TURKISH JOURNAL OF SCIENCE
VOLUME 6, ISSUE 3, 167–173
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Partial Sums of The Miller-Ross Function

Sercan Kazımoğlu[a]

[a]*Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars,Turkey*

Abstract. This article deals with the ratio of normalized Miller-Ross function $\mathbb{E}_{\nu,c}(z)$ and its sequence of partial sums $(\mathbb{E}_{\nu,c})_m(z)$. Several examples which illustrate the validity of our results are also given.

1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathcal{U}$. Consider the function $E_{\nu,c}(z)$ defined by

$$E_{\nu,c}(z) = z^{\nu} \sum_{n=0}^{\infty} \frac{(cz)^n}{\Gamma(\nu + n + 1)} \tag{2}$$

where Γ stands for the Euler gamma function and $\nu > -1$, $c \in \mathbb{C}$ and $z \in \mathcal{U}$. This function was introduced by Miller and Ross in 1993 [9] and is therefore known as the Miller-Ross function.
The function defined by (2) does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization of the Miller-Ross function $E_{\nu,c}(z)$: for $z \in \mathcal{U}$,

$$\mathbb{E}_{\nu,c}(z) = \Gamma(\nu + 1) z^{1-\nu} E_{\nu,c}(z) = \sum_{n=0}^{\infty} \frac{c^n \Gamma(\nu + 1)}{\Gamma(\nu + n + 1)} z^{n+1} \tag{3}$$

where $\nu > -1$ and $c \in \mathbb{C}$.
Note that some special cases of $\mathbb{E}_{\nu,c}(z)$ are:

$$\begin{cases} \mathbb{E}_{0,1}(z) = e^z z \\ \mathbb{E}_{1,1}(z) = e^z - 1 \\ \mathbb{E}_{3,1}(z) = \frac{3\left(2e^z - z^2 - 2z - 2\right)}{z^2} \\ \mathbb{E}_{\frac{1}{2},\frac{1}{2}}(z) = e^{\frac{z}{2}} \sqrt{\frac{\pi}{2}} \sqrt{z} \operatorname{Erf} \sqrt{\frac{z}{2}}, \end{cases} \tag{4}$$

Corresponding author: SK mail address: srcnkzmglu@gmail.com ORCID:0000-0002-1023-4500
Received: 24 November 2021; Accepted: 27 December 2021; Published: 30 December 2021
Keywords. Analytic functions, Partial sums, Miller-Ross function, Univalent function
2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 33C10
Cited this article as: Kazımoğlu S. Partial Sums of The Miller-Ross Function, Turkish Journal of Science. 2021, 6(3), 167-173.

where Erf $\sqrt{z}$ is the error function.
For various interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) refered to the works of Brickman et al. [1], Kazımoğlu et al. [7], Çağlar and Orhan [2], Lin and Owa [8], Deniz and Orhan [4, 5], Owa et al. [11], Sheil-Small [14], Silverman [15] and Silvia [16]. Recently, some researchers have studied on partial sums of special functions (see [3, 7, 10, 13, 17]).
By using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+n+1)$, we obtain the following series representation for the ratio of normalized Miller-Ross function $\mathbb{E}_{\nu,c}(z)$ given by (3):

$$\begin{cases} (\mathbb{E}_{\nu,c})_0(z) = z \\ (\mathbb{E}_{\nu,c})_m(z) = z + \sum\limits_{n=1}^{m} A_n z^{n+1},\ m \in \mathbb{N} = \{1,2,3,\ldots\}, \end{cases} \tag{5}$$

where

$$A_n = \frac{c^n \Gamma(\nu+1)}{\Gamma(\nu+n+1)} = \frac{c^n}{(\nu+1)_n},\ \nu > -1 \text{ and } c \in \mathbb{C}.$$

We obtain lower bounds on ratios like

$$\Re\left\{\frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_m(z)}\right\},\ \Re\left\{\frac{(\mathbb{E}_{\nu,c})_m(z)}{\mathbb{E}_{\nu,c}(z)}\right\},\ \Re\left\{\frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)}\right\},\ \Re\left\{\frac{(\mathbb{E}_{\nu,c})'_m(z)}{\mathbb{E}'_{\nu,c}(z)}\right\}.$$

Several examples will be also given.
Results concerning partial sums of analytic functions may be found in [6, 12] etc.

2. MAIN RESULTS

In order to obtain our results we need the following lemma.

Lemma 2.1. *Let $\nu > -1$, $c \in \mathbb{C}$ and $|c| < \nu + 1$. Then the function $\mathbb{E}_{\nu,c}(z)$ satisfies the next two inequalities:*

$$\left|\mathbb{E}_{\nu,c}(z)\right| \le \frac{\nu+1}{\nu-|c|+1} \quad (z \in \mathcal{U}) \tag{6}$$

$$\left|\mathbb{E}'_{\nu,c}(z)\right| \le 1 + \frac{2\nu|c| + 2|c| - |c|^2}{(\nu-|c|+1)^2} \quad (z \in \mathcal{U}). \tag{7}$$

Proof. By using the well-known triangle inequality:

$$|z_1 + z_2| \le |z_1| + |z_2|$$

and the inequality

$$(\nu+1)_n \ge (\nu+1)^n,\ n \in \mathbb{N}, \tag{8}$$

we have

$$\begin{aligned} \left|\mathbb{E}_{\nu,c}(z)\right| &= \left|z + \sum_{n=1}^{\infty} \frac{c^n\Gamma(\nu+1)}{\Gamma(\nu+n+1)} z^{n+1}\right| \le 1 + \sum_{n=1}^{\infty} \frac{|c|^n\Gamma(\nu+1)}{\Gamma(\nu+n+1)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{|c|^n}{(\nu+1)_n} \le 1 + \sum_{n=1}^{\infty}\left(\frac{|c|}{\nu+1}\right)^n = \frac{\nu+1}{\nu-|c|+1}, \quad (|c| < \nu+1) \end{aligned}$$

and thus, inequality (6) is proved.
To prove (7), using again (8) and the triangle inequality, for $z \in \mathcal{U}$, we obtain

$$\begin{aligned} \left|\mathbb{E}'_{\nu,c}(z)\right| &= \left|1+\sum_{n=1}^{\infty}\frac{(n+1)\,c^{n}\Gamma(\nu+1)}{\Gamma(\nu+n+1)}z^{n}\right| \leq 1+\sum_{n=1}^{\infty}\frac{(n+1)\,|c|^{n}\Gamma(\nu+1)}{\Gamma(\nu+n+1)} \\ &= 1+\sum_{n=1}^{\infty}\frac{(n+1)\,|c|^{n}}{(\nu+1)_{n}} \leq 1+\sum_{n=1}^{\infty}(n+1)\left(\frac{|c|}{\nu+1}\right)^{n} = 1+\frac{2\nu\,|c|+2\,|c|-|c|^{2}}{(\nu-|c|+1)^{2}}, \quad (|c|<\nu+1) \end{aligned}$$

and thus, inequality (7) is proved. □

Let $w(z)$ be an analytic function in $\mathcal{U}$. In the sequel, we will frequently use the following well-known result:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\}>0,\ z\in\mathcal{U} \text{ if and only if } |w(z)|<1,\ z\in\mathcal{U}.$$

Theorem 2.2. *Let* $\nu > -1$ *and* $0 < 2|c| \leq \nu + 1$. *Then*

$$\Re\left\{\frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_{m}(z)}\right\} \geq \frac{\nu-2|c|+1}{\nu-|c|+1},\ z\in\mathcal{U} \tag{9}$$

and

$$\Re\left\{\frac{(\mathbb{E}_{\nu,c})_{m}(z)}{\mathbb{E}_{\nu,c}(z)}\right\} \geq \frac{\nu-|c|+1}{\nu+1}. \tag{10}$$

Proof. From inequality (6) we get

$$1+\sum_{n=1}^{\infty}A_{n} \leq \frac{\nu+1}{\nu-|c|+1}, \text{where } A_{n}=\frac{c^{n}\Gamma(\nu+1)}{\Gamma(\nu+n+1)},\ \nu>-1,\ c\in\mathbb{C} \text{ and } n\in\mathbb{N}.$$

The last inequality is equivalent to

$$\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n_1}^{\infty}A_{n} \leq 1.$$

In order to prove the inequality (9), we consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)}=\left(\frac{\nu-|c|+1}{|c|}\right)\frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_{m}(z)}-\left(\frac{\nu-2|c|+1}{|c|}\right)$$

or

$$\frac{1+w(z)}{1-w(z)}=\frac{1+\sum_{n=1}^{m}A_{n}z^{n}+\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_{n}z^{n}}{1+\sum_{n=1}^{m}A_{n}z^{n}}. \tag{11}$$

From (11), we obtain

$$w(z)=\frac{\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_{n}z^{n}}{2+2\sum_{n=1}^{m}A_{n}z^{n}+\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_{n}z^{n}}$$

and

$$|w(z)| \leq \frac{\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_{n}}{2-2\sum_{n=1}^{m}A_{n}-\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_{n}}.$$

Now, $|w(z)| \leq 1$ if and only if

$$2\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_{n} \leq 2-2\sum_{n=1}^{m}A_{n}$$

which is equivalent to

$$\sum_{n=1}^{m} A_n + \left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n \leq 1. \tag{12}$$

To prove (12), it suffices to show that its left-hand side is bounded above by

$$\left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=1}^{\infty} A_n$$

which is equivalent to

$$\left(\frac{\nu - 2|c| + 1}{|c|}\right) \sum_{n=1}^{m} A_n \geq 0.$$

The last inequality holds true for $0 < 2|c| \leq \nu + 1$.
We use the same method to prove the inequality (10). Consider the function $w(z)$ given by

$$\begin{aligned} \frac{1 + w(z)}{1 - w(z)} &= \left(\frac{\nu + 1}{|c|}\right) \frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_m(z)} - \left(\frac{\nu - |c| + 1}{|c|}\right) \\ &= \frac{1 + \sum_{n=1}^{m} A_n z^n - \left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n z^n}{1 + \sum_{n=1}^{m} A_n z^n}. \end{aligned}$$

From the last equality we get

$$w(z) = \frac{-\left(\frac{\nu+1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n z^n}{2 + 2\sum_{n=1}^{m} A_n z^n - \left(\frac{\nu - 2|c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{\nu+1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n}{2 - 2\sum_{n=1}^{m} A_n - \left(\frac{\nu - 2|c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n}.$$

Then, $|w(z)| \leq 1$ if and only if

$$\sum_{n=1}^{m} A_n + \left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n \leq 1. \tag{13}$$

Since the left-hand side of (13) is bounded above by

$$\left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=1}^{\infty} A_n,$$

we have that the inequality (10) holds true. Now, the proof of our theorem is completed. □

Theorem 2.3. *Let* $\nu > -1$ *and* $0 < 2\nu|c| + 2|c| - |c|^2 \leq \frac{(\nu+1)^2}{2}$. *Then*

$$\Re\left\{\frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)}\right\} \geq 1 - \frac{2\nu|c| + 2|c| - |c|^2}{(\nu - |c| + 1)^2},\ z \in \mathcal{U} \tag{14}$$

and

$$\Re\left\{\frac{(\mathbb{E}_{\nu,c})'_m(z)}{\mathbb{E}'_{\nu,c}(z)}\right\} \geq \frac{(\nu - |c| + 1)^2}{(\nu - |c| + 1)^2 + 2\nu|c| + 2|c| - |c|^2},\ z \in \mathcal{U}. \tag{15}$$

Proof. From (7) we have

$$1+\sum_{n=1}^{\infty}(n+1)A_n \le 1+\frac{2\nu|c|+2|c|-|c|^2}{(\nu-|c|+1)^2},$$

where $A_n=\frac{c^n\Gamma(\nu+1)}{\Gamma(\nu+n+1)}$, $\nu>-1$, $c\in\mathbb{C}$ and $n\in\mathbb{N}$. The above inequality is equivalent to

$$\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=1}^{\infty}(n+1)A_n \le 1.$$

To prove (14), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)}=\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)}-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)$$

which gives

$$w(z)=\frac{\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n+\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)|\le\frac{\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The condition $|w(z)|\le 1$ holds true if and only if

$$\sum_{n=1}^{m}(n+1)A_n+\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_n\le 1. \tag{16}$$

The left-hand side of (16) is bounded above by

$$\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=1}^{\infty}(n+1)A_n$$

which is equivalent to

$$\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=1}^{m}(n+1)A_n\ge 0$$

which holds true for $0<2\nu|c|+2|c|-|c|^2\le\frac{(\nu+1)^2}{2}$.
The proof of (15) follows the same pattern. Consider the function $w(z)$ given by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}+1\right)\frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)}-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\\ &= \frac{1+\sum_{n=1}^{m}(n+1)A_nz^n-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{1+\sum_{n=1}^{\infty}(n+1)A_nz^n}.\end{aligned}$$

Consequently, we have that

$$w(z)=\frac{-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}+1\right)\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}+1\right)\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The last inequality implies that $|w(z)| \leq 1$ if and only if

$$\left(\frac{2(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=m+1}^{\infty}(n+1)A_n \leq 2-2\sum_{n=1}^{m}(n+1)A_n$$

or equivalently

$$\sum_{n=1}^{m}(n+1)A_n+\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=m+1}^{\infty}(n+1)A_n \leq 1. \tag{17}$$

It remains to show that the left-hand side of (17) is bounded above by

$$\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=1}^{\infty}(n+1)A_n.$$

This is equivalent to

$$\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=1}^{m}(n+1)A_n \geq 0,$$

which holds true for $0 < 2\nu|c|+2|c|-|c|^2 \leq \frac{(\nu+1)^2}{2}$. Now, the proof of our theorem is completed. □

3. Examples

In this section, we give several examples which illustrate our main theorems in Sections 2. In Theorem 2.2 and Theorem 2.3, we obtain the following corollaries for special cases of ν and c.

Corollary 3.1. *If we take $\nu = 3$ and $c = 1$, we have*

$$\mathbb{E}_{3,1}(z) = \frac{3\left(2e^z-z^2-2z-2\right)}{z^2}, \quad \mathbb{E}'_{3,1}(z) = \frac{6\left(e^z(z-2)+z+2\right)}{z^3}$$

and for $m = 0$ we get

$$\left(\mathbb{E}_{3,1}(z)\right)_0(z) = z, \quad \left(\mathbb{E}'_{3,1}(z)\right)_0(z) = 1,$$

so,

$$\begin{aligned}
\Re\left\{\frac{\left(2e^z-z^2-2z-2\right)}{z^3}\right\} &\geq \frac{2}{9} \approx 0.222, \quad z \in \mathcal{U}, \\
\Re\left\{\frac{z^3}{\left(2e^z-z^2-2z-2\right)}\right\} &\geq \frac{9}{4} \approx 2.25, \quad z \in \mathcal{U}, \\
\Re\left\{\frac{\left(e^z(z-2)+z+2\right)}{z^3}\right\} &\geq \frac{1}{27} \approx 0.037, \quad z \in \mathcal{U}, \\
\Re\left\{\frac{z^3}{\left(e^z(z-2)+z+2\right)}\right\} &\geq \frac{27}{8} \approx 3.375, \quad z \in \mathcal{U}.
\end{aligned}$$

Setting $m = 0$, $\nu = \frac{3}{2}$ and $c = \frac{1}{2}$ in Theorem 2.2 and Theorem 2.3 respectively, we obtain the next result involving the function $\mathbb{E}_{\frac{1}{2},\frac{1}{2}}(z)$, defined by (4), and its derivative.

Corollary 3.2. *The following inequıalities hold true:*

$$\Re\left\{\frac{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}{z\sqrt{z}}\right\} \geq \frac{1}{4}\approx 0.25,\ \ z\in\mathcal{U},$$

$$\Re\left\{\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}\right\} \geq \frac{12}{5}\approx 2.4,\ \ z\in\mathcal{U},$$

$$\Re\left\{\frac{e^{\frac{z}{2}}\sqrt{2\pi}\,(z-1)\,Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}{z\sqrt{z}}\right\} \geq \frac{7}{12}\approx 0.583,\ \ z\in\mathcal{U},$$

$$\Re\left\{\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{2\pi}\,(z-1)\,Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}\right\} \geq \frac{12}{25}\approx 0.48,\ \ z\in\mathcal{U}.$$

Example 3.3. *The image domains of* $f_1(z)=\frac{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}{z\sqrt{z}}$, $f_2(z)=\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}$, $f_3(z)=\frac{e^{\frac{z}{2}}\sqrt{2\pi}(z-1)Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}{z\sqrt{z}}$ *and* $f_4(z)=\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{2\pi}(z-1)Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}$ *are shown in Figure 1.*

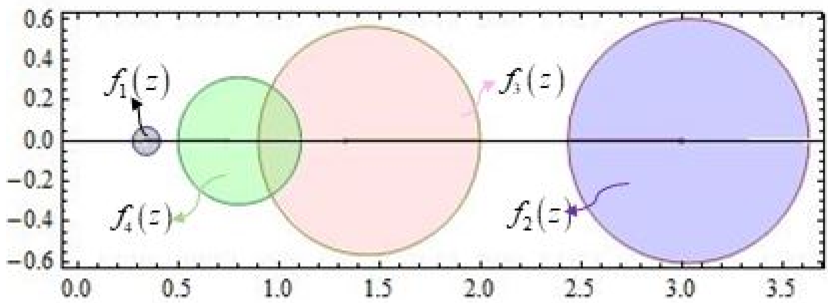

Figure1.

References

[1] Brickman L, Hallenbeck DJ, MacGregor TH, Wilken D. Convex hulls and extreme points of families of starlike and convex mappings. Trans. Amer. Math. Soc. 185, 1973, 413–428.
[2] Çağlar M, Orhan H. On neighborhood and partial sums problem for generalized Sakaguchi type functions. The Scientifc Annals of Al.I. Cuza University of Iasi. 1, 2014, 17–28.
[3] Çağlar M, Deniz E. Partial sums of the normalized Lommel functions. Math. Inequal. Appl. 18, 2015, 1189–1199.
[4] Deniz E, Orhan H. Some Properties Of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator. Czech. Math. J. 60, 2010, 699–713.
[5] Deniz E, Orhan H. Certain subclasses of multivalent functions defined by new multiplier transformations. Arab. J. Sci. Eng. 36, 2011, 1091–1112.
[6] Frasin BA. Generalization of partial sums of certain analytic and univalent functions. Appl. Math. Lett. 21, 2008, 735–741.
[7] Kazımoğlu S, Deniz E, Çağlar M. Partial Sums of The Bessel-Struve Kernel Function. 3rd International Conference on Mathematical and Related Sciences: Current Trend and Developments. 2020, 267–275.
[8] Lin LJ, Owa S. On partial sums of the Libera integral operator. J. Math. Anal. Appl. 213, 1997, 444–454.
[9] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. Wiley. 1993.
[10] Orhan H, Yağmur N. Partial Sums of generalized Bessel functions. J. Math. Inequal. 8, 2014, 863–877.
[11] Owa S, Srivastava HM, Saito N. Partial sums of certain classes of analytic functions. International Journal of Computer Mathematics.81, 2014, 1239-1256.
[12] Ravichandran V. Geometric properties of partial sums of univalent functions. Math. Newslett. 22, 2012, 208–221.
[13] Rehman MS, Ahmad QZ, Srivastava HM, Khan B, Khan N. Partial sums of generalized $q-$Mittag-Leffler functions. Aims Math. 5, 2019, 408–420.
[14] Sheil-Small T. A note on partial sums of convex schlicht functions. Bull. London Math. Soc. 2, 1970, 165–168.
[15] Silverman H. Partial sums of starlike and convex functions. J. Math. Anal. Appl. 209, 1997, 221–227.
[16] Silvia EM. On partial sums of convex functions of order α. Houston J. Math. 11, 1985, 397–404.
[17] Yağmur N, Orhan H. Partial sums of generalized Struve functions. Miskolc Mathematical Notes. 17, 2016, 657–670.