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On Midpoint Type Inequalities for Proportional Caputo-Hybrid Operator

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Abstract. The primary purpose of this paper is to introduce a newly developed generalized identity, which is rigorously proven, and to apply it in order to present some midpoint type inequalities via a proportional Caputo-hybrid operator.

1. INTRODUCTION

Definition 1.1. *The function* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, *is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ *and* $\lambda\in[0,1]$. *We say that* f *is concave if* $(-f)$ *is convex.*

The theory of convex functions is a crucial area of mathematics that has applications in a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. This theory is also highly relevant in other areas of science, such as economics, finance, engineering, and management sciences. One of the most well-known inequalities in the literature is the Hermite-Hadamard integral inequality (see, [5]), which is a fundamental tool for studying the properties of convex functions. This inequality has important implications in many areas of mathematics and has been extensively studied in recent years, leading to the development of new and powerful mathematical techniques for solving a broad range of problems.

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

These inequalities were first introduced independently by Charles Hermite and Jacques Hadamard in the late 19th century and has since found numerous applications in various fields of mathematics, including analysis, geometry, and probability theory. The inequalities states that if a function is convex on a given interval, then the average value of the function over that interval is bounded from above by the midpoint value of the function, multiplied by the length of the interval. This inequalities provide a powerful tool

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for estimating integrals and has become a standard result in the theory of convex functions. The Hermite-Hadamard inequalities have numerous applications in mathematics. For example, they can be used to solve problems in integral calculus, probability theory, statistics, optimization, and number theory. The inequalities are also useful in solving physical and engineering problems that require the determination of function averages. In general, the Hermite-Hadamard inequalities provide a powerful tool for solving a wide range of mathematical problems. They are widely studied and used in various fields of mathematics, and their applications continue to grow as new problems are encountered. One of the most widely applied inequalities for convex functions is Hadamard's inequality, which has significant geometric implications. This inequality has been extensively studied in the literature, leading to numerous directions for extension and a rich mathematical literature (see [4]-[10], [19]).

While fractional calculus has a rich historical background, recent developments in the field, particularly in the introduction of novel fractional derivative and integral operators by researchers, have revitalized interest in this area, particularly within applied sciences. This surge in interest has led to the introduction of numerous new fractional operators into the literature, driven by investigations into the properties of fractional derivative and associated integral operators, such as their singularity and locality, and modifications to their kernel structure.

Despite ongoing debates surrounding the efficacy of these operators, it is crucial to evaluate their contributions within the context of their respective problem domains. Though each operator serves a functional purpose, some may include a memory effect or a general kernel structure, which may make them more suitable for specific applications. As such, it is essential to consider the functionality of these operators, alongside their potential to improve the solutions of the problems in which they are employed. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see [12]. Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type, for some of them, please see ([1], [3], [11], [13]-[18], [21]-[24]). Let us begin by introducing this type of inequality.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

Now, let's recall the basic expressions of Hermite-Hadamard inequality for fractional integrals is proved by Sarikaya et al. in [16] as follows:

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a function with $a < b$ and $f \in L_1([a,b])$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$.

The following definition is very important for fractional calculus (see [12]).

Definition 1.4. *Let $\alpha > 0$ and $\alpha \notin \{1,2,..\}$, $n = [\alpha]+1$, $f \in AC^n[a,b]$, the space of functions having n-th derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:*

$$^{C}D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \quad x > a$$

and

$$^{C}D^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)dt, \quad x < b.$$

If $\alpha = n \in \{1,2,...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $^{C}D^{\alpha}_{a^+}f(x)$ coincides with $f^{(n)}(x)$ whereas $^{C}D^{\alpha}_{b^-}f(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$^{C}D^{0}_{a^+}f(x) = {}^{C}D^{0}_{b^-}f(x) = f(x)$$

where $n = 1$ and $\alpha = 0$.

The Caputo derivative operator is a fractional derivative operator that is widely used in the field of fractional calculus. It is defined as the fractional derivative of a function with respect to time, where the order of the derivative is a non-integer value. The Riemann-Liouville integral operator, on the other hand, is a fractional integral operator that is also commonly used in fractional calculus.

The proportional Caputo hybrid operator is a mathematical operation that has been proposed as a non-local and singular operator, incorporating both derivative and integral operator components in its definition. It can be expressed as a straightforward linear combination of the Riemann-Liouville integral and Caputo derivative operators, see ([2] and [6]).

Definition 1.5. *Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I° and $f, f' \in L^1(I)$. Then the proportional Caputo-hybrid operator may be defined as*

$$^{PC}_{0}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \left(K_1(\alpha,\tau)f(\tau) + K_0(\alpha,\tau)f'(\tau)\right)(t-\tau)^{-\alpha}d\tau$$

where $\alpha \in [0,1]$ and K_0 and K_1 are functions satisfing

$$\begin{aligned} \lim_{\alpha\to 0^+} K_0(\alpha,\tau) &= 0; \ \lim_{\alpha\to 1} K_0(\alpha,\tau) = 1; \ K_0(\alpha,\tau) \neq 0, \ \alpha \in (0,1]; \\ \lim_{\alpha\to 0} K_1(\alpha,\tau) &= 0; \ \lim_{\alpha\to 1^-} K_1(\alpha,\tau) = 0; \ K_1(\alpha,\tau) \neq 0, \ \alpha \in [0,1). \end{aligned}$$

In this study, let's redefine the above definition by new defining the K_0 and K_1 functions as follows:

Definition 1.6. *Let $f: I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I° and $f, f' \in L^1(I)$. The left-sided and right-sided proportional Caputo-hybrid operator of order α are defined as follows:*

$$^{PC}_{a^+}D^{\alpha}_{b}f(b) = \frac{1}{\Gamma(1-\alpha)}\int_a^b \left[K_1(\alpha,b-\tau)f(\tau) + K_0(\alpha,b-\tau)f'(\tau)\right](b-\tau)^{-\alpha}d\tau$$

and

$$^{PC}_{b^-}D^{\alpha}_{a}f(a) = \frac{1}{\Gamma(1-\alpha)}\int_a^b \left[K_1(\alpha,\tau-a)f(\tau) + K_0(\alpha,\tau-a)f'(\tau)\right](\tau-a)^{-\alpha}d\tau$$

where $\alpha \in [0,1]$ and $K_0(\alpha,t) = (1-\alpha)^2 t^{1-\alpha}$ and $K_1(\alpha,t) = \alpha^2 t^{\alpha}$.

In the following theorem, Hermite-Hadamard type inequality for the proportional Caputo-hybrid operator is proved by Sarikaya in [14] as follows:

Theorem 1.7. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$ and f, f' be convex functions on I. Then the following inequalities hold:*

$$\begin{aligned} & \alpha^2 (b-a)^\alpha f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha} f'\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}_a^{PC}D_b^\alpha f(b) + {}_b^{PC}D_a^\alpha f(a) \right] \\ \leq\ & \alpha^2 (b-a)^\alpha \left[\frac{f(a)+f(b)}{2}\right] + (1-\alpha)(b-a)^{1-\alpha}\left[\frac{f'(a)+f'(b)}{4}\right]. \end{aligned} \tag{3}$$

The main object of this paper is to present some midpoint type inequalities via proportional Caputo-hybrid operator using a newly developed generalized an identity, which is rigorously proven. Our findings not only expand upon previous research but also offer valuable insights and techniques for addressing a broad range of mathematical and scientific problems.

2. MAIN RESULTS

To prove our other main results, we require the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$, and $f', f'' \in L[a,b]$. Then the following identity holds,*

$$\begin{aligned} & F(a,b;\alpha) \\ =\ & \alpha^2 (b-a)^\alpha f\left(\frac{a+b}{2}\right) + \frac{(1-\alpha)(b-a)^{1-\alpha}}{2} f'\left(\frac{a+b}{2}\right) \\ & - \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}_a^{PC}D_b^\alpha f(b) + {}_b^{PC}D_a^\alpha f(a) \right] \end{aligned} \tag{4}$$

where

$$\begin{aligned} & F(a,b;\alpha) \\ =\ & \alpha^2 (b-a)^{1+\alpha} \left\{ \int_{\frac{1}{2}}^{1} (1-t) f'(ta+(1-t)b)\,dt - \int_0^{\frac{1}{2}} t f'(ta+(1-t)b)\,dt \right\} \\ & + \frac{(1-\alpha)(b-a)^{2-\alpha}}{4} \left\{ \int_{\frac{1}{2}}^{1} \left(1 - t^{2-2\alpha} + (1-t)^{2-2\alpha}\right) f''(ta+(1-t)b)\,dt \right. \\ & \left. - \int_0^{\frac{1}{2}} \left(1 - (1-t)^{2-2\alpha} + t^{2-2\alpha}\right) f''(ta+(1-t)b)\,dt \right\}. \end{aligned}$$

Proof. By integration by parts, we have

$$\int_0^{\frac{1}{2}} t f'((1-t)a+tb)\,dt = \frac{1}{2(b-a)} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_0^{\frac{1}{2}} f((1-t)a+tb)\,dt$$

and

$$\int_0^{\frac{1}{2}} t^{2-2\alpha} f''((1-t)a+tb)\,dt = \frac{1}{2^{2-2\alpha}(b-a)} f'\left(\frac{a+b}{2}\right) - \frac{2-2\alpha}{b-a}\int_0^{\frac{1}{2}} t^{1-2\alpha} f'((1-t)a+tb)\,dt.$$

Using the change of the variable, by multiplying the results by $\frac{\alpha^2(b-a)^{1+\alpha}}{2}$ and $\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}$ and adding by side to side we have

$$\begin{aligned} &\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_0^{\frac{1}{2}} t f'((1-t)a+tb)\,dt + \frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_0^{\frac{1}{2}} t^{2-2\alpha} f''((1-t)a+tb)\,dt \\ = &\frac{\alpha^2(b-a)^{\alpha}}{4} f\left(\frac{a+b}{2}\right) + \frac{(1-\alpha)(b-a)^{1-\alpha}}{2^{4-2\alpha}} f'\left(\frac{a+b}{2}\right) \\ &-\frac{1}{2(b-a)^{1-\alpha}}\int_a^{\frac{a+b}{2}}\left[\alpha^2(\tau-a)^{\alpha} f(\tau) + (1-\alpha)^2(\tau-a)^{1-\alpha} f'(\tau)\right](\tau-a)^{-\alpha}\,d\tau. \end{aligned} \tag{5}$$

Using a similar method, we have

$$\begin{aligned} &-\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_0^{\frac{1}{2}} t f'(ta+(1-t)b)\,dt - \frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_{\frac{1}{2}}^{1}\left(1-t^{2-2\alpha}\right) f''((1-t)a+tb)\,dt \\ = &\frac{\alpha^2(b-a)^{\alpha}}{4} f\left(\frac{a+b}{2}\right) + (1-\alpha)(b-a)^{1-\alpha}\left(\frac{1}{4}-\frac{1}{2^{4-2\alpha}}\right) f'\left(\frac{a+b}{2}\right) \\ &-\frac{1}{2(b-a)^{1-\alpha}}\int_{\frac{a+b}{2}}^{b}\left[\alpha^2(\tau-a)^{\alpha} f(\tau) + (1-\alpha)^2(\tau-a)^{1-\alpha} f'(\tau)\right](\tau-a)^{-\alpha}\,d\tau. \end{aligned} \tag{6}$$

By adding (5) and (6), we obtain that

$$\begin{aligned} &\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_0^{\frac{1}{2}} t f'((1-t)a+tb)\,dt + \frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_0^{\frac{1}{2}} t^{2-2\alpha} f''((1-t)a+tb)\,dt \\ &-\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_0^{\frac{1}{2}} t f'(ta+(1-t)b)\,dt - \frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_{\frac{1}{2}}^{1}\left(1-t^{2-2\alpha}\right) f''((1-t)a+tb)\,dt \\ = &\frac{\alpha^2(b-a)^{\alpha}}{2} f\left(\frac{a+b}{2}\right) + \frac{(1-\alpha)(b-a)^{1-\alpha}}{4} f'\left(\frac{a+b}{2}\right) \\ &-\frac{1}{2(b-a)^{1-\alpha}}\int_a^{b}\left[\alpha^2(\tau-a)^{\alpha} f(\tau) + (1-\alpha)^2(\tau-a)^{1-\alpha} f'(\tau)\right](\tau-a)^{-\alpha}\,d\tau. \end{aligned} \tag{7}$$

Using a similar method, we have

$$\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_{\frac{1}{2}}^{1}(t-1) f'((1-t)a+tb)\,dt \tag{8}$$

$$-\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_{\frac{1}{2}}^{1}(1-t)^{2-2\alpha}f''((1-t)a+tb)\,dt$$

$$=\frac{\alpha^2(b-a)^{\alpha}}{4}f\left(\frac{a+b}{2}\right)+\frac{(1-\alpha)(b-a)^{1-\alpha}}{2^{4-2\alpha}}f'\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{2(b-a)^{1-\alpha}}\int_{\frac{a+b}{2}}^{b}\left[\alpha^2(b-\tau)^{\alpha}f(\tau)+(1-\alpha)^2(b-\tau)^{1-\alpha}f'(\tau)\right](b-\tau)^{-\alpha}\,d\tau.$$

and

$$\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_{\frac{1}{2}}^{1}(1-t)f'(ta+(1-t)b)\,dt \tag{9}$$

$$+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_{\frac{1}{2}}^{1}\left(1-t^{2-2\alpha}\right)f''(ta+(1-t)b)\,dt$$

$$=\frac{\alpha^2(b-a)^{\alpha}}{4}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}\left(\frac{1}{4}-\frac{1}{2^{4-2\alpha}}\right)f'\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{2(b-a)^{1-\alpha}}\int_{a}^{\frac{a+b}{2}}\left[\alpha^2(b-\tau)^{\alpha}f(\tau)+(1-\alpha)^2(b-\tau)^{1-\alpha}f'(\tau)\right](b-\tau)^{-\alpha}\,d\tau.$$

By adding (8) and (9), we get

$$\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_{\frac{1}{2}}^{1}(t-1)f'((1-t)a+tb)\,dt \tag{10}$$

$$-\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_{\frac{1}{2}}^{1}(1-t)^{2-2\alpha}f''((1-t)a+tb)\,dt$$

$$+\frac{\alpha^2(b-a)^{1+\alpha}}{2}\int_{\frac{1}{2}}^{1}(1-t)f'(ta+(1-t)b)\,dt$$

$$+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\int_{\frac{1}{2}}^{1}\left(1-t^{2-2\alpha}\right)f''(ta+(1-t)b)\,dt$$

$$=\frac{\alpha^2(b-a)^{\alpha}}{2}f\left(\frac{a+b}{2}\right)+\frac{(1-\alpha)(b-a)^{1-\alpha}}{4}f'\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{2(b-a)^{1-\alpha}}\int_{a}^{b}\left[\alpha^2(b-\tau)^{\alpha}f(\tau)+(1-\alpha)^2(b-\tau)^{1-\alpha}f'(\tau)\right](b-\tau)^{-\alpha}\,d\tau.$$

Thus, by adding (7) and (10), we get desired equality (4). Note that

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}(t-1)\left[f'((1-t)a+tb)-f'(ta+(1-t)b)\right]dt \\
=&\int_{\frac{1}{2}}^{1}(1-t)f'(ta+(1-t)b)\,dt-\int_{0}^{\frac{1}{2}}tf'(ta+(1-t)b)\,dt,
\end{aligned}$$

$$\begin{aligned}
&\int_{0}^{\frac{1}{2}}t\left[f'((1-t)a+tb)-f'(ta+(1-t)b)\right]dt \\
=&\int_{\frac{1}{2}}^{1}(1-t)f'(ta+(1-t)b)\,dt-\int_{0}^{\frac{1}{2}}tf'(ta+(1-t)b)\,dt,
\end{aligned}$$

$$\begin{aligned}
&\int_{0}^{\frac{1}{2}}t^{2-2\alpha}f''((1-t)a+tb)\,dt-\int_{\frac{1}{2}}^{1}(1-t)^{2-2\alpha}f''((1-t)a+tb)\,dt \\
=&\int_{\frac{1}{2}}^{1}(1-t)^{2-2\alpha}f''(ta+(1-t)b)\,dt-\int_{0}^{\frac{1}{2}}t^{2-2\alpha}f''(ta+(1-t)b)\,dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left(1-t^{2-2\alpha}\right)\left[f''(ta+(1-t)b)-f''((1-t)a+tb)\right]dt \\
=&\int_{\frac{1}{2}}^{1}\left(1-t^{2-2\alpha}\right)f''(ta+(1-t)b)\,dt-\int_{0}^{\frac{1}{2}}\left(1-(1-t)^{2-2\alpha}\right)f''(ta+(1-t)b)\,dt.
\end{aligned}$$

□

Remark 2.2. *In Lemma 2.1,*

i) we choose $\alpha = 1$, then the equality (4) becomes the following equality,

$$\begin{aligned}
&\frac{1}{(b-a)}F(a,b;1)=f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx \\
=&(b-a)\left\{\int_{\frac{1}{2}}^{1}(1-t)f'(ta+(1-t)b)\,dt-\int_{0}^{\frac{1}{2}}tf'(ta+(1-t)b)\,dt\right\}
\end{aligned}$$

which is proved by Kirmaci in [8],

ii) we choose $\alpha = 0$, then the equality (4) becomes the following equality

$$F(a,b;0) = \frac{(b-a)}{2} f'\left(\frac{a+b}{2}\right) - \frac{f(b)-f(a)}{2}$$
$$= \frac{(b-a)^2}{2}\left\{\int_{\frac{1}{2}}^{1}(1-t) f''(ta+(1-t)b)\,dt - \int_0^{\frac{1}{2}}\left(t-t^2\right) f''(ta+(1-t)b)\,dt\right\}$$

iii) we choose $\alpha = \frac{1}{2}$, then the equality (4) becomes the following equality

$$\frac{4}{(b-a)^{\frac{1}{2}}} F\left(a,b;\frac{1}{2}\right) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right) - 2\int_a^b \left[f(\tau)+f'(\tau)\right] d\tau$$
$$= 2(b-a)\left\{\int_{\frac{1}{2}}^{1}(1-t) f'(ta+(1-t)b)\,dt - \int_0^{\frac{1}{2}} t f'(ta+(1-t)b)\,dt\right\}.$$

Theorem 2.3. *With the assumptations in Lemma 2.1. If $\left|f'\right|$ and $\left|f''\right|$ are convex on $[a,b]$, then we have the following inequality*

$$\begin{aligned} &\left|\alpha^2 (b-a)^\alpha f\left(\frac{a+b}{2}\right) + \frac{(1-\alpha)(b-a)^{1-\alpha}}{2} f'\left(\frac{a+b}{2}\right) \right. \\ &\left. - \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\left[{}_{a}^{PC}D_b^\alpha f(b) + {}_{b}^{PC}D_a^\alpha f(a)\right]\right| \\ \leq\ & \alpha^2 (b-a)^{1+\alpha}\left(\frac{\left|f'(a)\right| + \left|f'(b)\right|}{8}\right) \\ &+ \frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\left[\frac{1}{2} + \frac{1}{(3-2\alpha)}\left(1-\frac{1}{2^{2-2\alpha}}\right)\right]\left(\left|f''(a)\right| + \left|f''(b)\right|\right). \end{aligned} \tag{11}$$

Proof. We take absolute value of (4) and by using the convexities of $\left|f'\right|$ and $\left|f''\right|$, we have

$$\begin{aligned} &|F(a,b;\alpha)| \\ =\ & \left|\alpha^2 (b-a)^\alpha f\left(\frac{a+b}{2}\right) + \frac{(1-\alpha)(b-a)^{1-\alpha}}{2} f'\left(\frac{a+b}{2}\right) \right. \\ &\left. - \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\left[{}_{a}^{PC}D_b^\alpha f(b) + {}_{b}^{PC}D_a^\alpha f(a)\right]\right| \\ \leq\ & \alpha^2 (b-a)^{1+\alpha}\left\{\int_{\frac{1}{2}}^{1}(1-t)\left|f'(ta+(1-t)b)\right| dt + \int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right| dt\right\} \\ &+ \frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\left\{\int_{\frac{1}{2}}^{1}\left|f''(ta+(1-t)b)\right| dt \right. \\ &+ \int_{\frac{1}{2}}^{1}\left|t^{2-2\alpha} - (1-t)^{2-2\alpha}\right|\left|f''(ta+(1-t)b)\right| dt \end{aligned}$$

$$
\begin{aligned}
& + \int_0^{\frac{1}{2}} \left|f''(ta+(1-t)b)\right| dt + \int_0^{\frac{1}{2}} \left|(1-t)^{2-2\alpha} - t^{2-2\alpha}\right| \left|f''(ta+(1-t)b)\right| dt \Bigg\} \\
\leq\ & \alpha^2 (b-a)^{1+\alpha} \left\{ \int_{\frac{1}{2}}^{1} (1-t)\left[t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right] dt + \int_0^{\frac{1}{2}} t\left[t\left|f'(a)\right| + (1-t)\left|f'(b)\right|\right] dt \right\} \\
& + \frac{(1-\alpha)(b-a)^{2-\alpha}}{4} \left\{ \int_0^1 \left[t\left|f''(a)\right| + (1-t)\left|f''(b)\right|\right] dt \right. \\
& + \int_{\frac{1}{2}}^{1} \left|t^{2-2\alpha} - (1-t)^{2-2\alpha}\right| \left[t\left|f''(a)\right| + (1-t)\left|f''(b)\right|\right] dt \\
& \left. + \int_0^{\frac{1}{2}} \left|(1-t)^{2-2\alpha} - t^{2-2\alpha}\right| \left[t\left|f''(a)\right| + (1-t)\left|f''(b)\right|\right] dt \right\} \\
\leq\ & \alpha^2 (b-a)^{1+\alpha} \left(\frac{\left|f'(a)\right| + \left|f'(b)\right|}{8} \right) \\
& + \frac{(1-\alpha)(b-a)^{2-\alpha}}{4} \left[\frac{1}{2} + \frac{1}{(3-2\alpha)}\left(1 - \frac{1}{2^{2-2\alpha}}\right) \right] \left(\left|f''(a)\right| + \left|f''(b)\right| \right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_{\frac{1}{2}}^{1} \left|1 - t^{2-2\alpha} + (1-t)^{2-2\alpha}\right| t dt &\leq \int_{\frac{1}{2}}^{1} t dt + \int_{\frac{1}{2}}^{1} \left(t^{2-2\alpha} - (1-t)^{2-2\alpha}\right) t dt \\
&= \frac{3}{8} + \frac{1}{4-2\alpha} - \frac{1}{(3-2\alpha)2^{3-2\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
\int_{\frac{1}{2}}^{1} \left|1 - t^{2-2\alpha} + (1-t)^{2-2\alpha}\right| (1-t) dt &\leq \int_{\frac{1}{2}}^{1} (1-t) dt + \int_{\frac{1}{2}}^{1} \left[t^{2-2\alpha} - (1-t)^{2-2\alpha}\right] (1-t) dt \\
&= \frac{1}{8} + \frac{1}{(3-2\alpha)(4-2\alpha)} - \frac{1}{(3-2\alpha)2^{3-2\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\frac{1}{2}} \left|1 - (1-t)^{2-2\alpha} + t^{2-2\alpha}\right| t dt &\leq \int_0^{\frac{1}{2}} t dt + \int_0^{\frac{1}{2}} \left[(1-t)^{2-2\alpha} - t^{2-2\alpha}\right] t dt \\
&= \frac{1}{8} + \frac{1}{(3-2\alpha)(4-2\alpha)} - \frac{1}{(3-2\alpha)2^{3-2\alpha}},
\end{aligned}
$$

and

$$
\int_0^{\frac{1}{2}} \left|1 - (1-t)^{2-2\alpha} + t^{2-2\alpha}\right| (1-t) dt \leq \int_0^{\frac{1}{2}} (1-t) dt + \int_0^{\frac{1}{2}} \left[(1-t)^{2-2\alpha} - t^{2-2\alpha}\right] (1-t) dt
$$

$$= \frac{3}{8}+\frac{1}{4-2\alpha}-\frac{1}{(3-2\alpha)\,2^{3-2\alpha}}.$$

This proves the inequality (11).

Remark 2.4. *In Theorem 2.3,*

i) we choose $\alpha=1$ *, then the inequality (11) becomes the following inequality,*

$$\frac{1}{(b-a)}\left|F(a,b;1)\right| = \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right)$$

which is proved by Kirmaci in [8],

ii) we choose $\alpha=0$*, then the inequality (11) becomes the following inequality*

$$\left|F(a,b;0)\right| = \left|\frac{(b-a)}{2}f'\left(\frac{a+b}{2}\right)-\frac{f(b)-f(a)}{2}\right| \le \frac{3(b-a)^2}{8}\left(\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right)$$

iii) we choose $\alpha=\frac{1}{2}$*, then the inequality (11) becomes the following inequality*

$$\frac{4}{(b-a)^{\frac{1}{2}}}\left|F\left(a,b;\frac{1}{2}\right)\right| = \left|f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)-2\int_a^b\left[f(\tau)+f'(\tau)\right]d\tau\right|$$
$$\le \frac{(b-a)}{4}\left(\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right)+\frac{3(b-a)}{4}\left(\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right).$$

Theorem 2.5. *With the assumptations in Lemma 2.1. If* $\left|f'\right|^q$ *and* $\left|f''\right|^q$ *are convex on* $[a,b]$ *for some fixed* $q>1$*, then we have the following inequality*

$$\begin{aligned}
&\left|\alpha^2(b-a)^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{(1-\alpha)(b-a)^{1-\alpha}}{2}f'\left(\frac{a+b}{2}\right)\right. \qquad (12)\\
&\left.-\frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\left[{}^{PC}_{\ a}D^{\alpha}_{b}f(b)+{}^{PC}_{\ b}D^{\alpha}_{a}f(a)\right]\right|\\
\le\ & \alpha^2(b-a)^{1+\alpha}\frac{1}{(p+1)^{\frac{1}{p}}\,2^{1+\frac{1}{p}}}\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right]\\
&+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\left[\frac{1}{2^{\frac{1}{p}}}+\frac{1}{((2-2\alpha)p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{(2-2\alpha)p}}\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We take absolute value of (4), by using the convexities of $\left|f'\right|$ and $\left|f''\right|$ and the well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|F(a,b;\alpha)\right|\\
=\ &\left|\alpha^2(b-a)^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{(1-\alpha)(b-a)^{1-\alpha}}{2}f'\left(\frac{a+b}{2}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\left[{}_{a}^{PC}D_{b}^{\alpha}f(b)+{}_{b}^{PC}D_{a}^{\alpha}f(a)\right]\Bigg| \\
\leq\;& \alpha^{2}(b-a)^{1+\alpha}\left\{\left(\int_{\frac{1}{2}}^{1}(1-t)^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{0}^{\frac{1}{2}}t^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right\} \\
&+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\left\{\left(\int_{\frac{1}{2}}^{1}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
&+\left(\int_{\frac{1}{2}}^{1}\left|t^{2-2\alpha}-(1-t)^{2-2\alpha}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}} \\
&+\left(\int_{0}^{\frac{1}{2}}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{0}^{\frac{1}{2}}\left|(1-t)^{2-2\alpha}-t^{2-2\alpha}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right\} \\
\leq\;& \alpha^{2}(b-a)^{1+\alpha}\frac{1}{(p+1)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left\{\left(\int_{\frac{1}{2}}^{1}\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{0}^{\frac{1}{2}}\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right\} \\
&+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\left\{\frac{1}{2^{\frac{1}{p}}}\left(\int_{\frac{1}{2}}^{1}\left[t\left|f''(a)\right|^{q}+(1-t)\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{1}{(2-2\alpha)p+1}\left(1-\frac{1}{2^{(2-2\alpha)p}}\right)\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[t\left|f''(a)\right|^{q}+(1-t)\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}} \\
&+\frac{1}{2^{\frac{1}{p}}}\left(\int_{0}^{\frac{1}{2}}\left[t\left|f''(a)\right|^{q}+(1-t)\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\frac{1}{(2-2\alpha)p+1}\left(1-\frac{1}{2^{(2-2\alpha)p}}\right)\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[t\left|f''(a)\right|^q+(1-t)\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}$$

$$= \alpha^2(b-a)^{1+\alpha}\frac{1}{(p+1)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right]$$

$$+\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\left[\frac{1}{2^{\frac{1}{p}}}+\left(\frac{1}{(2-2\alpha)p+1}\left(1-\frac{1}{2^{(2-2\alpha)p}}\right)\right)^{\frac{1}{p}}\right]$$

$$\times\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right].$$

Note that

$$\int_{\frac{1}{2}}^{1}\left[t^{2-2\alpha}-(1-t)^{2-2\alpha}\right]^p dt$$

$$\leq \int_{\frac{1}{2}}^{1}\left[t^{(2-2\alpha)p}-(1-t)^{(2-2\alpha)p}\right]dt=\frac{1}{(2-2\alpha)p+1}\left(1-\frac{1}{2^{(2-2\alpha)p}}\right)$$

and

$$\int_0^{\frac{1}{2}}\left[(1-t)^{2-2\alpha}-t^{2-2\alpha}\right]^p dt$$

$$\leq \int_0^{\frac{1}{2}}\left[(1-t)^{(2-2\alpha)p}-t^{(2-2\alpha)p}\right]dt=\frac{1}{(2-2\alpha)p+1}\left(1-\frac{1}{2^{(2-2\alpha)p}}\right).$$

Here, we use

$$(A-B)^p\leq A^p-B^p$$

for any$A>B\geq 0$ and $p\geq 1$. This proves the inequality (12).

Remark 2.6. *In Theorem 2.5,*

i) we choose $\alpha = 1$, then the inequality (12) becomes the following inequality,

$$\frac{1}{(b-a)}\left|F(a,b;1)\right| = \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right]$$

which is proved by Kirmaci in [8],

ii) we choose $\alpha = 0$, then the inequality (12) becomes the following inequality

$$\left|F(a,b;0)\right| = \left|\frac{(b-a)}{2}f'\left(\frac{a+b}{2}\right)-\frac{f(b)-f(a)}{2}\right|$$

$$\leq \frac{(b-a)^2}{4}\left[\frac{1}{2^{\frac{1}{p}}}+\frac{1}{(2p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{2p}}\right)^{\frac{1}{p}}\right] \times\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right]$$

iii) we choose $\alpha=\frac{1}{2}$, then the inequality (12) becomes the following inequality

$$\begin{aligned}
\frac{4}{(b-a)^{\frac{1}{2}}}\left|F\left(a,b;\frac{1}{2}\right)\right| &= \left|f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)-2\int_a^b\left[f(\tau)+f'(\tau)\right]d\tau\right| \\
&\leq \frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right] \\
&\quad +\frac{(b-a)}{2}\left[\frac{1}{2^{\frac{1}{p}}}+\frac{1}{(p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{p}}\right)^{\frac{1}{p}}\right] \\
&\quad \times\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

References

[1] Budak H., Pehlivan E. and Kosem P., *On new extensions of Hermite-Hadamard inequalities for generalized fractional integrals*, Sahand Communications in Mathematical Analysis, 18(1), 73-88, (2021).

[2] Baleanu D., Fernandez A. and Akgul A., *On a fractional operator combining proportional and classical differintegrals,* Mathematics, 2020, 8, 360.

[3] Budak H., Bilişik C.C. and Sarikaya M. Z., *On some new extensions of inequalities of Hermite-Hadamard type for generalized fractional integrals*, Sahand Communications in Mathematical Analysis, 19(2), 65-79, (2022).

[4] Dragomir S.S. and Agarwal R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezodial formula*, Appl. Math. lett., 11(5) (1998), 91-95.

[5] Dragomir S.S.and Pearce C. E. M., Selected topics on Hermite–Hadamard inequalities and applications, RGMIA Monographs, Victoria University, 2000.

[6] Gürbüz M., Akdemir A. O. and Dokuyucu M. A., *Novel approaches for differentiable convex functions via the proportional Caputo-hybrid operators*, Fractal and Fractional, 6(5), 258, (2022).

[7] Kavurmaci H., Avci M. and Özdemir M.E., *New inequalities of Hermite-Hadamard type for convex functions with applications*, J. Inequalities Appl. 2011, 2011, 86.

[8] Kirmaci U.S., *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. 2004, 147, 137–146.

[9] Kirmaci U. S., Bakula M. K.,Özdemir M. E. and Pečarić J., *Hadamard-type inequalities for s-convex functions*, Applied Mathematics and Computation, 193(1), 26-35, (2007).

[10] Mitrinovic D. S., Pecaric J. E., and Fink A. M., Inequalities involving functions and their integrals and derivatives, Kluwer Academic Publishers, Dordrecht, 1994.

[11] Mohammed P. O. and Brevik I., *A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals,* Symmetry, 12(4), 610, (2020).

[12] Samko S. G., Kilbas A. A, Marichev O. I., *Fractional Integrals and Derivatives Theory and Application*, Gordan and Breach Science, New York, 1993.

[13] Öğülmüş H. and Sarikaya M. Z., *Some Hermite-Hadamard type inequalities for h-convex functions and their applications,* Iranian Journal of Science and Technology, Transactions A: Science, 44, 813-819, (2020).

[14] Sarikaya M.Z., *On Hermite-Hadamard type inequalities for proportional Caputo-hybrid operator*, Konuralp Journal of Mathematics,11(1)(2023)31-39.

[15] Sarikaya M.Z. and Yildirim H., *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Mathematical Notes, 17(2), 1049-1059, (2016).

[16] Sarikaya M.Z., Set E., Yaldiz H. and Basak N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model. 57, 2403-2407 (2013).

[17] Sarikaya M.Z., Ertugral F., *On the generalized Hermite-Hadamard inequalities*, Annals of the University of Craiova-Mathematics and Computer Science Series 47 (2020), no. 1, 193-213.

[18] Sarikaya M.Z., Budak H., *Generalized Hermite-Hadamard type integral inequalities for fractional integrals*, Filomat 30 (2016), no. 5, 1315–1326.

[19] Sarikaya M. Z. and Aktan N., *On the generalization of some integral inequalities and their applications*, Mathematical and computer Modelling, 54(9-10), 2175-2182, (2011).

[20] Zabandan G., *An extension and refinement of Hermite-Hadamard inequality and related results*, Int. J. Nonlinear Anal. Appl. 11 (2020) No. 2, 379-390.

[21] Zhang Y., Wang J., *On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals*, J. Inequal. Appl. 2013 (2013), Art. number 220.

[22] Wang J., Li X., Feckan M., Zhou Y., *Hermite–Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal. 92 (2012), no. 11, 2241–2253.

[23] Wang J., Li X., Zhu C., *Refinements of Hermite-Hadamard type inequalities involving fractional integrals*, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 655–666.

[24] Wang J., Zhu C. and Zhou Y., *New generalized Hermite-Hadamard type inequalities and applications to special means*, Journal of Inequalities and Applications, 2013(1), 1-15, (2013).