right] + \frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho(n_2) \end{aligned} \tag{9} $$

where $\xi \in (0,1]$, $B(\xi)$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

Proof. Since ρ is strongly convex function, we can write

$$ \rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) - ct(1-t)(n_2-n_1)^2 \tag{10} $$

for all $n_1, n_2 \in I$, $n_1 < n_2$ and $t \in [0,1]$. If we multiply the both sides of (10) with $t^{\xi-1}$, and after that if we integrate the resulting inequality on $[0,1]$ over t, we obtain

$$ \begin{aligned} & \int_0^1 t^{\xi-1}\rho(tn_1 + (1-t)n_2)dt \\ \leq\ & \int_0^1 t^{\xi-1}\left[t\rho(n_1) + (1-t)\rho(n_2) - ct(1-t)(n_2-n_1)^2\right]dt \\ =\ & \frac{\rho(n_1)}{\xi+1} + \frac{\rho(n_2)}{\xi(\xi+1)} - \frac{c(n_2-n_1)^2}{(\xi+1)(\xi+2)}. \end{aligned} \tag{11} $$

By changing the variable $tn_1 + (1-t)n_2 = u$, we can write the inequality in (11) as

$$\begin{aligned} & \frac{1}{(n_2-n_1)^{\xi}} \int_{n_1}^{n_2} (n_2-u)^{\xi-1} \rho(u)du \\ \leq \;& \frac{\rho(n_1)}{\xi+1} + \frac{\rho(n_2)}{\xi(\xi+1)} - \frac{c(n_2-n_1)^2}{(\xi+1)(\xi+2)}. \end{aligned} \tag{12}$$

If we multiply the both sides of (12) by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and if we add the term $\frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho(n_2)$ to the both sides of (12), we get the inequality in (9). □

Remark 2.2. *If we choose $\xi = 1$ in Theorem 2.1, we obtain the right hand side of (4).*

Theorem 2.3. *Suppose that $\rho, \sigma : I \subset \mathbb{R} \to [0,\infty)$ are convex and strongly convex (with modulus c, $c > 0$) functions on I respectively where $n_1, n_2 \in I$, $n_1 < n_2$. If $\rho\sigma \in L[n_1, n_2]$, we have the following inequality*

$$\begin{aligned} & \frac{1}{(n_2-n_1)^{\xi}} \left({}^{AB}_{n_1}I^{\xi}_{n_2} \{\rho\sigma(n_2)\} \right) \\ \leq \;& \frac{\xi}{B(\xi)\Gamma(\xi)} \left\{ \frac{1}{\xi+2} \left[\rho(n_1)\sigma(n_1) + \frac{2}{\xi(\xi+1)}\rho(n_2)\sigma(n_2) \right] \right. \\ & + \frac{1}{(\xi+1)(\xi+2)} \left[\rho(n_1)\sigma(n_2) + \rho(n_2)\sigma(n_1) \right] \\ & \left. -c(n_2-n_1)^2 \frac{1}{(\xi+2)(\xi+3)} \left[\rho(n_1) + \frac{2\rho(n_2)}{\xi+1} \right] \right\} + \frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho\sigma(n_2) \end{aligned} \tag{13}$$

where $\xi \in (0,1]$, $B(\xi)$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

Proof. If we consider the definitions of convex function and strongly convex function we can write

$$\rho(tn_1 + (1-t)n_2) \leq t\rho(n_1) + (1-t)\rho(n_2) \tag{14}$$

and

$$\sigma(tn_1 + (1-t)n_2) \leq t\sigma(n_1) + (1-t)\sigma(n_2) - ct(1-t)(n_2-n_1)^2 \tag{15}$$

for all $n_1, n_2 \in I$ and $t \in [0,1]$. If we multiply the inequalities in (14) and (15) side by side, we obtain

$$\begin{aligned} & \rho(tn_1 + (1-t)n_2)\sigma(tn_1 + (1-t)n_2) \\ \leq \;& t^2\rho(n_1)\sigma(n_1) + t(1-t)\rho(n_1)\sigma(n_2) - ct^2(1-t)(n_2-n_1)^2\rho(n_1) \\ & +t(1-t)\sigma(n_1)\rho(n_2) + (1-t)^2\rho(n_2)\sigma(n_2) - ct(1-t)^2(n_2-n_1)^2\rho(n_2). \end{aligned} \tag{16}$$

Similar to the steps in the proof of the Theorem 2.1, if we multiply both sides of (16) by $t^{\xi-1}$, and after that if we integrate the resulting inequality on $[0,1]$ over t, we obtain

$$\begin{aligned} & \int_0^1 t^{\xi-1}\rho(tn_1 + (1-t)n_2)\sigma(tn_1 + (1-t)n_2)dt \\ \leq \;& \frac{1}{\xi+2} \left[\rho(n_1)\sigma(n_1) + \frac{2}{\xi(\xi+1)}\rho(n_2)\sigma(n_2) \right] \\ & + \frac{1}{(\xi+1)(\xi+2)} \left[\rho(n_1)\sigma(n_2) + \rho(n_2)\sigma(n_1) \right] \\ & -c(n_2-n_1)^2 \frac{1}{(\xi+2)(\xi+3)} \left[\rho(n_1) + \frac{2\rho(n_2)}{\xi+1} \right]. \end{aligned} \tag{17}$$

If we change the variable for the left hand side of inequality in (17), and after this operation if we multiply the both sides of resulting inequality with $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and if we add the term $\frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho\sigma(n_2)$, we get the inequality in (13). □

Remark 2.4. *If we choose $\xi = 1$ in Theorem 2.3, we obtain the inequality in Theorem 2.14 in [41].*

Theorem 2.5. *Suppose that $\rho, \sigma : I \subset \mathbb{R} \to [0, \infty)$ are strongly convex functions with modulus c $(c > 0)$ on I respectively where $n_1, n_2 \in I$, $n_1 < n_2$. If $\rho\sigma \in L[n_1, n_2]$, we have the following inequality*

$$\begin{aligned}
& \frac{1}{(n_2-n_1)^{\xi}}\left({}^{AB}_{n_1}I^{\xi}_{n_2}\{\rho\sigma(n_2)\}\right) \\
\leq\ & \frac{\xi}{B(\xi)\Gamma(\xi)}\left\{\frac{1}{\xi+2}\left[\rho(n_1)\sigma(n_1)+\frac{2}{\xi(\xi+1)}\rho(n_2)\sigma(n_2)\right]\right. \\
& +\frac{1}{(\xi+1)(\xi+2)}\left[\rho(n_1)\sigma(n_2)+\rho(n_2)\sigma(n_1)\right] \\
& \left.-\frac{c(n_2-n_1)^2}{(\xi+2)(\xi+3)}\left[\rho(n_1)+\frac{2\rho(n_2)}{\xi+1}+\sigma(n_1)+\frac{2\sigma(n_2)}{\xi+1}\right]+c^2(n_2-n_1)^4\frac{2}{(\xi+2)(\xi+3)(\xi+4)}\right\} \\
& +\frac{1-\xi}{(n_2-n_1)^{\xi}B(\xi)}\rho\sigma(n_2)
\end{aligned} \tag{18}$$

where $\xi \in (0, 1]$, $B(\xi)$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

Proof. Theorem 2.5 can be proved similar to the proof of Theorem 2.3. It is left to the interested reader. □

Remark 2.6. *If we choose $\xi = 1$ in Theorem 2.5, we obtain the inequality in Theorem 2.11 in [41].*

Now, we will give some results by using the following lemma which is the first lemma of Ostrowski type that includes Atangana-Baleanu operator.

Lemma 2.7. *(See [10]) Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I°. If $\rho' \in L[n_1, n_2]$, the following identity for Atangana-Baleanu integral operators is valid for all $n \in [n_1, n_2]$, $\xi \in (0, 1]$ and $t \in [0, 1]$:*

$$\begin{aligned}
& \frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right] \\
& -\frac{1}{(n_2-n_1)}\left[{}^{AB}I^{\xi}_{n}\{\rho(n_1)\}+{}^{AB}_{n}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\
& +\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right] \\
=\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\rho'(tn+(1-t)n_1)dt \\
& -\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\rho'(tn+(1-t)n_2)dt.
\end{aligned} \tag{19}$$

Here $B(\xi) > 0$ and $\Gamma(\xi)$ are normalization function and Euler gamma function respectively.

By using this lemma, let us arrange results for first order differentiable strongly convex functions.

Theorem 2.8. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho' \in L[n_1, n_2]$. If $|\rho'|$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$, $|\rho'| \leq M$ and $\frac{M}{\xi+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}, \frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right\}$, for* *all $n \in [n_1, n_2]$, $\xi \in (0,1]$ we obtain the inequality below:*

$$
\begin{aligned}
&\left|\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]\right.\\
&-\frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\}+{}_n^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right]\\
&\left.+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right|\\
\le\ &\frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}\right)\\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right).
\end{aligned}
\tag{20}
$$

Here $B(\xi) > 0$.

Proof. By using the equality in (19), we have

$$
\begin{aligned}
&\left|\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]\right.\\
&-\frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\}+{}_n^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right]\\
&\left.+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right|\\
\le\ &\frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_1)\right|dt\\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_2)\right|dt.
\end{aligned}
\tag{21}
$$

If we use the strongly convexity of $|\rho'|$ and the fact that $|\rho'| \le M$ in (21), we can deduce

$$
\begin{aligned}
&\left|\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]\right.\\
&-\frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\}+{}_n^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right]\\
&\left.+\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right|\\
&\le\frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left[t\left|\rho'(n)\right|+(1-t)\left|\rho'(n_1)\right|-ct(1-t)(n-n_1)^2\right]dt\\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\int_0^1 t^{\xi}\left[t\left|\rho'(n)\right|+(1-t)\left|\rho'(n_2)\right|-ct(1-t)(n_2-n)^2\right]dt\\
\le\ &\frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}\right)\\
&+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right).
\end{aligned}
$$

The proof is done. □

Corollary 2.9. *In Theorem 2.8, if we choose* $n = \frac{n_1+n_2}{2}$, *we have the following inequality:*

$$\begin{aligned} &\left| \frac{(n_2-n_1)^{\xi-1}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\rho\left(\frac{n_1+n_2}{2}\right) \right. \\ &-\frac{1}{n_2-n_1}\left[{}^{AB}I^{\xi}_{\frac{n_1+n_2}{2}}\{\rho(n_1)\} + {}^{AB}_{\frac{n_1+n_2}{2}}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\ &\left. +\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right] \right| \\ \leq\ & \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{M}{\xi+1}-\frac{c\,(n_2-n_1)^2}{4\,(\xi+2)\,(\xi+3)}\right). \end{aligned}$$

In the rest of the this section, for the simplicity we will use the following notations:

$$\begin{aligned} N_1 \;=\; & \frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right] \\ & -\frac{1}{(n_2-n_1)}\left[{}^{AB}I^{\xi}_{n}\{\rho(n_1)\} + {}^{AB}_{n}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\ & +\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right], \end{aligned}$$

$$\begin{aligned} N_2 = & \frac{(n_2-n_1)^{\xi-1}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\rho\left(\frac{n_1+n_2}{2}\right) \\ & -\frac{1}{n_2-n_1}\left[{}^{AB}I^{\xi}_{\frac{n_1+n_2}{2}}\{\rho(n_1)\} + {}^{AB}_{\frac{n_1+n_2}{2}}I^{\xi}_{n_2}\{\rho(n_2)\}\right] \\ & +\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]. \end{aligned}$$

It will also not be repeated in the rest of the study that $B > 0$ is the normalization function and Γ is the gamma function.

Theorem 2.10. *Let* $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ *and* $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a differentiable function on* I° *and* $\rho' \in L[n_1, n_2]$. *If* $\left|\rho'\right|^q$ *is strongly convex function with modulus* $c > 0$ *on* $[n_1, n_2]$ *and* $\left|\rho'\right| \leq M$, $M^q \geq \max\left\{\frac{c(n-n_1)^2}{6}, \frac{c(n_2-n)^2}{6}\right\}$, *for all* $n \in [n_1, n_2]$, $\xi \in (0, 1]$ *we obtain the inequality below:*

$$\begin{aligned} |N_1| \;\leq\; & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q-\frac{c(n-n_1)^2}{6}\right)^{\frac{1}{q}} \\ & +\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q-\frac{c(n_2-n)^2}{6}\right)^{\frac{1}{q}} \end{aligned} \tag{22}$$

where $q > 1$ *and* $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. To prove Theorem 2.10; we will use Lemma 2.7, property of modulus, Hölder inequality, strongly convexity of $\left|\rho'\right|^q$ and the fact that $\left|\rho'\right| \leq M$. So, we can write

$$\begin{aligned}
|N_1| &\leq \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\rho'(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\rho'(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_0^1 \left[t\left|\rho'(n)\right|^q + (1-t)\left|\rho'(n_1)\right|^q - ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_0^1 \left[t\left|\rho'(n)\right|^q + (1-t)\left|\rho'(n_2)\right|^q - ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}} \\
&\leq \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q - \frac{c(n-n_1)^2}{6}\right)^{\frac{1}{q}} \\
&\quad + \frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q - \frac{c(n_2-n)^2}{6}\right)^{\frac{1}{q}}
\end{aligned}$$

which is the inequality in (22). □

Corollary 2.11. *In Theorem 2.10, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi p+1}\right)^{\frac{1}{p}}\left(M^q - \frac{c(n_2-n_1)^2}{24}\right)^{\frac{1}{q}}.$$

Theorem 2.12. *Let $n_1 < n_2$, $n_1, n_2 \in I^\circ$and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho' \in L[n_1, n_2]$. If $\left|\rho'\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho'\right| \leq M$, $\frac{M^q}{\xi+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}, \frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in (0, 1]$ we obtain the inequality below:*

$$\begin{aligned}
|N_1| &\leq \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\xi+1} - \frac{c(n-n_1)^2}{(\xi+2)(\xi+3)}\right)^{\frac{1}{q}} \\
&\quad + \frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\xi+1} - \frac{c(n_2-n)^2}{(\xi+2)(\xi+3)}\right)^{\frac{1}{q}}
\end{aligned} \tag{23}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. In the proof of Theorem 2.12, we will use the Hölder's inequality in a different way as following:

$$\begin{aligned}
|N_1| &\leq \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\xi}\left|\rho'(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

If we use the strongly convexity of $\left|\rho'\right|^q$ and the fact that $\left|\rho'\right| \leq M$ in above and if we make the necessary calculations in obtained new inequality we will reach the inequality in (23). □

Corollary 2.13. *In Theorem 2.12, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{1}{\xi+1}\right)^{\frac{1}{p}}\left(\frac{M^q}{\xi+1}-\frac{c(n_2-n_1)^2}{4(\xi+2)(\xi+3)}\right)^{\frac{1}{q}}.$$

Remark 2.14. *Theorems 2.8-2.12 and Corollaries 2.9-2.13 are generalizations of Theorem 2.2, Theorem 2.5, Theorem 2.8, Corollary 2.4, Corollary 2.7 and Corollary 2.10 respectively which are obtained by Set et al. in [41].*

Theorem 2.15. *Let $n_1 < n_2$, $n_1, n_2 \in I^\circ$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho' \in L[n_1, n_2]$. If $\left|\rho'\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho'\right| \leq M$, $\frac{M^q}{\xi p+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi p+2)(\xi p+3)}, \frac{c(n_2-n)^2}{(\xi p+2)(\xi p+3)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in (0, 1]$ we obtain the inequality below:*

$$\begin{aligned} &|N_1| \qquad (24)\\ \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{q-1}{\xi(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M^q}{\xi p+1}-\frac{c(n-n_1)^2}{(\xi p+2)(\xi p+3)}\right)^{\frac{1}{q}}\\ &+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\frac{q-1}{\xi(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M^q}{\xi p+1}-\frac{c(n_2-n)^2}{(\xi p+2)(\xi p+3)}\right)^{\frac{1}{q}} \end{aligned}$$

where $q \geq p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Applying Hölder's inequality in a different way, we have

$$\begin{aligned} &|N_1|\\ \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\xi p}\left|\rho'(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\xi p}\left|\rho'(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}}\\ \leq\ & \frac{(n-n_1)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1 t^{\xi p}\left[t\left|\rho'(n)\right|^q+(1-t)\left|\rho'(n_1)\right|^q-ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}}\\ &+\frac{(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left(\int_0^1 t^{\xi\left(\frac{q-p}{q-1}\right)}dt\right)^{1-\frac{1}{q}}\\ &\times\left(\int_0^1 t^{\xi p}\left[t\left|\rho'(n)\right|^q+(1-t)\left|\rho'(n_2)\right|^q-ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we use the fact that $\left|\rho'\right| \leq M$ and if we calculate the integrals above, we complete the proof. □

Corollary 2.16. *In Theorem 2.15, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\left(\frac{q-1}{\xi(q-p)+q-1}\right)^{1-\frac{1}{q}}\left(\frac{M^q}{\xi p+1}-\frac{c(n_2-n_1)^2}{4(\xi p+2)(\xi p+3)}\right)^{\frac{1}{q}}.$$

To obtain new results for second order differentiable strongly convex functions we will use the following Ostrowski-like lemma.

Lemma 2.17. *(See [10]) Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I°. If $\rho'' \in L[n_1, n_2]$, identity for Atangana-Baleanu integral operators in equation (25) is valid for all $n \in [n_1, n_2]$, $t, \xi \in [0,1]$:*

$$\begin{aligned} &\frac{1}{(n_2-n_1)}\left[{}^{AB}I_{n}^{\xi}\{\rho(n_1)\}+{}_{n}^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right]-\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right] \\ &-\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]+\frac{(n-n_1)^{\xi+1}-(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n) \\ =\ &\frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\rho''(tn+(1-t)n_1)dt \\ &+\frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\rho''(tn+(1-t)n_2)dt. \end{aligned} \tag{25}$$

Theorem 2.18. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho'' \in L[n_1, n_2]$. If $\left|\rho''\right|$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$, $\left|\rho''\right| \leq M_1$ and $\frac{M_1}{\xi+2} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}, \frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in [0,1]$ we obtain the inequality below:*

$$\begin{aligned} &\left|\frac{1}{(n_2-n_1)}\left[{}^{AB}I_{n}^{\xi}\{\rho(n_1)\}+{}_{n}^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right]-\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right. \\ &\left.-\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]+\frac{(n-n_1)^{\xi+1}-(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n)\right| \\ \leq\ &\frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{M_1}{\xi+2}-\frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}\right) \\ &+\frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{M_1}{\xi+2}-\frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right). \end{aligned} \tag{26}$$

Proof. By using the equality in (25), property of modulus and strongly convexity of $\left|\rho''\right|$ we have

$$\begin{aligned} &\left|\frac{1}{(n_2-n_1)}\left[{}^{AB}I_{n}^{\xi}\{\rho(n_1)\}+{}_{n}^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right]-\frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right]\right. \\ &\left.-\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right]+\frac{(n-n_1)^{\xi+1}-(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n)\right| \\ \leq\ &\frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right|+(1-t)\left|\rho''(n_1)\right|-ct(1-t)(n-n_1)^2\right]dt \\ &+\frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right|+(1-t)\left|\rho''(n_2)\right|-ct(1-t)(n_2-n)^2\right]dt. \end{aligned}$$

We complete the proof by making the necessary calculations in above and by taking into consideration that $\left|\rho''\right| \leq M_1$. □

Corollary 2.19. *In Theorem 2.18, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{M_1}{\xi+2}-\frac{c(n_2-n_1)^2}{4(\xi+3)(\xi+4)}\right).$$

In the rest of this section, for simplicity we will use

$$\begin{aligned} N_3 &= \frac{1}{(n_2-n_1)}\left[{}^{AB}I_n^{\xi}\{\rho(n_1)\} + {}_n^{AB}I_{n_2}^{\xi}\{\rho(n_2)\}\right] - \frac{1-\xi}{(n_2-n_1)B(\xi)}\left[\rho(n_1)+\rho(n_2)\right] \\ &\quad -\frac{\rho(n)}{(n_2-n_1)B(\xi)\Gamma(\xi)}\left[(n_2-n)^{\xi}+(n-n_1)^{\xi}\right] + \frac{(n-n_1)^{\xi+1}-(n_2-n)^{\xi+1}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\rho'(n). \end{aligned}$$

Theorem 2.20. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho'' \in L[n_1, n_2]$. If $\left|\rho''\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho''\right| \le M_1$, $M_1^q \ge \max\left\{\frac{c(n-n_1)^2}{6}, \frac{c(n_2-n)^2}{6}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in [0, 1]$ we obtain the inequality below:*

$$\begin{aligned} &|N_3| \\ &\le \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{(\xi+1)p+1}\right)^{\frac{1}{p}}\left(M_1^q - \frac{c(n-n_1)^2}{6}\right)^{\frac{1}{q}} \\ &\quad + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{(\xi+1)p+1}\right)^{\frac{1}{p}}\left(M_1^q - \frac{c(n_2-n)^2}{6}\right)^{\frac{1}{q}} \end{aligned} \tag{27}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. To prove this theorem, we will use similar operations that we used when proving Theorem 2.10. So, we have

$$\begin{aligned} &|N_3| \\ &\le \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{(\xi+1)p}dt\right)^{\frac{1}{p}} \\ &\quad \times\left(\int_0^1\left[t\left|\rho''(n)\right|^q + (1-t)\left|\rho''(n_1)\right|^q - ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}} \\ &\quad + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\int_0^1 t^{(\xi+1)p}dt\right)^{\frac{1}{p}} \\ &\quad \times\left(\int_0^1\left[t\left|\rho''(n)\right|^q + (1-t)\left|\rho''(n_2)\right|^q - ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we calculate the integrals above and if we consider the fact that $\left|\rho''\right| \le M_1$, we get the inequality in (27). □

Corollary 2.21. *In Teorem 2.20, if we choose $n = \frac{n_1+n_2}{2}$, we have the following inequality:*

$$|N_2| \le \frac{(n_2-n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{(\xi+1)p+1}\right)^{\frac{1}{p}}\left(M_1^q - \frac{c(n_2-n_1)^2}{24}\right)^{\frac{1}{q}}.$$

Theorem 2.22. *Let $n_1 < n_2$, $n_1, n_2 \in I^{\circ}$ and $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on I° and $\rho'' \in L[n_1, n_2]$. If $\left|\rho''\right|^q$ is strongly convex function with modulus $c > 0$ on $[n_1, n_2]$ and $\left|\rho''\right| \le M_1$, $\frac{M_1^q}{\xi+2} \ge \max\left\{\frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}, \frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right\}$, for all $n \in [n_1, n_2]$, $\xi \in [0, 1]$ we obtain the inequality below:*

$$\begin{aligned} &|N_3| \\ &\le \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{\xi+2}\right)^{\frac{1}{p}}\left(\frac{M_1^q}{\xi+2} - \frac{c(n-n_1)^2}{(\xi+3)(\xi+4)}\right)^{\frac{1}{q}} \\ &\quad + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{1}{\xi+2}\right)^{\frac{1}{p}}\left(\frac{M_1^q}{\xi+2} - \frac{c(n_2-n)^2}{(\xi+3)(\xi+4)}\right)^{\frac{1}{q}} \end{aligned} \tag{28}$$

where $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Via Hölder's inequality and strongly convexity of $\left|\rho''\right|^q$ we can write

$$\begin{aligned} |N_3| \leq & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)} \left(\int_0^1 t^{\xi+1} dt\right)^{\frac{1}{p}} \\ & \times\left(\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right|^q + (1-t)\left|\rho''(n_1)\right|^q - ct(1-t)(n-n_1)^2\right]dt\right)^{\frac{1}{q}} \\ & + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)} \left(\int_0^1 t^{\xi+1} dt\right)^{\frac{1}{p}} \\ & \times\left(\int_0^1 t^{\xi+1}\left[t\left|\rho''(n)\right|^q + (1-t)\left|\rho''(n_2)\right|^q - ct(1-t)(n_2-n)^2\right]dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we consider the fact that $\left|\rho''\right| \leq M_1$ and calculate the integrals, we get the inequality in (28). □

Corollary 2.23. *In Theorem 2.22, if we choose* $n = \frac{n_1+n_2}{2}$, *we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)} \left(\frac{1}{\xi+2}\right)^{\frac{1}{p}} \left(\frac{M_1^q}{\xi+2} - \frac{c(n_2-n_1)^2}{4(\xi+3)(\xi+4)}\right)^{\frac{1}{q}}.$$

Theorem 2.24. *Let* $n_1 < n_2$, $n_1, n_2 \in I^\circ$ *and* $\rho : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a differentiable function on* I° *and* $\rho'' \in L[n_1, n_2]$. *If* $\left|\rho''\right|^q$ *is strongly convex function with modulus* $c > 0$ *on* $[n_1, n_2]$ *and* $\left|\rho''\right| \leq M_1$, $\frac{M_1^q}{\xi p+p+1} \geq \max\left\{\frac{c(n-n_1)^2}{(\xi p+p+2)(\xi p+p+3)}, \frac{c(n_2-n)^2}{(\xi p+p+2)(\xi p+p+3)}\right\}$, *for all* $n \in [n_1, n_2]$, $\xi \in [0, 1]$ *we obtain the inequality below:*

$$\begin{aligned} & |N_3| \qquad (29) \\ \leq & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)} \left(\frac{q-1}{(\xi+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \left(\frac{M_1^q}{\xi p+p+1} - \frac{c(n-n_1)^2}{(\xi p+p+2)(\xi p+p+3)}\right)^{\frac{1}{q}} \\ & + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)} \left(\frac{q-1}{(\xi+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \left(\frac{M_1^q}{\xi p+p+1} - \frac{c(n_2-n)^2}{(\xi p+p+2)(\xi p+p+3)}\right)^{\frac{1}{q}} \end{aligned}$$

where $q \geq p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Via a version of the Hölder inequality that we have used in the proof of Theorem 2.15, we can write

$$\begin{aligned} |N_3| \leq & \frac{(n-n_1)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)} \left(\int_0^1 t^{(\xi+1)\left(\frac{q-p}{q-1}\right)} dt\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1 t^{(\xi+1)p}\left|\rho''(tn+(1-t)n_1)\right|^q dt\right)^{\frac{1}{q}} \\ & + \frac{(n_2-n)^{\xi+2}}{(n_2-n_1)B(\xi)\Gamma(\xi)(\xi+1)} \left(\int_0^1 t^{(\xi+1)\left(\frac{q-p}{q-1}\right)} dt\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1 t^{(\xi+1)p}\left|\rho''(tn+(1-t)n_2)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

If we use strongly convexity of $\left|\rho''\right|^q$ with $\left|\rho''\right| \leq M_1$, and if we calculate the necessary integrals, we obtain the inequality in (29). □

Corollary 2.25. *In Theorem 2.24, if we choose* $n = \frac{n_1+n_2}{2}$*, we have the following inequality:*

$$|N_2| \leq \frac{(n_2-n_1)^{\xi+1}}{2^{\xi+1}B(\xi)\Gamma(\xi)(\xi+1)}\left(\frac{q-1}{(\xi+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \times\left(\frac{M_1^q}{\xi p+p+1}-\frac{c(n_2-n_1)^2}{4(\xi p+p+2)(\xi p+p+3)}\right)^{\frac{1}{q}}.$$

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