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# Fixed Point Properties for Asymptotically Nonexpansive Mappings on Large Classes of Closed, Bounded and Convex Subsets in $\alpha$-Duals of Certain Difference Sequence Spaces

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**Abstract.** In 1970, Cesàro sequence spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for $\ell^\infty$, $c_0$ and c. Then, in 1983, Orhan introduced Cesàro Difference Sequence Spaces. Later, Et and Tripathy et. al. generalized the space introduced by Orhan. Moreover, in 1989, Çolak obtained new types of sequence spaces by generalizing Kızmaz's idea and using Çolak's structure, Et and Esi, in 2000, obtained generalized difference sequences. Using Et and Esi's structure, Ansari and Chaudhry, in 2012, introduced a new type of generalized difference sequence space. Et and Işık, in 2012, obtained new type of generalized difference sequence spaces which have equivalent norm to that of Ansari and Chaudhry's type Banach spaces. Then, Et and Işık found $\alpha$-duals of the Banach spaces they got and investigated geometric properties for them.

In this study, firstly, we recall that in 1979, Goebel and Kuczomow showed that there exist large classes of closed bounded and convex subsets in $\ell^1$ with fixed point property for nonexpansive mappings. It is notable that after Goebel and Kuczumow's study, Kaczor and Prus wanted to find large classes of closed bounded and convex subsets with fixed point property for asymptotically nonexpansive mappings; then indeed they gave positive answer in $\ell^1$. In this study, we study Kaczor and Prus analogy in the second and third order $\alpha$-duals of difference sequence spaces introduced by Et and Işık and show that affine asypmtotically nonexpansive mappings on large classes of closed, bounded and convex subsets of the Banach spaces taken have fixed points.

## 1. Introduction and Preliminaries

When a Banach space satisfies the condition that every invariant nonexpansive mappings defined on any closed, bounded and convex (cbc) nonemtpy subset has a fixed point, then it is said that the space has the fixed point property for nonexpansive mappings. We need to note that distances between images of distant points under nonexpansive mapping cannot exceed the distances between the points taken. Researchers have considered categorizing Banach spaces with this property.

Firstly, in 1965, Browder [3] shows that Hilbert spaces have the property and the result was generalized in [19] to reflexive Banach spaces with normal structure.

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Then, researchers have especially investigated nonreflexive classical Banach spaces and wondered if they can be renormable and falls in the same category with their equivalent norm while they fail to be members of the category with their usual norm but they were able to detect some nonreflexive Banach spaces which have equivalent norms and they become to have the fixed point property with those renormings. The first example was given in 2008 by Lin [20] for $\ell^1$. Then even it has been asked if the same could have been done for $c_0$, but the answer still remains open. Since the researchers have considered trying to obtain the analogous results for well-known other classical nonreflexive Banach spaces, in 2012, another experiment was done for Lebesgue integrable functions space $L_1[0,1]$ by Hernandes Lineares and Maria [21] but they were able to obtain the positive answer when they restricted the nonexpansive mappings by assuming they were affine as well. One can say that there is no doubt most tries have been inspired by the ideas of the study [16] where in 1979, Goebel and Kuczumow proved that while $\ell^1$ fails the fixed point property since one can easily find a cbc nonweakly compact subset there and a fixed point free invariant nonexpansive map, it is possible to find a very large class subsets in target such that invariant nonexpansive mappings defined on the members of the class have fixed points. In fact, it is easy to notice the traces of those ideas in Lin's work [20]. Even Goebel and Kuczumow's work has inspired many other researchers to investigate if there exist more example of nonreflexive Banach spaces with large classes satisfying fixed point property. For example, Kaczor and Prus [17] wanted to generalize Goebel and Kuczumow's findings by investigating if the same could be done for asymptotically nonexpansive mappings. Then, as their result, they proved that under affinity condition, asymptotically nonexpansive invariant mappings defined on a large class of cbc subsets in $\ell^1$ can have fixed points. Moreover, in 2013, Everest [13] extended Kaczor and Prus's results by finding larger classes satisfying the fixed point property for affine asymptotically nonexpansive mappings. Thus, affinity condition become an easiness tool for their works. In fact, as an another well-known nonreflexive Banach space, Lebesgue space $L_1[0,1]$ was studied in [21] and in their study they obtained an analogous result to [20] as they showed that $L_1[0,1]$ can be renormed to have the fixed point property for affine nonexpansive mappings.

In this study we will investigate some Banach spaces analogous to $\ell^1$. Our aim is to discuss the analogous results for Köthe-Toeplitz duals of certain generalized difference sequence spaces studied by Et and Işık [12]. We show that there exists a very large class of cbc subsets in those spaces with fixed point property for affine asymptotically nonexpansive mappings. Thus, first we will recall the definition of Cesàro sequence spaces introduced by Shiue [24] in 1970 and next we will give Kızmaz's construction in [18] in 1981 for difference sequence spaces since the dual space we work on is obtained from the generalizations of Kızmaz's idea which are derived differently by many researchers such as [8–11, 22, 23, 25]. But we need to note that Et and Esi's work [11] and the further study by Et and Çolak [10] used the new type of difference sequence definition from Çolak's work [8]. Then, using Et and Esi's structure, Ansari and Chaudhry [1], in 2012, introduced a new type of generalized difference sequence spaces. Changing Ansari and Chaudhry's construction slightly, Et and Işık [12], in 2012, obtained new type of generalized difference sequence spaces which have equivalent norm to that of Ansari and Chaudhry's type Banach spaces. Then, Et and Işık found $\alpha$-duals of the Banach spaces they got and investigated geometric properties for them.

Now, first we recall that Shiue [24], in 1970, introduced the Cesàro sequence spaces written as

$$\text{ces}_p = \left\{ (x_n)_n \subset \mathbb{R} \;\middle|\; \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

such that $\ell^p \subset \text{ces}_p$ and

$$\text{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

such that $\ell^\infty \subset ces_\infty$ where $1 \le p < \infty$. Their topological properties have been investigated and it has been seen that for $1 < p < \infty$, $\text{ces}_p$ is a seperable reflexive Banach space. Furthermore, many researchers such as Cui [5] in 1999, Cui, Hudzik, and Li [6] in 2000 and Cui, Meng, and Pluciennik [7] in 2000 were able to prove that for $1 < p < \infty$, Cesàro sequence space $\text{ces}_p$ has the fixed point property.

Easiest way to show that was due to both reflexivity by the fact the space has normal structure when $1 < p < \infty$ (using the fact via Kirk [19]) and the space having the weak fixed point property because of its Garcia-Falset coefficient is less than 2 (see for example the result by Falset [14] in 1997). A good reference about fixed point theory results for Cesàro sequence spaces can be a survey in Chen et. al. [4].

After the introduction of Cesàro sequence spaces, Kızmaz [18], denoting by $\ell^\infty(\triangle)$, $c(\triangle)$ and $c_0(\triangle)$, introduced difference sequence spaces for $\ell^\infty$, c and $c_0$ where they are the Banach spaces of bounded, convergent and null sequences, respectively. Here $\triangle$ represented the difference operator applied to the sequence $x = (x_n)_n$ with the rule given by $\triangle\ x = (x_k - x_{k+1})_k$. In 1981, Kızmaz [18] studied then Köthe-Toeplitz Duals and topological properties for them.

As earlier it was stated, Çolak [8] was one of the researchers generalizing Kızmaz's ideas in [18]. In his work [8] in 1989, Çolak obtained the generalized version of the difference sequence space in the following way by picking an arbitrary sequence of nonzero complex values $v = (v_n)_n$. The new difference operator is denoted by $\triangle_v$ and a sequence $x = (x_n)_n$ the difference sequence is written as $\triangle_v x = (v_k x_k - v_{k+1}x_{k+1})_k$. Then, in their study [11] in 2000, Et and Esi defined a generalized difference sequence space as below.

$$\triangle_v(\ell^\infty) = \left\{x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle_v x \in \ell^\infty\right\},$$

$$\triangle_v(c) = \left\{x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle_v x \in c\right\},$$

$$\triangle_v(c_0) = \left\{x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle_v x \in c_0\right\}.$$

Then, they also defined m$^{th}$ order generalized type difference sequence for any $m \in \mathbb{N}$ given by $\triangle_v^0 x = (v_k x_k)_k$, $\triangle_v^m x = (\triangle_v^m x_k)_k = \left(\triangle_v^{m-1}x_k - \triangle_v^{m-1}x_{k+1}\right)_k$ with $\triangle_v^m x_k = \sum_{i=0}^{m}(-1)^i\binom{m}{i}v_{k+i}x_{k+i}$ for each $k \in \mathbb{N}$.

In fact, Et and Esi [11] further generalized the above difference sequence spaces and Bektaş, Et and Çolak [2] in 2004 not only found the Köthe-Toeplitz duals for them but also obtained the duals for the generalized types of Et and Esi's. We may recall here that their 2$^{nd}$ order and 3$^{rd}$ order difference sequence spaces have the following norms respectively:

$$\|x\|_v^{(2)} = |v_1x_1| + |v_2x_2| + \left\|\triangle_v^m x\right\|_\infty$$

$$\|x\|_v^{(3)} = |v_1x_1| + |v_2x_2| + |v_3x_3| + \left\|\triangle_v^m x\right\|_\infty$$

Then the corresponding Köthe-Toeplitz duals were obtained as in [2] and [11] such that they are written as below:

$$D_1^2 := \left\{a = (a_n)_n \subset \mathbb{R} \;\middle|\; \left(n^2 v_n{}^{-1} a_n\right)_n \in \ell^1\right\} = \left\{a = (a_n)_n \subset \mathbb{R}:\ \|a\|^{(2)} = \sum_{k=1}^{\infty}\frac{k^2|a_k|}{|v_k|} < \infty\right\}$$

and

$$D_1^3 := \left\{a = (a_n)_n \subset \mathbb{R} \;\middle|\; \left(n^3 v_n{}^{-1} a_n\right)_n \in \ell^1\right\} = \left\{a = (a_n)_n \subset \mathbb{R}:\ \|a\|^{(3)} = \sum_{k=1}^{\infty}\frac{k^3|a_k|}{|v_k|} < \infty\right\}$$

Note that $D_1^m \subset \ell^1$ if $k^m\left|v_k{}^{-1}\right| > 1$ for each $k \in \mathbb{N}$ and $\ell^1 \subset D_1^m$ if $k^m\left|v_k{}^{-1}\right| < 1$ for each $k \in \mathbb{N}$ and $m = 2,\ 3$.

Ansari and Chaudhry [1], in 2012, introduced a new type of generalized difference sequence spaces by picking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ as Çolak [8] did and next by symbolizing the new difference sequence space as $\triangle_{v,r}^m(E)$ for arbitrary $r \in \mathbb{R}$, $m \in \mathbb{N}$ and writing that space as below where $X$ is any of the sequence spaces $\ell^\infty$, $c$ or $c_0$.

$$\triangle_{v,r}^m(X) = \left\{x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle_v^m x \in X\right\}$$

where Ansari and Chaudhry [1] defined the norm by

$$\|x\|_{\triangle,v}^m = \sum_{k=1}^{m}|v_k x_k| + \sup_{k\in\mathbb{N}}\left|k^r\triangle_v^m x_k\right|$$

Then, by obtaining an equivalent norm to Ansari and Chaudhry's Banach space, Et and Işık [12] defined m$^{th}$ order generalized type difference sequence for any $m \in \mathbb{N}$ given by

$$\Delta_{v,r}^{(m)}(X) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \Delta_v^m x \in X \right\}$$

where the norm is as follows:

$$\|x\|_{\Delta,v}^{(m)} = \sup_{k \in \mathbb{N}} \left| k^r \Delta_v^m x_k \right|$$

Then, Et and Işık found $\alpha$-duals of the Banach spaces they got and investigated geometric properties for them such that $m^{\text{th}}$ order $\alpha$-duals for their Banach spaces are written as

$$U_1^m := \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; \left( n^{m-r} v_n{}^{-1} a_n \right)_n \in \ell^1 \right\} = \left\{ a = (a_n)_n \subset \mathbb{R}\text{:}\ \|a\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^{m-r} |a_k|}{|v_k|} < \infty \right\}$$

In this study, we will consider second and third order $\alpha$-duals; that is, we study their special cases when $m = 2$ and $m = 3$. Note that $U_1^m \subset \ell^1$ if $k^{m-r} \left| v_k{}^{-1} \right| > 1$ for each $k \in \mathbb{N}$ and $\ell^1 \subset U_1^m$ if $k^{m-r} \left| v_k{}^{-1} \right| < 1$ for each $k \in \mathbb{N}$ and $m = 2, 3$.

We will need the below well-known preliminaries before giving our main results. [15] may be suggested as a good reference for these fundamentals.

**Definition 1.1.** *Consider that $(X, \|\cdot\|)$ is a Banach space and let $C$ be a non-empty cbc subset. Let $: C \to C$ be a mapping. We say that*
*1. $T$ is an affine mapping if for every $t \in [0,1]$ and $a, b \in C$, $T((1-t)a + tb) = (1-t)T(a) + t\,T(b)$.*
*2. $T$ is a nonexpansive mapping if for every $a, b \in C$, $\| T(a) - T(b) \| \leq \| a - b \|$.*
*3. $T$ is an asymptotically nonexpansive mapping if there exists a sequence of scalars $(\mathrm{k_n})_{\mathrm{n} \in \mathrm{N}}$ decreasingly converging to 1 such that for every $a, b \in C$, and for every $\mathrm{n} \in \mathbb{N}$, $\| \mathrm{T}^n(a) - \mathrm{T}^n(b) \| \leq k_n \| a - b \|$.*

Then, we will easily obtain an anologous key lemma from the below lemma in the work [16].

**Lemma 1.2.** *Let $\{u_n\}$ be a sequence in $\ell^1$ converging to $u$ in weak-star topology, then for every $w \in \ell^1$,*

$$r(w) = r(u) + \|w - u\|_1$$

*where*

$$r(w) = \limsup_n \|u_n - w\|_1.$$

Note that our scalar field in this study will be real numbers although Çolak [8] considers complex values of $v = (v_n)_n$ while introducing his structer of the difference sequence which is taken as the fundamental concept in this study.

## 2. Main Results

In this section, we will present our results. As earlier it has been mentioned in the first section, we investigate Kaczor and Prus' analogy for the spaces $U_1^2$ and $U_1^3$. We aim to show that there are large classes of cbc subsets in these spaces such that every nonexpansive invariant mapping defined on the subsets in the classes taken has a fixed point. Recall that the invariant mappings have the same domain and the range.

Firstly, due to isometric isomorphism, using Lemma 1.2, we will provide the straight analogous result as a lemma below which will be a key step as in the works such as [16] and [13] and in fact the methods in the study [13] will be our lead in this work.

**Lemma 2.1.** *Let $\{u_n\}$ be a sequence in a Banach space $Z$ which is either of the spaces $U_1^2$ or $U_1^3$ such that $\|.\|$ denotes the norm for each space and assume $\{u_n\}$ converges to $u$ in weak-star topology, then for every $w \in Z$,*

$$r(w) = r(u) + \|w - u\|$$

*where*

$$r(w) = \limsup_n \|u_n - w\|.$$

Then we prove the following theorems as our main results.

**Theorem 2.2.** *Fix $t \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := t\, v_1\, e_1$, and $f_n := \frac{v_n}{n^{2-r}} e_n$ for all integers $n \geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both $c_0$ and $\ell^1$. Then, consider the cbc subset $E^{(2)} = E_t{}^{(2)}$ of $U_1^2$ by*

$$E^{(2)} := \left\{ \sum_{n=1}^{\infty} \alpha_n f_n : \ \forall n \in \mathbb{N}, \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\}.$$

*Then, $E^{(2)}$ has the fixed point property for affine asymptotically $\|\,.\|^{(2)}$-nonexpansive mappings.*

*Proof.* Fix $t \in (0,1)$. Let $T{:}E^{(2)} \to E^{(2)}$ be an affine asymptotically mapping. Then, since $T$ is affine, by Lemma 1.1.2 in [13], there exists a sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(2)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|^{(2)} \underset{n}{\to} 0$. Due to isometric isomorphism $U_1^2$ shares common geometric properties with $\ell^1$ and so both $U_1^2$ and its predual have the same fixed point theory facts to $\ell^1$ and $c_0$, respectly. Thus, considering that on bounded subsets the weak star topology on $\ell^1$ is equivalent to the coardinate-wise convergence topology, and $c_0$ is separable, in $U_1^2$, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E^{(2)}$ by

$$C^{(2)} := \overline{E^{(2)}}^{w^*} = \left\{ \sum_{n=1}^{\infty} \alpha_n f_n : \text{each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n \leq 1 \right\}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C^{(2)}$ of $u^{(n)}$.

Then, by Lemma 2.1, we have a function $r{:}U_1^2 \to [0,\infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|^{(2)}, \ \forall w \in U_1^2$$

such that for every $w \in U_1^2$,

$$r(w) = r(u) + \|u - w\|^{(2)}.$$

Since $T$ is asymptotically nonexpansive mapping, there exists a decreasing sequence $(k_n)_{n\in\mathbb{N}} \in [1,\infty)$ decreasingly convergent to 1 such that $\forall a, b \in U_1^2$ and $\forall n \in \mathbb{N}$,

$$\|T^n a - T^n b\|^{(2)} \leq k_n \|a - b\|^{(2)}.$$

*Case 1:* $u \in E^{(2)}$.

Fix $s \in \mathbb{N}$ and take $k_0 = 1$. Then, we have $r(T^s u) = r(u) + \|T^s u - u\|^{(2)}$ and

$$\begin{aligned} r(T^s u) &= \limsup_n \left\|T^s u - u^{(n)}\right\|^{(2)} \leq \limsup_n \left\|T^s u - T^s\left(u^{(n)}\right)\right\|^{(2)} + \limsup_n \left\|T^s\left(u^{(n)}\right) - u^{(n)}\right\|^{(2)} \quad (1) \\ &\leq \limsup_n k_s \left\|u - u^{(n)}\right\|^{(2)} + \limsup_n \sum_{j=1}^{s} \left\|T^j\left(u^{(n)}\right) - T^{j-1}\left(u^{(n)}\right)\right\|^{(2)} \\ &\leq k_s \limsup_n \left\|u - u^{(n)}\right\|^{(2)} + \limsup_n \sum_{j=1}^{s} k_{j-1} \left\|T\left(u^{(n)}\right) - u^{(n)}\right\|^{(2)} \\ &= k_s r(u). \end{aligned}$$

Therefore, $\|T^s u - u\|^{(2)} \le r(u)(k_s - 1)$ and so by taking limit as $s \to \infty$, we have $\lim_s \|T^s u - u\|^{(2)} = 0$ but then since $\lim_s \left\|T^{s+1}u - Tu\right\|^{(2)} \le \lim_s \|T^s u - u\|^{(2)}$, $\lim_s \left\|T^{s+1}u - Tu\right\|^{(2)} = 0$ and so $T^s u$ converges both $Tu$ and $u$; thus, $Tu = u$ by the uniqueness of the limits.

*Case 2:* $u \in C^{(2)} \setminus E^{(2)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\delta_n \ge 0$, $\forall n \in \mathbb{N}$.

Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and next define

$$h := (\delta_1 + \xi) f_1 + \sum_{n=2}^{\infty} \delta_n f_n.$$

Then, $\|h - u\|^{(2)} = \|t\xi e_1\|^{(2)} = \mathrm{t}\xi$.

Now fix $w \in E^{(2)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ with $\alpha_n \ge 0$, $\forall n \in \mathbb{N}$. We may also write each $f_k$ with coefficients $\gamma_k$ for each $k \in \mathbb{N}$ where $\gamma_1 := t\, v_1$, and $\gamma_n := \frac{v_n}{n^{2-r}}$ for all integers $n \ge 2$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\begin{aligned}
\|\mathrm{w} - u\|^{(2)} &= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(2)} = \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(2)} \\
&= \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k) f_k\right\|^{(2)} \\
&= \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k) \frac{k^{2-r} f_k}{v_k}\right| = \sum_{k=1}^{\infty} |\alpha_k - \delta_k| \\
&\ge \left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right| \\
&= \left|1 - \sum_{k=1}^{\infty} \delta_k\right| \\
&= \xi.
\end{aligned}$$

Hence,

$$\|\mathrm{w} - u\|^{(2)} \ge t\xi = \|h - u\|^{(2)}.$$

Next, we have the following.

$$\begin{aligned} r(h) &= r(u) + \|h-u\|^{(2)} \le r(u) + \|T^s h-u\|^{(2)} = r(T^s h) \text{ but this follows} \\ &= \limsup_n \left\|T^s h-u^{(n)}\right\|^{(2)} \text{ then similarly to the inequality (1)} \\ &\le \limsup_n \left\|T^s h-T^s\left(u^{(n)}\right)\right\|^{(2)} + \limsup_n \left\|u^{(n)}-T^s\left(u^{(n)}\right)\right\|^{(2)} \\ &\le k_s \limsup_n \left\|h-u^{(n)}\right\|^{(2)} + \limsup_n \sum_{j=1}^{s} \left\|T^j\left(u^{(n)}\right) - T^{j-1}\left(u^{(n)}\right)\right\|^{(2)} \\ &\le k_s \limsup_n \left\|h-u^{(n)}\right\|^{(2)} + \limsup_n \sum_{j=1}^{s} k_{j-1} \left\|T\left(u^{(n)}\right) - u^{(n)}\right\|^{(2)} \\ &\le k_s \limsup_n \left\|h-u^{(n)}\right\|^{(2)} + 0 \\ &= k_s r(h). \end{aligned}$$

Hence, $r(h) \le r(T^s h) \le k_s r(h)$ ; thus, by taking limit as $s \to \infty$, we have $\lim_s r(T^s h) = r(h)$ ; that is,

$$\lim_s r(u) + \|T^s h-u\|^{(2)} = \lim_s r(u) + \|h-u\|^{(2)} \text{ which means } \lim_s \|T^s h-u\|^{(2)} = \|h-u\|^{(2)}. \tag{2}$$

Moreover, for any $w \in E^{(2)}$,

$$\begin{aligned} \|w-h\|^{(2)} &= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - (\delta_1+\xi) f_1 - \sum_{n=2}^{\infty} \delta_n f_n\right\|^{(2)} = \left\|\sum_{k=2}^{\infty} (\alpha_k-\delta_k) f_k + (\alpha_1-\delta_1-\xi) f_1\right\|^{(2)} \\ &\le \left\|\sum_{k=2}^{\infty} (\alpha_k-\delta_k) f_k\right\|^{(2)} + \left\|(\alpha_1-\delta_1-\xi) f_1\right\|^{(2)} = \sum_{k=2}^{\infty} \left|(\alpha_k-\delta_k) \frac{k^{2-r} f_k}{v_k}\right| + \left|(\alpha_1-\delta_1-\xi) \frac{f_1}{v_1}\right| \\ &\le \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + t|\alpha_1-\delta_1-\xi| = \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + t\left|\alpha_1 + \sum_{k=2}^{\infty} \alpha_k - \sum_{k=2}^{\infty} \alpha_k - \delta_1 - 1 + \sum_{k=1}^{\infty} \delta_k\right| \\ &= \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + t\left|\sum_{k=2}^{\infty} \delta_k - \sum_{k=2}^{\infty} \alpha_k\right| \\ &\le \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + \mathrm{t}\sum_{k=2}^{\infty} |\alpha_k-\delta_k| = (1+\mathrm{t})\sum_{k=2}^{\infty} |\alpha_k-\delta_k| = \frac{1+t}{1-t}(1-\mathrm{t})\sum_{k=2}^{\infty} |\alpha_k-\delta_k| \\ &= \frac{1+t}{1-t}\left[t\xi - t\xi + (1-\mathrm{t})\sum_{k=2}^{\infty} |\alpha_k-\delta_k|\right] = \frac{1+t}{1-t}\left[t(1-(1-\xi)) - t\xi + (1-\mathrm{t})\sum_{k=2}^{\infty} |\alpha_k-\delta_k|\right] \\ &= \frac{1+t}{1-t}\left[t(1-(1-\xi)) + (1-\mathrm{t})\sum_{k=2}^{\infty} |\alpha_k-\delta_k| - t\xi\right] \\ &= \frac{1+t}{1-t}\left[t\left(\sum_{k=1}^{\infty} \alpha_k - \sum_{k=1}^{\infty} \delta_k\right) + (1-\mathrm{t})\sum_{k=2}^{\infty} |\alpha_k-\delta_k| - t\xi\right] \\ &\le \frac{1+t}{1-t}\left[t\sum_{k=1}^{\infty} |\alpha_k-\delta_k| + (1-\mathrm{t})\sum_{k=2}^{\infty} |\alpha_k-\delta_k| - t\xi\right]. \end{aligned}$$

Hence,

$$\|w-h\|^{(2)} \leq \frac{1+t}{1-t}\left[t\,|\alpha_1-\delta_1| \;+\sum_{k=2}^{\infty}|\alpha_k-\delta_k|-t\xi\right]=\frac{1+t}{1-t}\left[\|w-u\|^{(2)} \;-\|h-u\|^{(2)}\right].$$

Now, fix $\varepsilon>0$ and recall that $t\in(0,1)$. Then, we can choose $\Gamma(\varepsilon):=\frac{1-t}{1+t}\varepsilon\in(0,\infty)$ such that for any $w=\sum_{k=1}^{\infty}\alpha_k f_k\in E^{(2)}$,

$$\left|\|w-u\|^{(2)} \;-\|h-u\|^{(2)}\right|\leq\|w-u\|^{(2)} \;-\|h-u\|^{(2)}<\Gamma.$$

Then, $\|w-h\|^{(2)}<\frac{1+t}{1-t}\Gamma=\varepsilon$.

Therefore, for every $\varepsilon>0$, there exists $\Gamma=\Gamma(\varepsilon)$ such that if $\left|\|w-u\|^{(2)} \;-\|h-u\|^{(2)}\right|<\Gamma$ then $\|w-h\|^{(2)}<\varepsilon$ so this implies for any sequence $(\vartheta_n)_n$ in $E^{(2)}$ with $\lim_n \|\vartheta_n-u\|^{(2)}=\|h-u\|^{(2)}$ implies $\lim_n \|\vartheta_n-h\|^{(2)}=0$. But then since from (2) we have $\lim_s \|T^s h-u\|^{(2)}=\|h-u\|^{(2)}$, we get $\lim_s \|T^s h-h\|^{(2)}=0$.

Furthermore,

$$\|h-Th\|^{(2)}\leq\lim_s \|T^s h-h\|^{(2)}+\lim_s \|T^s h-Th\|^{(2)}\leq k_1\lim_s \left\|T^{s-1}h-h\right\|^{(2)}=0$$

Hence, $Th=h$ and so $E^{(2)}$ has the fixed point property for asymptotically nonexpansive mappings as desired. □

**Theorem 2.3.** *Fix $t\in(0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1:=t\,v_1\,e_1$, and $f_n:=\frac{v_n}{n^{3-r}}e_n$ for all integers $n\geq 2$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both $c_0$ and $\ell^1$. Then, consider the cbc subset $E^{(3)}=E_t{}^{(3)}$ of $U_1^3$ by*

$$E^{(3)}:=\left\{\sum_{n=1}^{\infty}\alpha_n f_n:\ \forall n\in\mathbb{N},\alpha_n\geq 0 \text{ and } \sum_{n=1}^{\infty}\alpha_n=1\right\}.$$

*Then, $E^{(3)}$ has the fixed point property for asymptotically $\|\,.\|^{(3)}$-nonexpansive mappings.*

*Proof.* Fix $t\in(0,1)$. Let $T{:}E^{(3)}\to E^{(3)}$ be an affine asymptotically mapping. Then, since $T$ is affine, by Lemma 1.1.2 in [13], there exists a sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}}\in E^{(3)}$ such that $\left\|Tu^{(n)}-u^{(n)}\right\|^{(3)}\underset{n}{\to}0$. Due to isometric isomorphism $U_1^3$ shares common geometric properties with $\ell^1$ and so both $U_1^3$ and its predual have the same fixed point theory facts to $\ell^1$ and $c_0$, respectly. Thus, considering that on bounded subsets the weak star topology on $\ell^1$ is equivalent to the coardinate-wise convergence topology, and $c_0$ is separable, in $U_1^3$, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E^{(3)}$ by

$$C^{(3)}:=\overline{E^{(3)}}^{w^*}=\left\{\sum_{n=1}^{\infty}\alpha_n f_n\text{: each }\alpha_n\geq 0\text{ and }\sum_{n=1}^{\infty}\alpha_n\leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u\in C^{(3)}$ of $u^{(n)}$.

Then, by Lemma 2.1, we have a function $r{:}U_1^3\to[0,\infty)$ defined by

$$r(w)=\limsup_n\left\|u^{(n)}-w\right\|^{(3)},\ \forall w\in U_1^3$$

such that for every $w\in U_1^3$,

$$r(w)=r(u)+\|u-w\|^{(3)}.$$

Since $T$ is asymptotically nonexpansive mapping, there exists a decreasing sequence $(k_n)_{n\in\mathbb{N}}\in[1,\infty)$ decreasingly convergent to 1 such that $\forall a, b\in U_1^3$ and $\forall n\in\mathbb{N}$,

$$\|T^{n}a-T^{n}b\|^{(3)}\leq k_n\,\|a-b\|^{(3)}.$$

*Case 1:* $u\in E^{(3)}$.

Fix $s\in\mathbb{N}$ and take $k_0=1$. Then, we have $r(T^s u)=r(u)+\|T^s u-u\|^{(3)}$ and

$$\begin{aligned}
r(T^s u) &= \limsup_n \left\|T^s u-u^{(n)}\right\|^{(3)}\leq \limsup_n \left\|T^s u-T^s\left(u^{(n)}\right)\right\|^{(3)}+\limsup_n \left\|T^s\left(u^{(n)}\right)-u^{(n)}\right\|^{(3)} \qquad (3)\\
&\leq \limsup_n k_s\left\|u-u^{(n)}\right\|^{(3)}+\limsup_n \sum_{j=1}^{s}\left\|T^j\left(u^{(n)}\right)-T^{j-1}\left(u^{(n)}\right)\right\|^{(3)}\\
&\leq k_s\limsup_n \left\|u-u^{(n)}\right\|^{(3)}+\limsup_n \sum_{j=1}^{s}k_{j-1}\left\|T\left(u^{(n)}\right)-u^{(n)}\right\|^{(3)}\\
&= k_s r(u).
\end{aligned}$$

Therefore, $\|T^s u-u\|^{(3)}\leq r(u)(k_s-1)$ and so by taking limit as $s\to\infty$, we have $\lim_s\|T^s u-u\|^{(3)}=0$ but then since $\lim_s\left\|T^{s+1}u-Tu\right\|^{(3)}\leq\lim_s\|T^s u-u\|^{(3)}$, $\lim_s\left\|T^{s+1}u-Tu\right\|^{(3)}=0$ and so $T^s u$ converges both $Tu$ and $u$; thus, $Tu=u$ by the uniqueness of the limits.

*Case 2:* $u\in C^{(3)}\setminus E^{(3)}$.

Then, we may find scalars satisfying $u=\sum_{n=1}^{\infty}\delta_n f_n$ such that $\sum_{n=1}^{\infty}\delta_n<1$ *and* $\delta_n\geq 0$, $\forall n\in\mathbb{N}$.

Define $\xi:=1-\sum_{n=1}^{\infty}\delta_n$ and next define

$$h:=(\delta_1+\xi)f_1+\sum_{n=2}^{\infty}\delta_n f_n.$$

Then, $\|h-u\|^{(3)}=\|t\xi e_1\|^{(3)}=t\xi$.

Now fix $w\in E^{(3)}$. Then, we may find scalars satisfying $w=\sum_{n=1}^{\infty}\alpha_n f_n$ such that $\sum_{n=1}^{\infty}\alpha_n=1$ with $\alpha_n\geq 0$, $\forall n\in\mathbb{N}$.

Then,

$$\begin{aligned}
\|w-u\|^{(3)} &= \left\|\sum_{k=1}^{\infty}\alpha_k f_k-\sum_{k=1}^{\infty}\delta_k f_k\right\|^{(3)}=\left\|\sum_{k=1}^{\infty}\alpha_k f_k-\sum_{k=1}^{\infty}\delta_k f_k\right\|^{(3)}\\
&= \left\|\sum_{k=1}^{\infty}(\alpha_k-\delta_k)f_k\right\|^{(3)}\\
&= \sum_{k=1}^{\infty}\left|(\alpha_k-\delta_k)\frac{k^{3-r}f_k}{v_k}\right| = \sum_{k=1}^{\infty}|\alpha_k-\delta_k|\\
&\geq \left|\sum_{k=1}^{\infty}(\alpha_k-\delta_k)\right|\\
&= \left|1-\sum_{k=1}^{\infty}\delta_k\right|\\
&= \xi.
\end{aligned}$$

Hence,

$$\|w-u\|^{(3)} \geq t\xi = \|h-u\|^{(3)}.$$

Next, we have the following.

$$\begin{aligned}
r(h) &= r(u) + \|h-u\|^{(3)} \leq r(u) + \|T^s h-u\|^{(3)} = r(T^s h) \text{ but this follows} \\
&= \limsup_n \left\|T^s h-u^{(n)}\right\|^{(3)} \text{ then similarly to the inequality (3)} \\
&\leq \limsup_n \left\|T^s h-T^s\left(u^{(n)}\right)\right\|^{(3)} + \limsup_n \left\|u^{(n)}-T^s\left(u^{(n)}\right)\right\|^{(3)} \\
&\leq k_s \limsup_n \left\|h-u^{(n)}\right\|^{(3)} + \limsup_n \sum_{j=1}^{s} \left\|T^j\left(u^{(n)}\right) - T^{j-1}\left(u^{(n)}\right)\right\|^{(3)} \\
&\leq k_s \limsup_n \left\|h-u^{(n)}\right\|^{(3)} + \limsup_n \sum_{j=1}^{s} k_{j-1} \left\|T\left(u^{(n)}\right) - u^{(n)}\right\|^{(3)} \\
&\leq k_s \limsup_n \left\|h-u^{(n)}\right\|^{(3)} + 0 \\
&= k_s r(h).
\end{aligned}$$

Hence, $r(h) \leq r(T^s h) \leq k_s r(h)$ ; thus, by taking limit as $s \to \infty$, we have $\lim_s r(T^s h) = r(h)$ ; that is,

$$\lim_s r(u) + \|T^s h-u\|^{(3)} = \lim_s r(u) + \|h-u\|^{(3)} \text{ which means } \lim_s \|T^s h-u\|^{(3)} = \|h-u\|^{(3)}. \tag{4}$$

Moreover, for any $w \in E^{(3)}$,

$$\begin{aligned}
\|w-h\|^{(3)} &= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - (\delta_1+\xi) f_1 - \sum_{n=2}^{\infty} \delta_n f_n\right\|^{(3)} = \left\|\sum_{k=2}^{\infty} (\alpha_k-\delta_k) f_k + (\alpha_1-\delta_1-\xi) f_1\right\|^{(3)} \\
&\leq \left\|\sum_{k=2}^{\infty} (\alpha_k-\delta_k) f_k\right\|^{(3)} + \left\|(\alpha_1-\delta_1-\xi) f_1\right\|^{(3)} = \sum_{k=2}^{\infty} \left|(\alpha_k-\delta_k) \frac{k^{3-r} f_k}{v_k}\right| + \left|(\alpha_1-\delta_1-\xi) \frac{f_1}{v_1}\right| \\
&\leq \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + t|\alpha_1-\delta_1-\xi| = \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + t\left|\alpha_1 + \sum_{k=2}^{\infty} \alpha_k - \sum_{k=2}^{\infty} \alpha_k - \delta_1 - 1 + \sum_{k=1}^{\infty} \delta_k\right| \\
&= \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + t\left|\sum_{k=2}^{\infty} \delta_k - \sum_{k=2}^{\infty} \alpha_k\right| \\
&\leq \sum_{k=2}^{\infty} |\alpha_k-\delta_k| + \mathrm{t}\sum_{k=2}^{\infty} |\alpha_k-\delta_k| = (1+\mathrm{t}) \sum_{k=2}^{\infty} |\alpha_k-\delta_k| = \frac{1+t}{1-t} (1-\mathrm{t}) \sum_{k=2}^{\infty} |\alpha_k-\delta_k| \\
&= \frac{1+t}{1-t}\left[t\xi - t\xi + (1-\mathrm{t}) \sum_{k=2}^{\infty} |\alpha_k-\delta_k|\right] = \frac{1+t}{1-t}\left[t(1-(1-\xi)) - t\xi + (1-\mathrm{t}) \sum_{k=2}^{\infty} |\alpha_k-\delta_k|\right] \\
&= \frac{1+t}{1-t}\left[t(1-(1-\xi)) + (1-\mathrm{t}) \sum_{k=2}^{\infty} |\alpha_k-\delta_k| - t\xi\right] \\
&= \frac{1+t}{1-t}\left[t\left(\sum_{k=1}^{\infty} \alpha_k - \sum_{k=1}^{\infty} \delta_k\right) + (1-\mathrm{t}) \sum_{k=2}^{\infty} |\alpha_k-\delta_k| - t\xi\right] \\
&\leq \frac{1+t}{1-t}\left[t\sum_{k=1}^{\infty} |\alpha_k-\delta_k| + (1-\mathrm{t}) \sum_{k=2}^{\infty} |\alpha_k-\delta_k| - t\xi\right].
\end{aligned}$$

Hence,

$$\|w-h\|^{(3)} \leq \frac{1+t}{1-t}\left[t\,|\alpha_1-\delta_1| + \sum_{k=2}^{\infty} |\alpha_k-\delta_k| - t\xi\right] = \frac{1+t}{1-t}\left[\|w-u\|^{(3)} - \|h-u\|^{(3)}\right].$$

Now, fix $\varepsilon > 0$ and recall that $t \in (0,1)$. Then, we can choose $\Gamma(\varepsilon) := \frac{1-t}{1+t}\varepsilon \in (0,\infty)$ such that for any $w = \sum_{k=1}^{\infty} \alpha_k f_k \in E^{(3)}$,

$$\left|\|w-u\|^{(3)} - \|h-u\|^{(3)}\right| \leq \|w-u\|^{(3)} - \|h-u\|^{(3)} < \Gamma.$$

Then, $\|w-h\|^{(3)} < \frac{1+t}{1-t}\Gamma = \varepsilon$.

Therefore, for every $\varepsilon > 0$, there exists $\Gamma = \Gamma(\varepsilon)$ such that if $\left|\|w-u\|^{(3)} - \|h-u\|^{(3)}\right| < \Gamma$ then $\|w-h\|^{(3)} < \varepsilon$ so this implies for any sequence $(\vartheta_n)_n$ in $E^{(3)}$ with $\lim_n \|\vartheta_n - u\|^{(3)} = \|h-u\|^{(3)}$ implies $\lim_n \|\vartheta_n - h\|^{(3)} = 0$. But then since from (4) we have $\lim_s \|T^s h - u\|^{(3)} = \|h-u\|^{(3)}$, we get $\lim_s \|T^s h - h\|^{(3)} = 0$.

Furthermore,

$$\|h-Th\|^{(3)} \leq \lim_s \|T^s h - h\|^{(3)} + \lim_s \|T^s h - Th\|^{(3)} \leq k_1 \lim_s \left\|T^{s-1}h - h\right\|^{(3)} = 0$$

Hence, $Th = h$ and so $E^{(3)}$ has the fixed point property for asymptotically nonexpansive mappings as desired. □

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