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Tubular Hypersurfaces According to Extended Darboux Frame Field of First Kind in E^4

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Abstract. In this paper, we study tubular hypersurfaces according to one of the extended Darboux frame field in Euclidean 4-space. We obtain the Gaussian and mean curvatures of tubular hypersurfaces according to extended Darboux frame field of first kind and give some results for them. Also, we prove a theorem about linear Weingarten tubular hypersurface and construct an example.

1. INTRODUCTION

A canal surface is formed by the envelope of the spheres whose centers lie on a curve and radii vary depending on this curve [18]. In case of a constant radius function, the envelope is called tubular or pipe surface [19]. Also for a canal surface, if the center curve is a straight line, then it becomes a revolution surface. Canal surfaces (especially tubular surfaces) have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape re-construction and so on. In this context, canal and tubular (hyper)surfaces have been studied by many geometers in Euclidean, Minkowskian, Galilean or pseudo-Galilean spaces (see [7], [14], [20]-[24], [28]-[30], [32], [34]-[37], and etc).

On the other hand, Frenet frame has been used in lots of studies about curves and surfaces, but sometimes scienticists have needed alternative frames because Frenet frame cannot be identified at the points where the curvature is zero. Therefore, new alternative frames to the Frenet frame such as Bishop frame, Darboux frame or extended Darboux frame have been defined by geometers and the theories of curves and surfaces have been started to handle according to these alternative frames (see [2], [3], [9]-[13], [25], [27], [33], and etc).

After recalling some basic notions about one type of extended Darboux frame field and the curvatures of hypersurfaces in E^4 in the second section of this paper, we deal with tubular hypersurfaces according to extended Darboux frame field of first kind in E^4 in the third section. We obtain the Gaussian and mean curvatures of tubular hypersurface according to extended Darboux frame field of first kind and give some results when the curve which constructs the tubular hypersurfaces is (unit speed) asymptotic or line of curvature on tubular hypersurface. Finally, we prove a theorem that states the tubular hypersurface according to extended Darboux frame field of first kind in E^4 is a linear Weingarten hypersurface.

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2. PRELIMINARIES

Let $\{e_1, e_2, e_3, e_4\}$ be the standart basis of Euclidean 4-space E^4. If $\overrightarrow{s} = (s_1, s_2, s_3, s_4)$, $\overrightarrow{t} = (t_1, t_2, t_3, t_4)$ and $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ are three vectors in E^4, then the inner product and vector product are given by

$$\left\langle \overrightarrow{s}, \overrightarrow{t} \right\rangle = s_1t_1 + s_2t_2 + s_3t_3 + s_4t_4$$

and

$$\overrightarrow{s} \times \overrightarrow{t} \times \overrightarrow{v} = \det \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ s_1 & s_2 & s_3 & s_4 \\ t_1 & t_2 & t_3 & t_4 \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix},$$

respectively. Also, the norm of the vector $\overrightarrow{s}$ is $\left\| \overrightarrow{s} \right\| = \sqrt{\left\langle \overrightarrow{s}, \overrightarrow{s} \right\rangle}$. Let $M \subset E^4$ denote a regular hypersurface and $\alpha : I \subset R \longrightarrow M$ be a unit speed curve. If $\{T, n, b_1, b_2\}$ is the moving Frenet frame along α, then the Frenet formulas are given by [15]

$$\begin{bmatrix} T' \\ n' \\ b_1' \\ b_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ b_1 \\ b_2 \end{bmatrix},$$

where T, n, b_1 and b_2 denote the unit tangent, the principal normal, the first binormal and the second binormal vector fields; k_1, k_2 and k_3 are the curvature functions of the curve α.

Here, we will recall the extended Darboux frame field of first kind (for simplicity, we'll call it *ED^1-frame field* throughout this paper) and for details about the construction of extended Darboux frame fields, we refer to [13].

We consider an embedding $\Psi : U \subset E^3 \longrightarrow E^4$, where U is an open subset of E^3. Now, we denote $M = \Psi(U)$ and identify M and U through the embedding Ψ. Let $\bar{\alpha} : I \longrightarrow U$ be a regular curve and we have a curve $\alpha : I \longrightarrow M \subset E^4$ defined by $\alpha(s) = \Psi(\bar{\alpha}(s))$ and so, the curve α is on the hypersurface M. If M is an orientable hypersurface oriented by the unit normal vector field $\mathcal{N}$ in E^4 and α is a Frenet curve of class $C^n (n \geq 4)$ with arc-length parameter s lying on M, then we denote the unit tangent vector field of the curve by T and denote the hypersurface unit normal vector field restricted to the curve by N, i.e.

$$T(s) = \alpha'(s) \text{ and } N(s) = \mathcal{N}(\alpha(s)).$$

The differential equations of ED-frame fields of first kind $\{T, E, D, N\}$ of the curve α on M in E^4 by matrix notation can be given as

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix}, \tag{1}$$

where $\langle E', N \rangle = \tau_g^1$, $\langle D', N \rangle = \tau_g^2$, $\langle T', E \rangle = \kappa_g^1$, $\langle E', D \rangle = \kappa_g^2$ and τ_g^i and κ_g^i are called the geodesic torsions and geodesic curvatures of order i, respectively. Also, $\langle T', N \rangle = \kappa_n$ is the normal curvature of the hypersurface in the direction of the tangent vector T [13].

Now, the relation matrix may be expressed as [13]

$$\begin{bmatrix} T \\ n \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_1 & \cos\phi_2 & \cos\phi_3 \\ 0 & \cos\psi_1 & \cos\psi_2 & \cos\psi_3 \\ 0 & \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} \tag{2}$$

and

$$\begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_1 & \cos\psi_1 & \cos\theta_1 \\ 0 & \cos\phi_2 & \cos\psi_2 & \cos\theta_2 \\ 0 & \cos\phi_3 & \cos\psi_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} T \\ n \\ b_1 \\ b_2 \end{bmatrix}. \tag{3}$$

Also, we have

$$\left.\begin{array}{l} \kappa_g^1 = \langle T', E\rangle = k_1\cos\phi_1,\ \kappa_n = \langle T', N\rangle = k_1\cos\phi_3, \\ \tau_g^1 = -\phi_1'\sin\phi_1\cos\phi_3 - \psi_1'\sin\psi_1\cos\psi_3 - \theta_1'\sin\theta_1\cos\theta_3 \\ \quad + k_2(\cos\phi_1\cos\psi_3 - \cos\psi_1\cos\phi_3) + k_3(\cos\psi_1\cos\theta_3 - \cos\theta_1\cos\psi_3), \\ \tau_g^2 = -\phi_2'\sin\phi_2\cos\phi_3 - \psi_2'\sin\psi_2\cos\psi_3 - \theta_2'\sin\theta_2\cos\theta_3 \\ \quad + k_2(\cos\phi_2\cos\psi_3 - \cos\psi_2\cos\phi_3) + k_3(\cos\psi_2\cos\theta_3 - \cos\theta_2\cos\psi_3), \\ \kappa_g^2 = -\phi_1'\sin\phi_1\cos\phi_2 - \psi_1'\sin\psi_1\cos\psi_2 - \theta_1'\sin\theta_1\cos\theta_2 \\ \quad + k_2(\cos\phi_1\cos\psi_2 - \cos\psi_1\cos\phi_2) + k_3(\cos\psi_1\cos\theta_2 - \cos\theta_1\cos\psi_2). \end{array}\right\} \tag{4}$$

Furthermore, the differential geometry of different types of (hyper)surfaces in 4-dimensional spaces has been a popular topic for geometers, recently ([1], [4], [5], [6], [8], [16], [17], [26], and etc). If

$$\begin{array}{l} \Psi : U \subset E^3 \longrightarrow E^4 \\ \quad (s,t,v) \longrightarrow \Psi(s,t,v) = (\Psi_1(s,t,v), \Psi_2(s,t,v), \Psi_3(s,t,v), \Psi_4(s,t,v)) \end{array} \tag{5}$$

is a hypersurface in E^4, then the unit normal vector field, the matrix forms of the first and second fundamental forms are

$$\mathcal{N}_\Psi = \frac{\Psi_s \times \Psi_t \times \Psi_v}{\|\Psi_s \times \Psi_t \times \Psi_v\|}, \tag{6}$$

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{7}$$

and

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{8}$$

respectively. Here $g_{ij} = \left\langle \Psi_{v_i}, \Psi_{v_j} \right\rangle$, $h_{ij} = \left\langle \Psi_{v_i v_j}, \mathcal{N}_\Psi \right\rangle$, $\Psi_{v_i} = \frac{\partial\Psi(v_1,v_2,v_3)}{\partial v_i}$, $\Psi_{v_i v_j} = \frac{\partial^2\Psi(v_1,v_2,v_3)}{\partial v_i v_j}$, $i, j \in \{1,2,3\}$. Also, the shape operator of the hypersurface (5) is

$$S = [a_{ij}] = [g^{ij}].[h_{ij}], \tag{9}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$.

With the aid of (6)-(9), the Gaussian and mean curvatures of a hypersurface in E^4 are given by

$$K = \det(S) = \frac{\det[h_{ij}]}{\det[g_{ij}]} \tag{10}$$

and

$$H = \frac{1}{3}tr(S), \tag{11}$$

respectively [31]. We say that a hypersurface is flat or minimal, if it has zero Gaussian curvature or zero mean curvature, respectively.

3. TUBULAR HYPERSURFACES ACCORDING TO ED^1-FRAME FIELD IN EUCLIDEAN 4-SPACE

In this section, we obtain the Gaussian and mean curvatures of tubular hypersurfaces according to ED^1-frame field in Euclidean 4-space E^4 and give some results for these curvatures when the curve α which constructs the tubular hypersurface is an asymptotic curve, a unit-speed asymptotic curve and a line of curvature lying on M.

Let $\alpha : I \longrightarrow M$ be a unit speed curve lying on a regular hypersurface M and we consider the tubular hypersurface $\mathcal{T}$ according to ED^1-frame field of α in E^4 given by

$$\mathcal{T}(s,t,v) = \alpha(s) \pm \rho\left[(\cos t \cos v)\, E(s) + (\sin t \cos v)\, D(s) + (\sin v)\, N(s)\right], \tag{12}$$

where $\alpha(s)$ is the center curve of tubular hypersurface $\mathcal{T}$, $\rho \in \mathbb{R}$ is constant radius, $s \in [0,l]$ and $t, v \in [0, 2\pi)$. From now on, we state $\alpha = \alpha(s)$, $T = T(s)$, $E = E(s)$, $D = D(s)$, $N = N(s)$ and we will consider the "±" as "+".

Firstly, from (1) and (12) the first derivatives of the tubular hypersurface (12) are obtained as

$$\left.\begin{aligned}
\mathcal{T}_s &= \left(1 - \rho(\kappa_g^1 \cos t \cos v + \kappa_n \sin v)\right) T - \rho\left(\kappa_g^2 \cos v \sin t + \tau_g^1 \sin v\right) E \\
&\quad + \rho\left(\kappa_g^2 \cos t \cos v - \tau_g^2 \sin v\right) D + \rho \cos v \left(\tau_g^1 \cos t + \tau_g^2 \sin t\right) N, \\
\mathcal{T}_t &= -\left(\rho \sin t \cos v\right) E + \left(\rho \cos t \cos v\right) D, \\
\mathcal{T}_v &= -\left(\rho \cos t \sin v\right) E - \left(\rho \sin t \sin v\right) D + \left(\rho \cos v\right) N.
\end{aligned}\right\} \tag{13}$$

From (6) and (13), the unit normal vector field of $\mathcal{T}$ in E^4 is

$$\mathcal{N} = (\cos t \cos v)\, E + (\sin t \cos v)\, D + (\sin v)\, N. \tag{14}$$

Also, the coefficients of the first fundamental form are

$$\left.\begin{aligned}
g_{11} &= \left(\rho\left(\kappa_g^2 \cos v \sin t + \tau_g^1 \sin v\right)\right)^2 + \left(\rho \cos v \left(\tau_g^1 \cos t + \tau_g^2 \sin t\right)\right)^2 \\
&\quad + \left(\rho\left(\kappa_g^2 \cos t \cos v - \tau_g^2 \sin v\right)\right)^2 + \left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right)^2, \\
g_{12} &= g_{21} = \rho^2 \cos v \left(\kappa_g^2 \cos v + \sin v(\tau_g^1 \sin t - \tau_g^2 \cos t)\right), \\
g_{13} &= g_{31} = \rho^2 \left(\tau_g^2 \sin t + \tau_g^1 \cos t\right), \\
g_{22} &= \rho^2 \cos^2 v, \; g_{23} = g_{32} = 0, \; g_{33} = \rho^2
\end{aligned}\right\} \tag{15}$$

and it follows that

$$\det[g_{ij}] = \rho^4(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v)^2 \cos^2 v. \tag{16}$$

Now, for obtaining the coefficients of the second fundamental form, we give the second derivatives

$\mathcal{T}_{v_i v_j} = \frac{\partial^2 \mathcal{T}}{\partial v_i v_j}$ of the tubular hypersurface (12):

$$\left.\begin{aligned}
\mathcal{T}_{ss} &= \mathcal{T}_{ss}^{1} T + \mathcal{T}_{ss}^{2} E + \mathcal{T}_{ss}^{3} D + \mathcal{T}_{ss}^{4} N,\\
\mathcal{T}_{st} &= \mathcal{T}_{ts} = \left(\rho\kappa_g^1 \sin t\cos v\right) T - \left(\rho\kappa_g^2 \cos t\cos v\right) E - \left(\rho\kappa_g^2 \sin t\cos v\right) D\\
&\quad - \left(\rho(\tau_g^1 \sin t - \tau_g^2 \cos t)\cos v\right) N,\\
\mathcal{T}_{sv} &= \mathcal{T}_{vs} = \left(\rho\left(\kappa_g^1 \sin v\cos t - \kappa_n \cos v\right)\right) T + \left(\rho\left(\kappa_g^2 \sin v\sin t - \tau_g^1 \cos v\right)\right) E\\
&\quad - \left(\rho\left(\kappa_g^2 \sin v\cos t + \tau_g^2 \cos v\right)\right) D - \left(\rho\left(\tau_g^1 \cos t + \tau_g^2 \sin t\right)\sin v\right) N,\\
\mathcal{T}_{tt} &= -\left(\rho\cos t\cos v\right) E - \left(\rho\sin t\cos v\right) D,\\
\mathcal{T}_{tv} &= \mathcal{T}_{vt} = \left(\rho\sin t\sin v\right) E - \left(\rho\cos t\sin v\right) D,\\
\mathcal{T}_{vv} &= -\left(\rho\cos t\cos v\right) E - \left(\rho\sin t\cos v\right) D - \left(\rho\sin v\right) N,
\end{aligned}\right\} \tag{17}$$

where

$$\mathcal{T}_{ss}^{1} = \rho\begin{pmatrix} \kappa_g^1\left(\kappa_g^2 \cos v\sin t + \tau_g^1 \sin v\right) - (\kappa_n)'\sin v\\ -\left(\tau_g^1\kappa_n \cos t + \tau_g^2\kappa_n \sin t + (\kappa_g^1)'\cos t\right)\cos v \end{pmatrix},$$

$$\begin{aligned}
\mathcal{T}_{ss}^{2} &= -\rho\left((\kappa_g^1)^2 + (\kappa_g^2)^2 + (\tau_g^1)^2\right)\cos t\cos v + \kappa_g^1(1 - \rho\kappa_n \sin v)\\
&\quad - \rho((-\kappa_g^2\tau_g^2 + (\tau_g^1)')\sin v + (\tau_g^2\tau_g^1 + (\kappa_g^2)')\sin t\cos v),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{ss}^{3} &= -\rho\left((\kappa_g^2)^2 + (\tau_g^2)^2\right)\sin t\cos v + (\tau_g^2\tau_g^1 - (\kappa_g^2)')\cos t\cos v\\
&\quad + (\kappa_g^2\tau_g^1 + (\tau_g^2)')\sin v),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{ss}^{4} &= -\rho\tau_g^1(\kappa_g^2 \sin t\cos v + \tau_g^1 \sin v) + \rho\tau_g^2(\kappa_g^2 \cos t\cos v - \tau_g^2 \sin v)\\
&\quad - \kappa_n\left(-1 + \rho\kappa_g^1 \cos t\cos v + \rho\kappa_n \sin v\right) + \rho\left((\tau_g^1)'\cos t + (\tau_g^2)'\sin t\right)\cos v.
\end{aligned}$$

Thus, from (8), (14) and (17), the coefficients of the second fundamental form are

$$\left.\begin{aligned}
h_{11} &= -\rho\left(\left((\kappa_g^1)^2 + (\tau_g^1)^2\right)\cos^2 t + (\kappa_g^2)^2 + 2\tau_g^1\tau_g^2 \sin t\cos t + (\tau_g^2)^2\sin^2 t\right)\cos^2 v\\
&\quad - \rho\left((\tau_g^1)^2 + (\tau_g^2)^2 + (\kappa_n)^2\right)\sin^2 v - \kappa_g^1\left(-1 + 2\rho\kappa_n \sin v\right)\cos t\cos v\\
&\quad - \rho\kappa_g^2\left(\tau_g^1 \sin t - \tau_g^2 \cos t\right)\sin(2v) + \kappa_n \sin v,\\
h_{12} &= h_{21} = -\rho\left(\kappa_g^2 \cos v + \sin v(\tau_g^1 \sin t - \tau_g^2 \cos t)\right)\cos v,\\
h_{13} &= h_{31} = -\rho\left(\tau_g^1 \cos t + \tau_g^2 \sin t\right),\\
h_{22} &= -\rho\cos^2 v,\ h_{23} = h_{32} = 0,\ h_{33} = -\rho
\end{aligned}\right\} \tag{18}$$

and it implies that

$$\det[h_{ij}] = -\rho^2\cos^2 v\left(\kappa_g^1 \cos t\cos v + \kappa_n \sin v\right)\left(-1 + \rho\kappa_g^1 \cos t\cos v + \rho\kappa_n \sin v\right). \tag{19}$$

So, from (10), (16) and (19), we have

Proposition 3.1. *The Gaussian curvature of the tubular hypersurfaces (12) in E^4 is*

$$K = -\frac{\kappa_g^1 \cos t\cos v + \kappa_n \sin v}{\rho^2(-1 + \rho\kappa_g^1 \cos t\cos v + \rho\kappa_n \sin v)}. \tag{20}$$

Corollary 3.2. *The Gaussian curvature of the tubular hypersurfaces (12) in E^4 does not depend on the geodesic curvature of order* 2 *and geodesic torsions of order* 1 *and order* 2.

Corollary 3.3. *The tubular hypersurfaces (12) in E^4 is flat if and only if*

$$\kappa_g^1 \cos t \cos v = -\kappa_n \sin v$$

holds.

Corollary 3.4. *If $\kappa_g^1 = \kappa_n = 0$, then the tubular hypersurfaces (12) in E^4 is flat.*

Also, after finding the inverse of the matrix of the first fundamental form and using this and (18) in (9), the shape operator of the tubular hypersurface (12) is obtained by

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}, \tag{21}$$

where

$$S_{11} = -\frac{\kappa_g^1 \cos t \cos v + \kappa_n \sin v}{-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v}, \ S_{12} = S_{13} = 0,$$

$$S_{21} = \frac{\sec v \left(\kappa_g^2 + \tan v \left(\tau_g^1 \sin t - \tau_g^2 \cos t\right)\right)}{\rho\left(-\sec v + \rho\kappa_g^1 \cos t + \rho\kappa_n \tan v\right)}, \ S_{22} = -\frac{1}{\rho}, \ S_{23} = 0,$$

$$S_{31} = \frac{\tau_g^1 \cos t + \tau_g^2 \sin t}{\rho\left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right)}, \ S_{32} = 0, S_{33} = -\frac{1}{\rho}.$$

Hence from (11) and (21), we get

Proposition 3.5. *The mean curvature of the tubular hypersurfaces (12) in E^4 is*

$$H = \frac{2 - 3\rho\left(\kappa_g^1 \cos t \cos v + \kappa_n \sin v\right)}{3\rho\left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right)}. \tag{22}$$

Corollary 3.6. *The mean curvature of the tubular hypersurfaces (12) in E^4 does not depend on the geodesic curvature of order* 2 *and geodesic torsions of order* 1 *and order* 2.

Corollary 3.7. *The tubular hypersurfaces (12) in E^4 is minimal if and only if*

$$\kappa_g^1 \cos t \cos v + \kappa_n \sin v = \frac{2}{3\rho} \tag{23}$$

holds.

Corollary 3.8. *If $\kappa_g^1 = \kappa_n = 0$, then the tubular hypersurface (12) in E^4 has negative constant mean curvature with* $\frac{-2}{3\rho}$.

Here, from (20) and (22), we can state the following theorem which gives an important relation between Gaussian and mean curvatures:

Theorem 3.9. *The Gaussian curvature K and the mean curvature H of tubular hypersurfaces (12) in E^4 satisfy*

$$3H = \rho^2 K - \frac{2}{\rho}. \tag{24}$$

Also, from (21) we have

$$\det(S - \lambda I_3) = -\frac{(1+\lambda\rho)^2 \left(-\lambda + (1+\lambda\rho)\,\kappa_g^1 \cos t \cos v + (1+\lambda\rho)\,\kappa_n \sin v\right)}{\rho^2 \left(-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v\right)}. \tag{25}$$

By solving the equation $\det(S - \lambda I_3) = 0$ from (25), we obtain the principal curvatures of the tubular hypersurfaces (12) in E^4 as follows:

Proposition 3.10. *The principal curvatures of the tubular hypersurfaces (12) in E^4 are*

$$\lambda_1 = \lambda_2 = -\frac{1}{\rho} \text{ and } \lambda_3 = -\frac{\kappa_g^1 \cos t \cos v + \kappa_n \sin v}{-1 + \rho\kappa_g^1 \cos t \cos v + \rho\kappa_n \sin v}. \tag{26}$$

Furthermore, if a curve α is a unit-speed asymptotic curve parametrized by arc-length on an oriented hypersurface M in E^4, then we have

$$\kappa_n = 0,\ \kappa_g^1 = k_1,\ \kappa_g^2 = k_2 \cos\varphi,\ \tau_g^1 = -k_2 \sin\varphi,\ \tau_g^2 = k_3 + \frac{d\varphi}{ds}, \tag{27}$$

where φ denotes the angle between D and B_1 [13]. Thus using (27), we have

Corollary 3.11. *If the curve α is a unit-speed asymptotic curve lying on M, then the Gaussian and mean curvatures of tubular hypersurface (12) in E^4 are*

$$K = -\frac{k_1 \cos t \cos v}{\rho^2 \left(-1 + \rho k_1 \cos t \cos v\right)} \tag{28}$$

and

$$H = \frac{2 - 3\rho k_1 \cos t \cos v}{3\rho \left(-1 + \rho k_1 \cos t \cos v\right)}, \tag{29}$$

respectively.

Corollary 3.12. *If the curve α is a unit-speed asymptotic curve lying on M, then the Gaussian and mean curvatures of tubular hypersurface (12) in E^4 are independent of the angle φ.*

Also in [24], the authors have studied on canal and tubular hypersurfaces according to the Frenet frame in E^4 and they have obtained the Gaussian and mean curvatures of tubular hypersurface

$$\mathcal{T}(s,t,v) = \alpha(s) + \rho\left[(\cos t \cos v)\, n(s) + (\sin t \cos v)\, b_1(s) + (\sin v)\, b_2(s)\right] \tag{30}$$

as (28) and (29). Therefore

Theorem 3.13. *If the curve α is a unit-speed asymptotic curve lying on M, then the Gaussian and mean curvatures of tubular hypersurfaces (12) according to ED^1-frame field and (30) according to Frenet frame coincide.*

On the other hand, the curve α lying on M is a line of curvature if and only if $\tau_g^1 = \tau_g^2 = 0$ [13]. So, we have

Corollary 3.14. *If the curve α is line of curvature lying on M, then the Gaussian and mean curvatures of tubular hypersurface (12) in E^4 are (20) and (22) respectively.*

Finally, we will give a theorem about linear Weingarten tubular hypersurface according to ED^1-frame field of unit speed curve α lying on M in E^4. We know that, a hypersurface is called a linear Weingarten hypersurface, if it satisfies

$$aH + bK = c, \tag{31}$$

where a, b, c are not all zero constants. Thus, we have

Theorem 3.15. *The tubular hypersurface (12) in E^4 is a linear Weingarten hypersurface.*

Proof. We know that, the relation between the mean and Gaussian curvatures of the tubular hypersurface (12) in E^4 is given by (24). So, if we take $a = 3,\ b = -\rho^2$ and $c = \frac{-2}{\rho}$ in (31), the proof completes. □

Example 3.16. *We take the unit speed curve*

$$\alpha(s) = \left(\sin\left(\frac{3s}{5}\right), \cos\left(\frac{3s}{5}\right), \sin\left(\frac{4s}{5}\right), \cos\left(\frac{4s}{5}\right)\right) \tag{32}$$

on the hypersphere $M...x^2 + y^2 + z^2 + t^2 = 2$ in E^4. The Frenet apparatus of this curve is

$$\left.\begin{aligned}
T &= \tfrac{1}{5}\left(3\cos\left(\tfrac{3s}{5}\right), -3\sin\left(\tfrac{3s}{5}\right), 4\cos\left(\tfrac{4s}{5}\right), -4\sin\left(\tfrac{4s}{5}\right)\right),\\
n &= -\tfrac{1}{\sqrt{337}}\left(9\sin\left(\tfrac{3s}{5}\right), 9\cos\left(\tfrac{3s}{5}\right), 16\sin\left(\tfrac{4s}{5}\right), 16\cos\left(\tfrac{4s}{5}\right)\right),\\
b_1 &= \tfrac{1}{5}\left(4\cos\left(\tfrac{3s}{5}\right), -4\sin\left(\tfrac{3s}{5}\right), -3\cos\left(\tfrac{4s}{5}\right), 3\sin\left(\tfrac{4s}{5}\right)\right),\\
b_2 &= -\tfrac{1}{\sqrt{337}}\left(16\sin\left(\tfrac{3s}{5}\right), 16\cos\left(\tfrac{3s}{5}\right), -9\sin\left(\tfrac{4s}{5}\right), -9\cos\left(\tfrac{4s}{5}\right)\right)
\end{aligned}\right\} \tag{33}$$

and

$$k_1 = \frac{\sqrt{337}}{25},\ k_2 = \frac{84}{25\sqrt{337}},\ k_3 = \frac{12}{\sqrt{337}}. \tag{34}$$

Also, we have the ED^1-frame fields of unit speed curve α as

$$\left.\begin{aligned}
T &= \tfrac{1}{5}\left(3\cos\left(\tfrac{3s}{5}\right), -3\sin\left(\tfrac{3s}{5}\right), 4\cos\left(\tfrac{4s}{5}\right), -4\sin\left(\tfrac{4s}{5}\right)\right),\\
E &= \tfrac{1}{\sqrt{2}}\left(\sin\left(\tfrac{3s}{5}\right), \cos\left(\tfrac{3s}{5}\right), -\sin\left(\tfrac{4s}{5}\right), -\cos\left(\tfrac{4s}{5}\right)\right),\\
D &= \tfrac{1}{5}\left(-4\cos\left(\tfrac{3s}{5}\right), 4\sin\left(\tfrac{3s}{5}\right), 3\cos(\tfrac{4s}{5}), -3\sin(\tfrac{4s}{5})\right),\\
N &= \tfrac{1}{\sqrt{2}}\left(\sin\left(\tfrac{3s}{5}\right), \cos\left(\tfrac{3s}{5}\right), \sin\left(\tfrac{4s}{5}\right), \cos\left(\tfrac{4s}{5}\right)\right)
\end{aligned}\right\} \tag{35}$$

and the normal curvature, geodesic curvatures and geodesic torsions of order 1 *and* 2 *are obtained as*

$$\kappa_n = -\frac{1}{\sqrt{2}},\ \kappa_g^1 = \frac{7}{25\sqrt{2}},\ \kappa_g^2 = -\frac{12\sqrt{2}}{25},\ \tau_g^1 = 0,\ \tau_g^2 = 0, \tag{36}$$

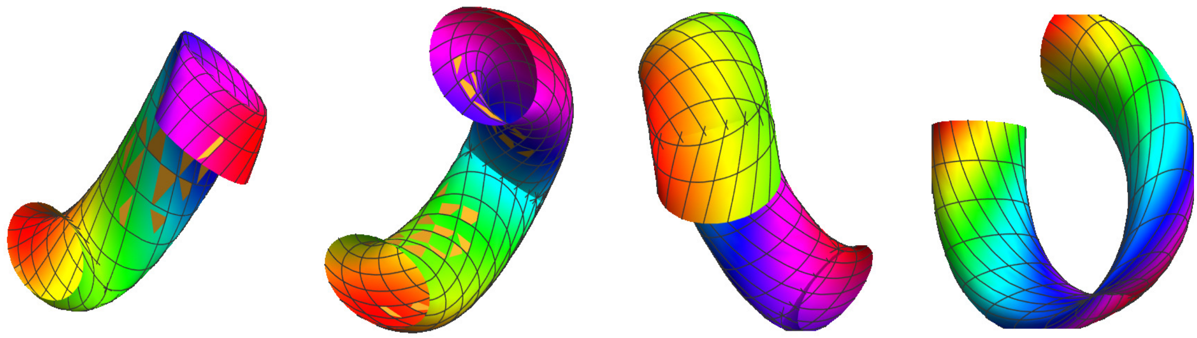
respectively. Hence using (35) in (12), we get the tubular hypersurface according to ED^1-frame field in E^4 as

$$\mathcal{T}(s,t,v) = \begin{pmatrix}
-\frac{4}{5}\rho\cos(\frac{3s}{5})\cos v\sin t + \frac{1}{2}\sin(\frac{3s}{5})\left(2 + \sqrt{2}\rho\left(\cos v\sin t + \sin v\right)\right),\\
\frac{4}{5}\rho\sin(\frac{3s}{5})\cos v\sin t + \frac{1}{2}\cos(\frac{3s}{5})\left(2 + \sqrt{2}\rho\left(\cos v\sin t + \sin v\right)\right),\\
\frac{3}{5}\rho\cos(\frac{4s}{5})\cos v\sin t + \frac{1}{2}\sin(\frac{4s}{5})\left(2 + \sqrt{2}\rho\left(-\cos v\sin t + \sin v\right)\right),\\
-\frac{3}{5}\rho\sin(\frac{4s}{5})\cos v\sin t + \frac{1}{2}\cos(\frac{4s}{5})\left(2 + \sqrt{2}\rho\left(-\cos v\sin t + \sin v\right)\right)
\end{pmatrix} \tag{37}$$

and from (20), (22) and (36), we obtain the Gaussian and mean curvatures of the tubular hypersurface (37) as

$$K = -\frac{7\cos t\cos v - 25\sin v}{\rho^2\left(7\rho\cos t\cos v - 25\left(\sqrt{2}+\rho\sin v\right)\right)} \text{ and } H = \frac{100 - 3\sqrt{2}\rho\,(7\cos t\cos v - 25\sin v)}{3\rho\left(-50 + \sqrt{2}\rho\,(7\cos t\cos v - 25\sin v)\right)}, \tag{38}$$

respectively. In the following figures, one can see the projections of the tubular hypersurface (37) for $v = \frac{\pi}{3}$ and $\rho = 3$ into $x_1x_2x_3$ (A), $x_1x_2x_4$ (B), $x_1x_3x_4$ (C) and $x_2x_3x_4$-spaces (D).



(a) **(b)** **(c)** **(d)**

Figure 1

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