TURKISH JOURNAL OF SCIENCE
VOLUME 8, ISSUE 3, 107–113
ISSN: 2587–0971

https://www.tjoscience.com

Inequalities Involving Conformable Fractional Integrals for η-Convex Functions

Erhan SET[a]

[a] *Department of Mathematics, Faculty of Sciences and Arts, Ordu University, Ordu, Turkey*

Abstract. In the literature, both fractional integral operators and different types of convexity are used to obtain generalizations, extensions and different versions of existing inequalities. Also in this paper, new Hermite Hadamard type inequalities involving conformable fractional integral operators for η-convex functions are obtained.

1. Introduction

Definition 1.1. *A function $f : [a, b] \to \mathbb{R}$ is said to be $\eta-$convex (or convex with respect to η) if the inequality*

$$f(tx + (1 - t)y) \leq f(y) + t\eta(f(x), f(y))$$

holds for all $x, y \in [a, b]$, $t \in [0, 1]$ and η is defined by $\eta : f([a, b]) \times f([a, b]) \to \mathbb{R}$.

In the above definition, if we set $\eta(x, y) = x - y$, then we can directly obtain the classical definition of a convex function. To see more results and details on η-convex functions see [5, 7, 8].

A useful inequality that holds, the Hermite-Hadamard inequality, is embodied in the following theorem. This inequality provides upper and lower bounds for the mean value of a convex function.

Theorem 1.2. *(see, e.g, [3]) If $f : I \to \mathbb{R}$ is a convex function, where $I = [a, b]$ and $\mathbb{R}$ are a set of real numbers, then the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

are valid.

In mathematical analysis and many applied sciences, the concepts of classical analysis have been used effectively for a long time. Over time, however, it has become clear that in the description of physical problems and dynamical processes of nature, the concepts of fractional analysis, whose origins are as old as classical analysis, are more effective. The effective use of fractional analysis in mathematics and other applied sciences has brought a new field of study to the literature. This development in fractional analysis has also had an impact on inequality theory and led to remarkable studies. Fractional analysis

Corresponding author: ES erhanset@yahoo.com ORCID:0000-0003-1364-5396
Received: 10 October 2023; Accepted: 5 December 2023; Published: 31 December 2023
Keywords. η-convex function, Hermite-Hadamard inequality.
2010 *Mathematics Subject Classification.* 26A09, 26D10, 26D15
Cited this article as: Set, E. (2023). Inequalities Involving Conformable Fractional Integrals for η-Convex Functions, Turkish Journal of Science, 8(3), 107-113.

has basically developed as the introduction of fractional order derivative and integral operators based on the fact that the order of the derivative is not an integer. In this context, integral operators such as Riemann-Liouville, Hadamard, (k-s) fractional integral have been presented. Recently, a new definition of the fractional integral presented by Khalil et al. [9] and extended version of this operator provided by Abdeljawad as the following:

Definition 1.3. *[1] Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by*

$$(I_\alpha^a f)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx$$

Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^bI_\alpha f)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n + 1 - n = 1$ and hence $(I_\alpha^a f)(t) = (J_{n+1}^a f)(t)$. Some recent result and properties concerning the fractional integral operators can be found in [1, 2, 4, 6, 9–12].

The Beta function $B(\alpha, \beta)$ is defined by (see, e.g., [14, Section 1.1])

$$B(\alpha,\, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \end{cases}$$

where $\Gamma(\alpha)$ is Gamma function. The Incomplete Beta function is defined by

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt.$$

For $x = 1$, the Incomplete Beta function coincides with the Beta function.

The main motivation of this paper is to obtain some novel inequalities for differentiable $\eta-$convex functions via conformable integral operators.

2. Main Results

Lemma 2.1. *[13] Assume that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $f' \in (L[a, b])$ then the following equality holds:*

$$\begin{aligned} \Psi_\alpha(a,b) \quad = \quad & \frac{-(b-a)\alpha}{16}\Big[\int_0^1 B_t(n+1, \alpha-n) f'\left(ta + (1-t)\frac{3a+b}{4}\right)dt \\ & - \int_0^1 B_{1-t}(\alpha-n, n+1) f'\left(t\frac{3a+b}{4} + (1-t)\frac{a+b}{2}\right)dt \\ & + \int_0^1 B_t(n+1, \alpha-n) f'\left(t\frac{a+b}{2} + (1-t)\frac{a+3b}{4}\right)dt \\ & - \int_0^1 B_{1-t}(\alpha-n, n+1) f'\left(t\frac{a+3b}{4} + (1-t)b\right)dt\Big] \end{aligned}$$

for $\alpha > 0$, $n = 0, 1, 2, \ldots$ where $B_t(.,.)$ is incomplete Beta function and

$$\Psi_\alpha(a,b) \tag{1}$$

$$
\begin{aligned}
&= \frac{\alpha}{4}\Bigg[B(n+1,\alpha-n)\left(f(a)+f\left(\frac{a+b}{2}\right)\right) \\
&\quad +B(\alpha-n,n+1)\left(f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{\alpha 4^{\alpha-1}n!}{(b-a)^{\alpha}} \\
&\quad \times\left(I_{\alpha}^{a}f\left(\frac{3a+b}{4}\right)+I_{\alpha}^{\frac{3a+b}{4}}f\left(\frac{a+b}{2}\right)+I_{\alpha}^{\frac{a+b}{2}}f\left(\frac{a+3b}{4}\right)+I_{\alpha}^{\frac{a+3b}{4}}f(b)\right)\Bigg].
\end{aligned}
$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b) such that $f'\in L([a,b])$ with $a,b\in I$, $a<b$ and $\alpha>0$. If $|f'|^q$ is $\eta-$convex function on $[a,b]$ and $q\geq 1$, then we have the following inequality:*

$$
\begin{aligned}
&|\Psi_{\alpha}(a,b)| \\
&\leq \frac{(b-a)\alpha}{16}(B(n+1,\alpha-n+1))^{1-\frac{1}{q}}\left(\kappa_1^{\frac{1}{q}}+\kappa_3^{\frac{1}{q}}\right) \\
&\quad +(B(n+2,\alpha-n))^{1-\frac{1}{q}}\left(\kappa_2^{\frac{1}{q}}+\kappa_4^{\frac{1}{q}}\right),
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
\kappa_1 &= B(n+1,\alpha-n+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&\quad +\left(\frac{1}{2}B(n+1,\alpha-n)-\frac{1}{2}B(n+3,\alpha-n)\right)\eta\left(|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right), \\
\kappa_2 &= B(n+2,\alpha-n)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{2}B(\alpha-n+2,n+1)\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right), \\
\kappa_3 &= B(n+1,\alpha-n+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q \\
&\quad +\left(\frac{1}{2}B(n+1,\alpha-n)-\frac{1}{2}B(n+3,\alpha-n)\right)\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right), \\
\kappa_4 &= B(n+2,\alpha-n)\left|f'(b)\right|^q+\frac{1}{2}B(\alpha-n+2,n+1)\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'(b)\right|^q\right)
\end{aligned}
$$

and $\alpha\in(n,n+1]$, $n=0,1,2,\ldots$, $B(a,b)$ is Euler Beta function.

Proof. From Lemma 2.1 and by applying power mean inequality with considering the $\eta-$convexity of $|f'|^q$ on $[a,b]$, we can write

$$
\begin{aligned}
&|\Psi_{\alpha}(a,b)| \\
&\leq \frac{(b-a)\alpha}{16}\Bigg[\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|dt \\
&\quad +\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \\
&\quad +\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|dt \\
&\quad +\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|dt\Bigg] \\
&\leq \frac{(b-a)\alpha}{16}\Bigg[\left(\int_0^1 B_t(n+1,\alpha-n)dt\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times\left(\int_0^1 B_t(n+1,\alpha-n)\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q + t\eta\left(\left|f'(a)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)\right)dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 B_{1-t}(\alpha-n,n+1)dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 B_{1-t}(\alpha-n,n+1)\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + t\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right)dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 B_t(n+1,\alpha-n)dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 B_t(n+1,\alpha-n)\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q + t\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)\right)dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 B_{1-t}(\alpha-n,n+1)dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 B_{1-t}(\alpha-n,n+1)\left(\left|f'(b)\right|^q + t\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q, \left|f'(b)\right|^q\right)\right)dt\right)^{\frac{1}{q}}\Bigg].$$

By using integration by parts and relations between Beta function and incomplete Beta function, we have

$$\int_0^1 B_t(n+1,\alpha-n)dt = B(n+1,\alpha-n+1), \tag{3}$$
$$\int_0^1 B_{1-t}(\alpha-n,n+1)dt = B(n+2,\alpha-n). \tag{4}$$

Substituting

$$\begin{aligned}
\kappa_1 &= \int_0^1 B_t(n+1,\alpha-n)\left|f'\left(\frac{3a+b}{4}\right)\right|^q dt \\
&\quad + \int_0^1 B_t(n+1,\alpha-n)t\eta\left(|f'(a)|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)dt \\
&= B(n+1,\alpha-n+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&\quad + \left(B_t(n+1,\alpha-n)\frac{t^2}{2}\Big|_0^1 - \int_0^1 t^n(1-t)^{\alpha-n-1}\frac{t^2}{2}dt\right)\eta\left(|f'(a)|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right) \\
&= B(n+1,\alpha-n+1)\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&\quad + \left(\frac{1}{2}B(n+1,\alpha-n) - \frac{1}{2}B(n+3,\alpha-n)\right)\eta\left(|f'(a)|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right),
\end{aligned}$$

$$\begin{aligned}
\kappa_2 &= \int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q dt \\
&\quad + \int_0^1 B_{1-t}(\alpha-n,n+1)t\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt \\
&= B(n+2,\alpha-n)\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&\quad + \left(B_{1-t}(\alpha-n,n+1)\frac{t^2}{2}\Big|_0^1 + \int_0^1 (1-t)^n t^{\alpha-n-1}\frac{t^2}{2}dt\right)\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)
\end{aligned}$$

$$= B(n+2,\alpha-n)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{2}B(\alpha-n+2,n+1)\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right),$$

$$\begin{aligned}\kappa_3 &= \int_0^1 B_t(n+1,\alpha-n)\left|f'\left(\frac{a+3b}{4}\right)\right|^q dt\\ &+\int_0^1 B_t(n+1,\alpha-n)t\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)dt\\ &= B(n+1,\alpha-n+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q\\ &+\left(B_t(n+1,\alpha-n)\frac{t^2}{2}\Big|_0^1-\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{t^2}{2}dt\right)\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)\\ &= B(n+1,\alpha-n+1)\left|f'\left(\frac{a+3b}{4}\right)\right|^q\\ &+\left(\frac{1}{2}B(n+1,\alpha-n)-\frac{1}{2}B(n+3,\alpha-n)\right)\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right),\end{aligned}$$

$$\begin{aligned}\kappa_4 &= \int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'(b)\right|^q dt\\ &+\int_0^1 B_{1-t}(\alpha-n,n+1)t\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'(b)\right|^q\right)dt\\ &= B(n+2,\alpha-n)\left|f'(b)\right|^q\\ &+\left(B_{1-t}(\alpha-n,n+1)\frac{t^2}{2}\Big|_0^1+\int_0^1(1-t)^n t^{\alpha-n-1}\frac{t^2}{2}dt\right)\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'(b)\right|^q\right)\\ &= B(n+2,\alpha-n)\left|f'(b)\right|^q+\frac{1}{2}B(\alpha-n+2,n+1)\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'(b)\right|^q\right)\end{aligned}$$

and by arrangement via the equalities (3) and (4) in the desired inequality. This completes the proof.

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b) such that $f'\in L([a,b])$ with $a,b\in I$, $a<b$ and $\alpha>0$. If $|f'|^q$ is $\eta-$convex function on $[a,b]$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality:*

$$\begin{aligned}&|\Psi_\alpha(a,b)|\\ \leq\ & \frac{(b-a)\alpha}{16}\left\{\left(\int_0^1|B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\right.\\ &\times\left[\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)}{2}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)}{2}\right)^{\frac{1}{q}}\right]\\ &+\left(\int_0^1|B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\end{aligned}$$

$$\times\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|f'(b)\right|^q+\frac{\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'(b)\right|^q\right)}{2}\right)^{\frac{1}{q}}\right]\Bigg\}.$$

for $\alpha>0$, $n=0,1,2,\dots$ where $B_t(.,.)$ is incomplete Beta function.

Proof. Using Lemma 2.1, well-known Hölder inequality and the $\eta-$convexity of $|f'|^q$ on $[a,b]$, we get

$$\begin{aligned}
&|\Psi_\alpha(a,b)|\\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|dt\\
&+\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt\\
&+\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|dt\\
&+\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|dt\Bigg]\\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{a+3b}{4}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]\\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\left(\int_0^1 (B_t(n+1,\alpha-n))^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+t\eta\left(|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)\right)dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (B_{1-t}(\alpha-n,n+1))^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+t\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right)dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (B_t(n+1,\alpha-n))^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+t\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)\right)dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (B_{1-t}(\alpha-n,n+1))^p dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left(\int_0^1\left(\left|f'(b)\right|^q+t\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'(b)\right|^q\right)\right)dt\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(b-a)\alpha}{16}\left[\left(\int_0^1(B_t(n+1,\alpha-n))^p dt\right)^{\frac{1}{p}}\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{\eta\left(|f'(a)|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)}{2}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1(B_{1-t}(\alpha-n,n+1))^p dt\right)^{\frac{1}{p}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\eta\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)}{2}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(B_t(n+1,\alpha-n))^p dt\right)^{\frac{1}{p}}\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{\eta\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)}{2}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(B_{1-t}(\alpha-n,n+1))^p dt\right)^{\frac{1}{p}}\left(\left|f'(b)\right|^q+\frac{\eta\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'(b)\right|^q\right)}{2}\right)^{\frac{1}{q}}.$$

So, the proof is completed.

References

[1] Abdeljawad, T. (2015). On conformable fractional calculus. Journal of Computational and Applied Mathematics, 279, 57–66.

[2] Akdemir, A.O., Ekinci, A., Set, E. (2017). Conformable fractional integrals and related new integral inequalities. Journal of Nonlinear and Convex Analysis, 18(4), 661–674.

[3] Cerone, P., Dragomir, S.S. (2011). Mathematical inequalities, CRS Press.

[4] Dahmani, Z. (2010). New inequalities in fractional integrals. Int. J. Nonlinear Sci., 9(4), 493–497.

[5] Delavar, M.R., Dragomir, S.S. (2017). On η–convexity. Mathematical Inequalities and Applications, 20(1), 203–216.

[6] Gorenflo, R., Mainardi, F. (1997). Fractional Calculus: Integral and Differential Equations of Fractional Order, Springer Verlag, Wien., 223–276.

[7] Gordji, M.E., Dragomir, S.S., Delavar, M.R. (2015). An inequality related to η-convex functions (II). Int. J. Nonlinear Anal. Appl., 6(2), 27–33.

[8] Gordji, M.E., Delavar, M.R., De La Sen, M. (2016). On φ–convex functions. Journal of Mathematical Inequalities, 10(1), 173–183.

[9] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M. (2014). A new definition of fractional derivative. Journal of Computational and Applied Mathematics, 264, 65–70.

[10] Miller, S., Ross, B. (1993). An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley. Soons. USA.

[11] Sarıkaya, M.Z., Ogunmez, H. (2012). On new inequalities via Riemann-Liouville fractional integration. Abst. Appl. Anal. 2012, Art.ID 428983, 10 pages http://dx.doi.org./10.1155/2012/428983.

[12] Sarıkaya, M.Z., Set, E., Yaldız, H., Başak, N. (2013). Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model., 57, 2403–2407.

[13] Set, E., Çelik, B. (2017). Certain Hermite-Hadamard type inequalities associated with conformable fractional integral operators. Creat. Math. Inform., 26(3), 321–330.

[14] Srivastava H.M., Choi, J. (2012). Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier Science Publishers, Amsterdam, London and New York.