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Non-null Translation-Homothetical surfaces in four-dimensional Minkowski space

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Abstract. In the present work, we deal with non-null translation-homothetical surfaces in Minkowski 4−space. Initially, we describe non-null $TH-$type surface (Translation-Homothetical surface). Then, we yield the normal curvature, mean curvature vector and Gaussian curvature functions. Using these concepts, we characterize the non-null semiumbilical, minimal and flat translation-homothetical surfaces in $\mathbb{E}_1^4$.

1. Introduction

In physics literature, special relativity is a scientific theory that explains the relationship between space and time. According to the theory, all objects and physical phenomena are relative. Time, space and motion are not independent of each other. Minkowski space-time is the geometry that mathematically describes the four-dimensional structure of special relativity. Minkowski 4−space (or Minkowski space-time) is defined with the help of a Lorentzian metric as

$$g(x,y) = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3. \tag{1}$$

Any arbitrary vector is known as spacelike, lightlike or timelike, if the Lorentzian metric $g(x,x)$ is positive definite, zero or negative definite, respectively. In Minkowski space-time, all surfaces are also divided into three categories in a similar way. Any surface in $\mathbb{E}_1^4$ is known as a spacelike surface (or timelike surface), given that its all tangent vectors are spacelike (timelike).

Let $M : \psi = \psi(s,t)$ be a non-lightlike (spacelike or timelike) surface in $\mathbb{E}_1^4$. Four-dimensional Minkowski space can be decomposed into tangent space and normal space of M, at each point p as:

$$\mathbb{E}_1^4 = T_p^{\perp}M \oplus T_pM. \tag{2}$$

Levi-Civita connections are indicated by $\tilde{\nabla}$ and ∇ on $\mathbb{E}_1^4$ and M. Assume: X and Y are tangent vector fields and N is a normal vector field of M. The vector fields $\tilde{\nabla}_XN$ and $\tilde{\nabla}_XY$ are decomposed into normal and

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tangent components by Weingarten and Gauss formulas:

$$\begin{aligned} \widetilde{\nabla}_X N &= -A_N X + D_X N, \\ \widetilde{\nabla}_X Y &= \nabla_X Y + h(X,Y), \end{aligned} \tag{3}$$

where D, h and A_N are the normal connection, the second fundamental form and the shape operator, respectively. [7, 12].

Let $\psi = \psi(s,t)$ be a parametrization for a non-null surface M in $\mathbb{E}^4_1$. Then, $T_pM = span\{\psi_s, \psi_t\}$ corresponds to the tangent space at a point p of M. The standard indications $E = g(\psi_s, \psi_s)$, $F = g(\psi_s, \psi_t)$, $G = g(\psi_t, \psi_t)$ are known as first fundamental form coefficients

$$I = Eds^2 + 2Fdsdt + Gdt^2, \tag{4}$$

We can choose the tangent vector fields, for the timelike surface, as $g(\psi_s, \psi_s) < 0$, $g(\psi_t, \psi_t) > 0$. In addition, we settle a normal frame field $\{N_1, N_2\}$ for the spacelike surface as $g(N_1, N_1) = -1$, $g(N_2, N_2) = 1$, i.e. $\{\psi_s, \psi_t, N_1, N_2\}$ is oriented positively in $\mathbb{E}^4_1$. For the later use, we set

$$\xi = \begin{cases} 1, & \text{if } M \text{ is spacelike} \\ -1, & \text{if } M \text{ is timelike} \end{cases} \tag{5}$$

Thus, we present $W = \sqrt{\xi(EG - F^2)}$. It means; $EG - F^2$ is positive or negative definite with respect to being the surface spacelike or timelike.

H:the mean curvature vector field can be computed by $H = \frac{1}{2}trh$. In other words, using the tangent bundle's orthonormal frame $\{X, Y\}$, it can be written as $H = \frac{1}{2}(\xi h(X,X) + h(Y,Y))$. The second fundamental form coefficients can be calculated as

$$\begin{aligned} c^1_{11} &= g(\psi_{ss}, N_1), \quad c^1_{12} = g(\psi_{st}, N_1), \quad c^1_{22} = g(\psi_{tt}, N_1), \\ c^2_{11} &= g(\psi_{ss}, N_2), \quad c^2_{12} = g(\psi_{st}, N_2), \quad c^1_{22} = g(\psi_{tt}, N_2). \end{aligned} \tag{6}$$

One can write the second fundamental tensor as

$$\begin{aligned} h(\psi_s, \psi_s) &= -\xi c^1_{11} N_1 + c^2_{11} N_2, \\ h(\psi_s, \psi_t) &= -\xi c^1_{12} N_1 + c^2_{12} N_2, \\ h(\psi_t, \psi_t) &= -\xi c^1_{22} N_1 + c^2_{22} N_2. \end{aligned} \tag{7}$$

Another way of representing it;

$$h(X,Y) = -\xi g(A_{N_1}(X), Y) N_1 + g(A_{N_2}(X), Y) N_2. \tag{8}$$

H_k is used for $k-$th mean curvature function and calculated by $H_k = g(H, N_k) = \frac{tr(A_{N_k})}{2}$, hence we obtain

$$H_k = \frac{c^k_{11}G - 2c^k_{12}F + c^k_{22}E}{2(EG - F^2)}. \tag{9}$$

According to the basis $\{N_1, N_2\}$, the mean curvature vector field H turns into

$$H = -\xi H_1 N_1 + H_2 N_2, \tag{10}$$

(see, [7, 12])

The mean curvature of M is congruent to the norm of the mean curvature vector ($\left\|\vec{H}\right\|$). The surface is called as minimal, if the mean curvature vector of it is identically zero [5].

Gaussian curvature of $M : \psi(s,t)$ can be stated by using the first and second fundamental forms' coefficients:

$$K = \frac{-\xi \det(A_{N_1}) + \det(A_{N_2})}{W^2}. \tag{11}$$

In case of zero Gaussian curvature, M is called as a flat surface.

Furthermore, with the help of orthonormal tangent vectors $\{\psi_1, \psi_2\}$ and unit normal vectors $\{N_1, N_2\}$, the normal curvature of a surface is

$$K_N = g\left(R^{\perp}(\psi_1, \psi_2) N_2,\ N_1\right). \tag{12}$$

This relation can be given by the entries of shape operator matrices:

$$K_N = h_{12}^1 \left(h_{22}^2 - h_{11}^2\right) + h_{12}^2 \left(h_{11}^1 - h_{22}^1\right).$$

Regarding the previous equation, a surface M is known as semiumbilical surface if the normal curvature is zero, for all points on M [8].

In [1], Yu A. Aminov focused on the notion of Monge Patch in $\mathbb{E}^4$ with the representation

$$f = f(s,t),\ g = g(s,t). \tag{13}$$

Also, in [3], the authors studied some surfaces given by the parametrization

$$\psi(s,t) = (s, t, f(s,t), g(s,t)). \tag{14}$$

Two special surfaces, called translation surfaces and homothetical (factorable) surfaces are interesting classes in differential geometry. These surfaces have been studied from many viewpoints, theoretically [2, 4, 9, 10, 13].

A new surface named $TH-$ type surface (or translation-homothetical surface) is first handled by Difi et. al. in $3-$dimensional Euclidean spaces [6]. The parameterization of this surface is given with the help of the sum and multiplication of differentiable functions. Some studies on $TH-$type surfaces can be found in [6, 11]. Recently, the authors have defined $TF-$ type ($TH-$ type) surface in $4-$dimensional Euclidean space[11]. They investigated the structure of this type of surface in $\mathbb{E}^4$.

In this study, we deal with the non-null translation-homothetical surfaces in 4-dimensional Minkowski space. First, we describe the non-lightlike (non-null) Translation-Homothetical surface in $\mathbb{E}_1^4$. Then, we yield the normal curvature, mean curvature vector and Gaussian curvature for spacelike and timelike surfaces. Further, we characterize some non-null semiumbilical, minimal and flat $TH-$type surfaces in Minkowski space-time.

2. Classification of Non-null Translation-Homothetical Surfaces in $\mathbb{E}_1^4$

Definition 2.1. *[2] The surface which is defined by the sum of two curves $\alpha(s) = (s, 0, z_1(s), z_2(s))$ and $\beta(t) = (0, t, w_1(t), w_2(t))$ is called as translation surface. Thus, the translation surface in $4-$dimensional space has the parametrization*

$$\psi(s,t) = (s, t, z_1(s) + w_1(t), z_2(s) + w_2(t)). \tag{15}$$

Definition 2.2. *[4] The surface which is given by an explicit form $f(s,t) = z_1(s) w_1(t)$, $g(s,t) = z_2(s) w_2(t)$ is called as homothetical (or factorable) surface where s, t, f, g are Cartesian coordinates. Thus, the homothetical surface in $4-$dimensional space has the parametrization*

$$\psi(s,t) = (s, t, z_1(s) w_1(t), z_2(s) w_2(t)). \tag{16}$$

With respect to these definitions, the translation-homothetical surface is defined as the following:

Definition 2.3. *The surface is called $TH-$type surface (or translation-homothetical surface) if it is given by the Monge patch*

$$\psi(s,t) = (s, t, \lambda(z_1(s)+w_1(t)) + \mu(z_1(s)w_1(t)), \sigma(z_2(s)+w_2(t)) + \rho(z_2(s)w_1(t))) \tag{17}$$

where λ, μ, σ and ρ are non-zero real constants.

$TH-$type surface in Minkowski space-time can be considered by the representation

$$\psi(s,t) = (s, t, z_1(s) + w_1(t) + z_1(s)w_1(t), z_2(s) + w_2(t) + z_2(s)w_2(t)). \tag{18}$$

Thus, in this study, we investigate some properties of non-null (spacelike and timelike) $TH-$type surfaces given by the parameterization (18). Let M be a non-null $TH-$type surface in $\mathbb{E}_1^4$, then we have the followings:

The following vector fields span the tangent space of M:

$$\begin{aligned} \psi_s &= \left(1, 0, z_1'(s) + z_1'(s)w_1(t), z_2'(s) + z_2'(s)w_2(t)\right), \\ \psi_t &= \left(0, 1, w_1'(t) + z_1(s)w_1'(t), w_2'(t) + z_2(s)w_2'(t)\right). \end{aligned} \tag{19}$$

Therefore, the first fundamental form coefficients can be yielded by the Lorentzian inner product

$$\begin{aligned} E &= -1 + \left(z_1' + z_1'w_1\right)^2 + \left(z_2' + z_2'w_2\right)^2, \\ F &= \left(z_1' + z_1'w_1\right)\left(w_1' + z_1w_1'\right) + \left(z_2' + z_2'w_2\right)\left(w_2' + z_2w_2'\right), \\ G &= 1 + \left(w_1' + z_1w_1'\right)^2 + \left(w_2' + z_2w_2'\right)^2. \end{aligned} \tag{20}$$

Choosing the surface as timelike or spacelike with respect to being $E < 0$ (or $E > 0$), one can determine $W = \sqrt{\xi(EG - F^2)}$.

Two times derivatives of $\psi(s,t)$ are

$$\begin{aligned} \psi_{ss} &= \left(0, 0, z_1''(s) + z_1''(s)w_1(t), z_2''(s) + z_2''(s)w_2(t)\right), \\ \psi_{st} &= \left(0, 0, z_1'(s)\, w_1'(t), z_2'(s)\, w_2'(t)\right), \\ \psi_{tt} &= \left(0, 0, w_1''(t) + z_1(s)w_1''(t), w_2''(t) + z_2(s)w_2''(t)\right). \end{aligned} \tag{21}$$

The orthonormal vector fields $\{N_1, N_2\}$ spans the normal space of non-null surface:

$$N_1 = \frac{1}{\sqrt{|A_1|}}\left(z_1' + z_1'w_1, -\left(w_1' + z_1w_1'\right), 1, 0\right), \tag{22}$$

$$N_2 = \frac{1}{\sqrt{A_1W^*}}\left(A_1\left(z_2'+z_2'w_2\right) - A_3\left(z_1'+z_1'w_1\right), A_3\left(w_1'+z_1w_1'\right) - A_1\left(w_2'+z_2w_2'\right), -A_3, A_1\right),$$

where

$$\begin{aligned} A_1 &= 1 - \left(z_1' + z_1'w_1\right)^2 + \left(w_1' + z_1w_1'\right)^2, \\ A_2 &= 1 - \left(z_2' + z_2'w_2\right)^2 + \left(w_2' + z_2w_2'\right)^2 \\ A_3 &= \left(w_1' + z_1w_1'\right)\left(w_2' + z_2w_2'\right) - \left(z_1' + z_1'w_1\right)\left(z_2' + z_2'w_2\right), \\ W^* &= A_1A_2 - (A_3)^2. \end{aligned}$$

and by using (29) and (30), c_{ij}^k, $(i, j, k = 1, 2)$ are given as

$$\begin{aligned} c_{11}^1 &= \frac{z_1'' + z_1'' w_1}{\sqrt{|A_1|}}, \qquad c_{11}^2 = \frac{\left(z_2'' + z_2'' w_2\right) A_1 - \left(z_1'' + z_1'' w_1\right) A_3}{\sqrt{A_1 W^*}}, \\ c_{12}^1 &= \frac{z_1' w_1'}{\sqrt{|A_1|}}, \qquad c_{12}^2 = \frac{z_2' w_2' A_1 - z_1' w_1' A_3}{\sqrt{A_1 W^*}}, \\ c_{22}^1 &= \frac{w_1'' + z_1 w_1''}{\sqrt{|A_1|}}, \qquad c_{22}^2 = \frac{\left(w_2'' + z_2 w_2''\right) A_1 - \left(w_1'' + z_1 w_1''\right) A_3}{\sqrt{A_1 W^*}}. \end{aligned} \tag{23}$$

we can write the orthonormal tangent vector by using Gram-Schmidt orthonormalization method for ψ_s and ψ_t,

$$\begin{aligned} X &= \frac{\psi_s}{\sqrt{|E|}}, \\ Y &= \frac{\sqrt{|E|}}{W}\left(\psi_t - \frac{F}{E}\psi_s\right). \end{aligned} \tag{24}$$

By the use of (6), (7), (24) and (8), the shape operator matrices can be presented as

$$\begin{bmatrix} h_{11}^1 & h_{12}^1 \\ h_{12}^1 & h_{22}^1 \end{bmatrix}, \qquad \begin{bmatrix} h_{11}^2 & h_{12}^2 \\ h_{12}^2 & h_{22}^1 \end{bmatrix}, \tag{25}$$

where the functions h_{ij}^k are given by

$$\begin{aligned} h_{11}^1 &= \xi \frac{\left(z_1'' + z_1'' w_1\right)}{E\sqrt{|A_1|}}, \quad h_{12}^1 = \frac{E z_1' w_1' - F\left(z_1'' + z_1'' w_1\right)}{EW\sqrt{|A_1|}}, \\ h_{22}^1 &= \xi \frac{\left(w_1'' + z_1 w_1''\right) E^2 - 2 z_1' w_1' EF + \left(z_1'' + z_1'' w_1\right) F^2}{E\sqrt{|A_1|}}, \\ h_{11}^2 &= \xi \frac{A_1\left(z_2'' + z_2'' w_2\right) - A_3\left(z_1'' + z_1'' w_1\right)}{E\sqrt{A_1 W^*}}, \\ h_{12}^2 &= \frac{\left(z_2' w_2' A_1 - z_1' w_1' A_3\right) E - \left[A_1\left(z_2'' + z_2'' w_2\right) - A_3\left(z_1'' + z_1'' w_1\right)\right] F}{EW\sqrt{A_1 W^*}}, \\ h_{22}^2 &= \xi \frac{\begin{array}{c}\left[\left(w_2'' + z_2 w_2''\right) A_1 - \left(w_1'' + z_1 w_1''\right) A_3\right] E^2 - 2\left(z_2' w_2' A_1 - z_1' w_1' A_3\right) EF \\ + \left[\left(z_2'' + z_2'' w_2\right) A_1 - A_3\left(z_1'' + z_1'' w_1\right)\right] F^2\end{array}}{EW^2\sqrt{A_1 W^*}}. \end{aligned} \tag{26}$$

2.1. *Non-null Flat Translation-Homothetical Surfaces*

Theorem 2.4. *Let M be a non-null translation-homothetical surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, its Gaussian curvature is given as*

$$K = \frac{\begin{array}{c} A_1\left(\left(z_2'' + z_2'' w_2\right)\left(w_2'' + z_2 w_2''\right) - \left(z_2' w_2'\right)^2\right) \\ + A_2\left(\left(z_1'' + z_1'' w_1\right)\left(w_1'' + z_1 w_1''\right) - \left(z_1' w_1'\right)^2\right) \\ - A_3\left(\left(z_2'' + z_2'' w_2\right)\left(w_1'' + z_1 w_1''\right) + \left(z_1'' + z_1'' w_1\right)\left(w_2'' + z_2 w_2''\right) - 2 z_1' w_1' z_2' w_2'\right) \end{array}}{W^* W^2}.$$

where W and W^* are defined as $W^2 = \xi\left(EG - F^2\right)$, $W^* = A_1 A_2 - A_3^2$, respectively.

Proof. By using (11) and (26), we obtain the desired result. □

Theorem 2.5. *Let M be a non-null translation-homothetical surface with the parameterization (18) in* $\mathbb{E}_1^4$*. Then M has zero Gaussian curvature if and only if*

$$\begin{aligned} 0 &= A_1\left(\left(z_2''+z_2''w_2\right)\left(w_2''+z_2w_2''\right)-\left(z_2'w_2'\right)^2\right) \\ &+A_2\left(\left(z_1''+z_1''w_1\right)\left(w_1''+z_1w_1''\right)-\left(z_1'w_1'\right)^2\right) \\ &-A_3\left(\left(z_2''+z_2''w_2\right)\left(w_1''+z_1w_1''\right)+\left(z_1''+z_1''w_1\right)\left(w_2''+z_2w_2''\right)-2z_1'w_1'z_2'w_2'\right). \end{aligned} \tag{27}$$

Theorem 2.6. *Let M be a non-null translation-homothetical surface with the parameterization (18) in* $\mathbb{E}_1^4$*. If M is given by one of the following parametrizations, then it is flat surface:*

(1) $f(s,t) = a_1w_1(t) + a_1 + w_1(t)$, $g(s,t) = a_2w_2(t) + a_2 + w_2(t)$;
(2) $f(s,t) = a_1z_1(s) + a_1 + z_1(s)$, $g(s,t) = a_2z_2(s) + a_2 + z_2(s)$;
(3) $f(s,t) = a_1w_1(t) + a_1 + w_1(t)$, $g(s,t) = a_2z_2(s) + a_2 + z_2(s)$;
(4) $f(s,t) = a_1z_1(s) + a_1 + z_1(s)$, $g(s,t) = a_2w_2(t) + a_2 + w_2(t)$;
(5) $f(s,t) = a_1$, $g(s,t) = a_2w_2(t) + a_2 + w_2(t)$;
(6) $f(s,t) = a_1$, $g(s,t) = a_2z_2(s) + a_2 + z_2(s)$;
(7) $f(s,t) = a$, $g(s,t) = be^{a_1s}e^{a_2t} - 1$;
(8) $f(s,t) = a$, $g(s,t) = z_2(s) + w_2(t) + z_2w_2(t)$ satisfying

$$\begin{aligned} z_2(s) &= \left((1-c)(a_3s+a_4)^{\frac{1}{1-c}}-1\right), \\ w_2(t) &= \left(\frac{(c-1)(a_5t+a_6)}{c}\right)^{\frac{c}{c-1}}-1, \end{aligned} \tag{28}$$

(9) $f(s,t) = a_1a_5e^{a_2s}e^{a_6t} - 1$, $g(s,t) = a_3a_7e^{a_4s}e^{a_8t} - 1$; $a_4a_6 = a_2a_8$,
(10) $f(s,t) = a_1a_5e^{a_2s}e^{a_6t} - 1$, $g(s,t) = a_3a_7e^{a_4s}e^{a_8t} - 1$; $a_2a_4 = a_6a_8$,
where a, b, c, a_i are real constants, $i = 1,..,8$, $c \neq 0, 1$.

Proof. Let M be a non-null $TH-$type surface given by the parametrization (18) in $\mathbb{E}_1^4$. If $z_1'(s) = 0$, $z_2'(s) = 0$ or $w_1'(t) = 0$, $w_2'(t) = 0$ or $z_1'(s) = 0$, $w_2'(t) = 0$ ($z_2'(s) = 0$, $w_1'(t) = 0$) in (27), then we obtain the cases (1), (2), (3) and (4). If $z_1'(s) = 0$ and $w_1'(t) = 0$,then we have

$$\left(z_2''+z_2''w_2\right)\left(w_2''+z_2w_2''\right)-\left(z_2'w_2'\right)^2=0. \tag{29}$$

In this equation, if $z_2' = 0$ (or $w_2' = 0$), then we obtain the surfaces (5) and (6). If $z_1'(s)w_1'(t) \neq 0$, then we get

$$\frac{z_2''(s)z_2(s)+z_2''(s)}{\left(z_2'(s)\right)^2}=\frac{\left(w_2'(t)\right)^2}{w_2''(t)w_2(t)+w_2''(t)}=c, \tag{30}$$

where $c \in IR$. If $c = 1$, from (30) we get the differential equations $z_2''(s)z_2(s)+z_2''(s)=\left(z_2'(s)\right)^2$ and $w_2''(t)w_2(t)+w_2''(t)=\left(w_2'(t)\right)^2$ which have the solutions $z_2(s) = a_3e^{a_4s} - 1$ and $w_2(t) = a_5e^{a_6t} - 1$. Then, we obtain the surface parameterization (7).

If $c \neq 1$, we yield the solution of the differential equation (30) as $z_2(s) = (1-c)(a_3s+a_4)^{\frac{1}{1-c}} - 1$ and $w_2(t) = \frac{(c-1)(a_5t+a_6)^{\frac{c}{c-1}}-1}{c}$ -1. Hence, we get the surface (8). Also, in equation (27) we suppose

$$\left(z_1''+z_1''w_1\right)\left(w_1''+z_1w_1''\right)-\left(z_1'w_1'\right)^2=0. \tag{31}$$

$$\left(z_2'' + z_2''w_2\right)\left(w_2'' + z_2w_2''\right) - \left(z_2'w_2'\right)^2 = 0. \tag{32}$$

and

$$\left(z_2'' + z_2''w_2\right)\left(w_1'' + z_1w_1''\right) + \left(z_1'' + z_1''w_1\right)\left(w_2'' + z_2w_2''\right) - 2z_1'w_1'z_2'w_2' = 0 \text{ or } A_3 = 0. \tag{33}$$

In a similar way, we obtain the solutions of the differential equations (31) and (32) as

$$\begin{aligned} z_1(s) &= a_1e^{a_2s} - 1, \quad w_1(t) = a_5e^{a_6t} - 1, \\ z_2(s) &= a_3e^{a_4s} - 1, \quad w_2(t) = a_7e^{a_8t} - 1. \end{aligned} \tag{34}$$

Substituting these functions into (33), we yield the surface parametrizations (9) and (10). □

2.2. *Non-null Minimal Translation-Homothetical Surfaces*

Theorem 2.7. *Let M be a non-null translation-homothetical surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, its mean curvature vector is given as*

$$H = -\xi \frac{\left(z_1'' + z_1''w_1\right)G - 2z_1'w_1'F + \left(w_1'' + z_1w_1''\right)E}{2W^2\sqrt{|A_1|}}N_1 + \frac{\left[\left(z_2'' + z_2''w_2\right)G - 2z_2'w_2'F + \left(w_2'' + z_2w_2''\right)E\right]A_1 - \left[\left(z_1'' + z_1''w_1\right)G - 2z_1'w_1'F + w_1'' + z_1w_1''E\right]A_3}{2W^2\sqrt{A_1W^*}}N_2. \tag{35}$$

Proof. By the use of (9), (10) and (23), we obtain the desired result. □

Theorem 2.8. *Let M be a non-null translation-homothetical surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, M has zero mean curvature if and only if*

$$\left(z_i'' + z_i''w_i\right)G - 2z_i'w_i'F + \left(w_i'' + z_iw_i''\right)E = 0. \tag{36}$$

Theorem 2.9. *Let M be a non-null TH−type surface with the parameterization (18) in $\mathbb{E}_1^4$. Then, M is minimal if it is given by one of the following parametrizations :*

(1) $f(s,t) = a_1t + a_2, \quad g(s,t) = a_3t + a_4,$
(2) $f(s,t) = a_1s + a_2, \quad g(s,t) = a_3s + a_4,$
(3) $f(s,t) = a_1s + a_2, \quad g(s,t) = a_3t + a_4,$
(4) $f(s,t) = a_1t + a_2, \quad g(s,t) = a_3s + a_4,$
(5) $f(s,t) = a, \quad g(s,t) = (s+b)\tan(ct+d) - 1,$
(6) $f(s,t) = a, \quad g(s,t) = (t+b)\tan(cs+d) - 1,$
(7) $f(s,t) = (s+b)\tan(ct+d) - 1, \quad g(s,t) = (s+b)\tan(ct+d) - 1,$
(8) $f(s,t) = (t+b)\tan(cs+d) - 1, \quad g(s,t) = (t+b)\tan(cs+d) - 1,$
(9) $f(s,t) = z_1(s) + w_1(t) + z_1(s)w_1(t), \quad g(s,t) = z_2(s) + w_2(t) + z_2(s)w_2(t)$
where the functions satisfy

$$\begin{aligned} s &= \pm\int \frac{dz_i(s)}{\sqrt{2c\ln(z_i(s)+1) - 2ca_1}}, \\ t &= \pm\int \frac{dw_i(t)}{\sqrt{a_2(w_i(t)+1)^4 - \frac{d}{2}}}, \end{aligned} \tag{37}$$

or

$$\begin{aligned} s &= \pm\int \frac{dz_i(s)}{\sqrt{a_1(z_i(s)+1)^4 - \frac{c}{2}}}, \\ t &= \pm\int \frac{dw_i(t)}{\sqrt{2d\ln(w_i(s)+1) - 2da_2}}, \end{aligned} \tag{38}$$

or

$$s = \pm\int \frac{dz_i(s)}{\sqrt{a_1(z_i(s)+1)^{2(1+k)} - a_2}}, \tag{39}$$
$$t = \pm\int \frac{dw_i(t)}{\sqrt{a_3(w_i(t)+1)^{2(1+k)} + a_4}}.$$

Proof. Let M is $TH-$type surface given by the parameterization (18) in $\mathbb{E}^4_1$. If M is minimal, then the equation (36) is hold. Hence, we write

$$\begin{aligned} 0 = {} & \left(z_1'' + z_1''w_1\right)\left(1 + \left(w_1' + z_1w_1'\right)^2 + \left(w_2' + z_2w_2'\right)^2\right) \\ & -2z_1'w_1'\left(\left(z_1' + z_1'w_1\right)\left(w_1' + z_1w_1'\right) + \left(z_2' + z_2'w_2\right)\left(w_2' + z_2w_2'\right)\right) \\ & + \left(w_1'' + z_1w_1''\right)\left(-1 + \left(z_1' + z_1'w_1\right)^2 + \left(z_2' + z_2'w_2\right)\right), \end{aligned} \tag{40}$$

and

$$\begin{aligned} 0 = {} & \left(z_2'' + z_2''w_2\right)\left(1 + \left(w_1' + z_1w_1'\right)^2 + \left(w_2' + z_2w_2'\right)^2\right) \\ & -2z_2'w_2'\left(\left(z_1' + z_1'w_1\right)\left(w_1' + z_1w_1'\right) + \left(z_2' + z_2'w_2\right)\left(w_2' + z_2w_2'\right)\right) \\ & + \left(w_2'' + z_2w_2''\right)\left(-1 + \left(z_1' + z_1'w_1\right)^2 + \left(z_2' + z_2'w_2\right)\right), \end{aligned} \tag{41}$$

The surface parametrizations (1), (2), (3) and (4) are obtained by taking $z_1'(s) = 0$, $z_2'(s) = 0$ or $w_1'(t) = 0$, $w_2'(t) = 0$ or $z_1'(s) = 0, w_2'(t) = 0$ or $z_2'(s) = 0, w_1'(t) = 0$, respectively. By taking $z_1'(s) = 0$ and $w_1'(t) = 0$, then we get

$$-\frac{w_2''}{w_2+1} + \frac{z_2''}{z_2+1} + \left(z_2'\right)^2\left(w_2''(w_2+1) - \left(w_2'\right)^2\right) + \left(w_2'\right)^2\left(z_2''(z_2+1) - \left(z_2'\right)^2\right) = 0. \tag{42}$$

In this equation, if we suppose $z_2''(s) = 0$ or $w_2''(t) = 0$, then we yield $w_2(t) = \frac{\tan(ct+d)}{a_1} - 1$ and $z_2(s) = \frac{\tan(ct+d)}{a_2} - 1$. Hence, the surfaces (5) and (6) are obtained. Also, in (42), if $z_2''(s)w_2''(t) \neq 0$, the derivatives of (42) with regards to s and t, one after another are obtained as

$$\frac{\left(z_2''(z_2+1) - \left(z_2'\right)^2\right)'}{\left(\left(z_2'\right)^2\right)'} = -\frac{\left(w_2''(w_2+1) - \left(w_2'\right)^2\right)'}{\left(\left(w_2'\right)^2\right)'} = c \tag{43}$$

where $c \in IR$. Therefore, integrating this equation regarding s or t, we get

$$\begin{aligned} z_2''(z_2+1) - (1+c)\left(z_2'\right)^2 &= k, \\ w_2''(w_2+1) - (1-c)\left(w_2'\right)^2 &= l, \end{aligned} \tag{44}$$

where $k, l \in IR$. By taking $c = 1$ and $c = -1$ in (44) respectively, we get

$$\begin{aligned} z_2''(z_2+1) &= k, \\ w_2''(w_2+1) - 2\left(w_2'\right)^2 &= l, \end{aligned} \tag{45}$$

and

$$\begin{aligned} z_2''(z_2+1) - 2\left(z_2'\right)^2 &= k, \\ w_2''(w_2+1) &= l. \end{aligned} \tag{46}$$

Thus, if we solve these differential equations, the results (37) and (38) are obtained. If $c \neq 1$ in (44), the solution of the differential equation is congruent to the last result of (9). Finally, by taking $z_1(s) = z_2(s)$ and $w_1(t) = w_2(t)$, then we have the surfaces (7) and (8). This completes the proof. □

2.3. Non-null Semiumbilical Translation-Homothetical Surfaces

Theorem 2.10. *Let M be a non-null translation-homothetical surface in $\mathbb{E}_1^4$. Then, M has the normal curvature as*

$$K_N = -\xi \frac{\begin{array}{c} E\left(z_1' w_1' \left(w_2'' + z_2 w_2''\right) - z_2' w_2' \left(w_1'' + z_1 w_1''\right)\right) \\ -F\left(\left(z_1'' + z_1'' w_1\right)\left(w_2'' + z_2 w_2''\right) - \left(z_2'' + z_2'' w_2\right)\left(w_1'' + z_1 w_1''\right)\right) \\ +G\left(z_2' w_2' \left(z_1'' + z_1'' w_1\right) - z_1' w_1' \left(z_2'' + z_2'' w_2\right)\right) \end{array}}{W^3 \sqrt{W^*}}. \tag{47}$$

Proof. Let M be a non-null $TH-$type surface with (15) in $\mathbb{E}_1^4$. Substituting the second fundamental form coefficients h_{ij}^k into (12), we get the result. □

Corollary 2.11. *Let M be a non-null translation-homothetical surface with the parameterization (15). If the functions z_i, w_i, $(i = 1, 2)$ are linear polynomial functions, then M corresponds to semiumbilical surface in $\mathbb{E}_1^4$.*

Proof. Let M be a non-null $TH-$type surface and suppose z_i, w_i, $(i = 1, 2)$ are the linear polynomial functions as

$$\begin{aligned} z_i &= a_i s + b_i, \\ w_i &= c_i t + d_i. \end{aligned} \tag{48}$$

Thus, by the use of (47) and (48), we get $z_i''(s) = 0$, $w_i''(t) = 0$, i.e , $K_N = 0$. This completes the proof. □

Example. The surface given by the parameterization

$$\psi(s,t) = (s, t, 2st - 2s + 3t - 3, -st + 2s + 4t - 8)$$

is a semiumbilical $TH-$type surface and can be plotted by projection in 3−dimension with command $plot3d\left([s+t, f(s,t), g(s,t)] : \ s = -2..2, \ t = 0..1\right)$:

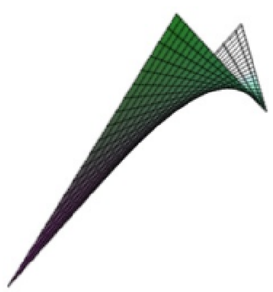

Figure 1: Semiumbilical $TH-$type surface

3. Conclusion

TH−type surfaces (or Translation-Homothetical surfaces) have been previously discussed by Difi et al.(2018) and Pamuk et al.(2021). They considered 3−dimensional spaces and 4−dimensional Euclidean space. In this article, we define non-null translation-homothetical surfaces in Minkowski space-time and classify these surfaces with respect to being flat, minimal and especially semiumbilical. The results provide valuable insights into the nature of surfaces in Minkowski space and will be of interest to researchers and scholars in the fields of mathematics, physics, and astronomy.

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