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Some Novel Results for Chebyshev Type Inequalities via Generalized Proportional Fractional Integral Operators

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Abstract. Some novel estimations for Chebyshev type inequalities have been presented via generalized proportional fractional integral operators for integrable functions. The results are more general estimations by using the expansion of exponential function.

1. Introduction

Integral inequalities, a branch of mathematical analysis, play a crucial role in extending the principles of classical inequalities to functions involving integrals. These inequalities offer powerful tools for analyzing and bounding the behavior of integral expressions, providing insights into the properties of functions and their relationships. Their importance extends across various mathematical disciplines, making them indispensable in fields such as analysis, differential equations, optimization, and applied mathematics. Integral inequalities involve the study of relationships between integrals of functions and their corresponding bounds. They provide a framework for comparing the size of integrals and offer valuable information about the behavior of functions over intervals. Some well-known integral inequalities include the Cauchy-Schwarz inequality, Chebyshev inequality, Grüss inequality, Hölder's inequality, and Minkowski's inequality, each serving specific purposes in mathematical analysis. Integral inequalities have practical significance in numerical analysis, where they are employed in the development and analysis of numerical methods. They help establish error estimates and convergence rates, guiding the design of efficient algorithms for approximating solutions to mathematical problems.
We will start with the expression of an inequality that has come to the fore with its applications and is the subject of many articles. Chebyshev inequality was given by Čebyšev in [12] as follows.

$$|T(\Psi,\Phi)| \leq \frac{1}{12}(\kappa_2-\kappa_1)^2 \|\Psi'\|_\infty \|\Phi'\|_\infty, \tag{1}$$

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where $\Psi, \Phi : [\kappa_2, \kappa_1] \rightarrow \mathbb{R}$ are absolutely continuous functions whose derivatives $\Psi', \Phi' \in L_\infty[\kappa_2, \kappa_1]$ and

$$T(\Psi, \Phi) = \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \Psi(x)\Phi(x)\,dx - \left(\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \Psi(x)\,dx \right) \left(\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \Phi(x)\,dx \right), \tag{2}$$

which is called the Chebyshev functional, provided the integrals in (2) exist. With the help of this famous functional, numerous new integral inequalities have been proved and several variants of Chebyshev's inequality have been established. Various generalizations, refinements and extensions can be found in [12]-[27].
Fractional calculus, a branch of mathematical analysis that extends the traditional concepts of differentiation and integration to non-integer orders, has gained increasing importance in various scientific and engineering disciplines. Initially introduced in the 17th century by mathematicians like Leibniz and Euler, fractional calculus has evolved into a powerful tool with applications in physics, engineering, biology, finance, and more. Its unique ability to capture non-local and memory-dependent phenomena makes it a crucial framework for understanding complex systems. Classical calculus deals with integer-order derivatives and integrals, representing the rate of change and accumulation of quantities, respectively. In fractional calculus, these operations are extended to non-integer orders, introducing fractional derivatives and integrals. The fractional derivative of a function describes its rate of change with respect to a non-integer order, providing a deeper insight into intricate behaviors that classical calculus may overlook. The importance of fractional calculus lies in its ability to bridge the gap between classical calculus and the real-world complexities of dynamic systems. As technology advances and our understanding of intricate phenomena deepens, fractional calculus continues to find new applications and challenges. Researchers are exploring its potential in artificial intelligence, machine learning, and data science, highlighting its adaptability to diverse domains. For various results and properties of fractional integral and derivative operators, we refer the papers [1]-[11] for interested readers. Due to the intensive work on it, the Riemann-Liouville integral operator is a prominent operator and is defined as follows.

Definition 1.1. *(See [1]) Let $\Psi \in L_1[\kappa_2, \kappa_1]$. The Riemann-Liouville integrals $J^\alpha_{\kappa_1+}\Psi$ and $J^\alpha_{\kappa_2-}\Psi$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by*

$$J^\alpha_{\kappa_1+}\Psi(t) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{t} (t-x)^{\alpha-1}\Psi(x)dx, \qquad t > \kappa_1$$

and

$$J^\alpha_{\kappa_2-}\Psi(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{\kappa_2} (x-t)^{\alpha-1}\Psi(x)dx, \qquad t < \kappa_2$$

respectively. Here $\Gamma(t)$ is the Gamma function and its definition is $\Gamma(t) = \int_0^\infty e^{-t}t^{x-1}dx$. It is to be noted that $J^0_{\kappa_1+}\Psi(t) = J^0_{\kappa_2-}\Psi(t) = \Psi(t)$ in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

We will continue with the generalized proportional fractional integral operator, which has been described recently and has been the main source of motivation for many studies in the literature with its use in many areas, especially inequality theory. In [5], Jarad et al. identified the proportional generalized fractional integrals that satisfy many important features as follows:

Definition 1.2. *The left and right generalized proportional fractional integral operators are respectively defined by*

$${}_{\kappa_1+}\mathfrak{I}^{\alpha,\lambda}\Psi(t) = \frac{1}{\lambda^\alpha\Gamma(\alpha)} \int_{\kappa_1}^{t} e^{\left[\frac{\lambda-1}{\lambda}(t-x)\right]}(t-x)^{\alpha-1}\Psi(x)dx, \qquad t > \kappa_1$$

and

$${}_{\kappa_2-}\mathfrak{I}^{\alpha}\Psi(t) = \frac{1}{\lambda^\alpha\Gamma(\alpha)} \int_{t}^{\kappa_2} e^{\left[\frac{\lambda-1}{\lambda}(x-t)\right]}(x-t)^{\alpha-1}\Psi(x)dx, \qquad t < \kappa_2$$

where $\lambda \in (0,1]$ and $\alpha \in \mathbb{C}$ and $\mathbb{R}(\alpha) > 0$.

In [17], Belarbi and Dahmani established following theorems related to the Chebyshev inequalities involving Riemann-Liouville fractional integral operator.

Theorem 1.3. *(See [17]) Let Ψ and Φ be two synchronous functions on $[0,\infty)$. Then for all $t>0,\ \alpha>0$, we have:*

$$J^{\alpha}(\Psi\Phi) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha}\Psi(t) J^{\alpha}\Phi(t). \tag{3}$$

Theorem 1.4. *(See [17]) Let Ψ and Φ be two synchronous functions on $[0,\infty)$. Then for all $t>0,\ \alpha>0,\ \beta>0$, we have:*

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}(\Psi\Phi)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)} J^{\alpha}(\Psi\Phi)(t) \geq J^{\alpha}\Psi(t) J^{\beta}\Phi(t) + J^{\beta}\Psi(t) J^{\alpha}\Phi(t). \tag{4}$$

Theorem 1.5. *(See [17]) Let $(\Psi_i)_{i=1,\dots,n}$ be n positive increasing functions on $[0,\infty)$. Then for any $t>0,\ \alpha>0$, we have*

$$J^{\alpha}\left(\prod_{i=1}^{n}\Psi_i\right)(t) \geq \left(J^{\alpha}(1)\right)^{1-n} \prod_{i=1}^{n} J^{\alpha}\Psi_i(t). \tag{5}$$

Theorem 1.6. *(See [17]) Let Ψ and Φ be two functions defined on $[0,\infty)$, such that Ψ is increasing, Φ is differentiable and there exist a real number $m := \inf_{t\geq 0}\Phi(t)'$. Then the inequality*

$$J^{\alpha}(\Psi\Phi)(t) \geq \left(J^{\alpha}(1)\right)^{-1} J^{\alpha}\Psi(t) J^{\alpha}\Phi(t) - \frac{mt}{\alpha+1} J^{\alpha}\Psi(t) + mJ^{\alpha}(t\Psi(t)) \tag{6}$$

is valid for all $t>0,\ \alpha>0$.

The following Theorems have been proved by Set et al. and they include some new inequalities of Chebyshev type via conformable and generalized fractional integral operators.

Theorem 1.7. *(See [24]) Let Ψ and Φ be two integrable functions which are synchronous on $[0,\infty)$. Then*

$$\begin{aligned} & \frac{x^{\alpha\tau}}{\Gamma(\tau+1)\alpha^{\tau}} ({}^{\beta}\mathfrak{J}^{\alpha}\Psi\Phi)(x) + \frac{x^{\alpha\beta}}{\Gamma(\beta+1)\alpha^{\beta}} ({}^{\tau}\mathfrak{J}^{\alpha}\Psi\Phi)(x) \\ \geq\ & ({}^{\beta}\mathfrak{J}^{\alpha}\Psi)(x)({}^{\tau}\mathfrak{J}^{\alpha}\Phi)(x) + ({}^{\tau}\mathfrak{J}^{\alpha}\Psi)(x)({}^{\beta}\mathfrak{J}^{\alpha}\Phi)(x) \end{aligned} \tag{7}$$

where $\alpha,\beta,\tau>0$ and Γ is Euler Gamma function.

Theorem 1.8. *(See [26]) Let t be a positive function on $[0,\infty]$ and let Ψ and Φ be two differentiable functions on $[0,\infty]$. If $\Psi' \in L_r([0,\infty])$, $\Phi' \in L_s([0,\infty])$, $r>1$, $r^{-1}+s^{-1}=1$, then for all $x>0,\ \alpha>0,\ \beta>0,\ \lambda>0,\ \theta>0$, we have*

$$\begin{aligned} & \Big|(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\Psi\Phi)(x;p) + (\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\Psi\Phi)(x;p) \\ - & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\Psi)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\Phi)(x;p) - (\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\Psi)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\Phi)(x;p)\Big| \\ \leq & \|\Psi'\|_r\|\Phi'\|_s \int_0^x\int_0^x (x-\tau)^{(\beta-1)}(x-\rho)^{(\theta-1)}|\tau-\rho|t(\tau)t(\rho) \\ \times & E^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}\Big(\omega(x-\tau)^{\alpha};p\Big)E^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}\Big(\omega(x-\rho)^{\lambda};p\Big)d\tau d\rho \\ \leq & \|\Psi'\|_r\|\Phi'\|_s x(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(x;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t)(x;p) \end{aligned} \tag{8}$$

The main purpose of this paper is to establish several Chebyshev type inequalities by using the generalized proportional fractional integral operators. The results have been performed by a different way comparing to the previous studies via using the expansion of exponential function in Taylor sense.

2. Main Results

Theorem 2.1. *Let* $\Psi, \Phi : [0,\infty) \to \mathbb{R}$ *be two integrable functions which are synchronous on* $[0,\infty)$. *For* $\alpha, \beta > 0$, $0 < \rho_1 \leq 1$, *then one has the following inequality:*

$$\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2)\geq{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2) \tag{9}$$

where

$$\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}=\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2}e^{a_1(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}du$$

and $a_1 = \frac{\rho_1-1}{\rho_1}$.

Proof. Since Ψ and Φ are synchronous functions on $[0,\infty)$, it can be written

$$(\Psi(u)-\Psi(v))(\Phi(u)-\Phi(v))\geq 0, \qquad u,v\in[0,\infty) \tag{10}$$

or equivalently,

$$\Psi(u)\Phi(u)+\Psi(v)\Phi(v)\geq\Psi(u)\Phi(v)+\Psi(v)\Phi(u). \tag{11}$$

If we product both sides of (11) by $\frac{1}{\rho_1\Gamma(\alpha)}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}$, it yields

$$\begin{aligned}&\frac{1}{\rho_1\Gamma(\alpha)}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\Psi(u)\Phi(u)+\frac{1}{\rho_1\Gamma(\alpha)}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\Psi(v)\Phi(v)\\ \geq\;&\frac{1}{\rho_1\Gamma(\alpha)}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\Psi(u)\Phi(v)+\frac{1}{\rho_1\Gamma(\alpha)}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\Psi(v)\Phi(u).\end{aligned}$$

Integrating both sides of the above equality with respect to u over $[\kappa_2, \kappa_1]$, we get

$$\begin{aligned}&\frac{1}{\rho_1\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\Psi(u)\Phi(u)du\\ &+\Psi(v)\Phi(v)\frac{1}{\rho_1\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}du\\ \geq\;&\Phi(v)\frac{1}{\rho_1\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\Psi(u)du\\ &+\Psi(v)\frac{1}{\rho_1\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2}e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\Phi(u).\end{aligned}$$

Let $a_1 = \frac{\rho_1-1}{\rho_1}$. By using the facts that

$$\begin{aligned}e^{a_1(\kappa_2-u)}&=\sum_{k_1=0}^{\infty}\frac{(a_1(\kappa_2-u))^{k_1}}{k_1!},\\ \frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2}e^{a_1(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}du&=\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}.\end{aligned}$$

We can conclude that

$$\begin{aligned}&{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2)+\Psi(v)\Phi(v)\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\\ \geq\;&\Phi(v){}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+\Psi(v){}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Phi)(\kappa_2).\end{aligned} \tag{12}$$

If we proceed a similar argument, by multiplying the above inequality by $\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)} e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-v)}(\kappa_2-v)^{\alpha-1}$ and integrating with respect to v over $[\kappa_2,\kappa_1]$, we obtain

$$\begin{aligned}
& {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2)\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1} \\
& +\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2} e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-v)}(\kappa_2-v)^{\alpha-1}\Psi(v)\Phi(v)\,dv \\
\geq\ & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi)(\kappa_2)\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2} e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-v)}(\kappa_2-v)^{\alpha-1}\Phi(v)\,dv \\
& +{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Phi)(\kappa_2)\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2} e^{\frac{\rho_1-1}{\rho_1}(\kappa_2-v)}(\kappa_2-v)^{\alpha-1}\Psi(v)\,dv.
\end{aligned}$$

By computing the above integrals, one can see that

$$ {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2)\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1} \geq {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2). $$

Which completes the proof. □

Remark 2.2. *Similar calculations as above shows that for any Ψ,Φ which synchronous functions on $[0,\infty)$, one can obtain*

$$ {}^{GPF}I_{\kappa_2}^{\alpha,\rho_1}(\Psi\Phi)(\kappa_1)\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1} \geq {}^{GPF}I_{\kappa_2}^{\alpha,\rho_1}\Psi(\kappa_1)+{}^{GPF}I_{\kappa_2}^{\alpha,\rho_1}\Phi(\kappa_1). $$

Theorem 2.3. *Let $\Psi,\Phi:[0,\infty)\to\mathbb{R}$ be two integrable functions which are synchronous on $[0,\infty)$. For all $\alpha,\beta>0$, $0<\rho_1\leq 1$, $0<\rho_2\leq 1$, one has the following inequality:*

$$\begin{aligned}
& \frac{1}{\rho_2^{\beta}\Gamma(\alpha)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_2}}{\beta+k_2}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2) \\
& +\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\times{}_{\kappa_1}^{GPF}I^{\beta,\rho_2}(\Psi\Phi)(\kappa_2) \\
\geq\ & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2){}_{\kappa_1}^{GPF}I^{\beta,\rho_2}\Phi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2){}_{\kappa_1}^{GPF}I^{\beta,\rho_2}\Psi(\kappa_2)
\end{aligned} \tag{13}$$

where

$$ \frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}=\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2} e^{a_1(\kappa_2-u)}(\kappa_2-u)^{\alpha-1}\,du,\quad a_1=\frac{\rho_1-1}{\rho_1} $$

and

$$ \frac{1}{\rho_2^{\beta}\Gamma(\alpha)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_2}}{\beta+k_2}=\frac{1}{\rho_2^{\beta}\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2} e^{a_2(\kappa_2-v)}(\kappa_2-v)^{\alpha-1}\,dv,\quad a_2=\frac{\rho_2-1}{\rho_2}. $$

Proof. We will start by multiplying both sides of (12) by $\frac{1}{\rho_2^{\beta}\Gamma(\alpha)} e^{a_2(\kappa_2-v)} (\kappa_2-v)^{\alpha-1}$, then we can write

$$\begin{aligned}
& {}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2) \times \frac{1}{\rho_2^{\beta}\Gamma(\alpha)} e^{a_2(\kappa_2-v)} (\kappa_2-v)^{\alpha-1} \\
& + \frac{1}{\rho_1^{\alpha}\Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1} \times \frac{1}{\rho_2^{\beta}\Gamma(\alpha)} e^{a_2(\kappa_2-v)} (\kappa_2-v)^{\alpha-1} \Psi(v)\Phi(v) \\
\geq\ & {}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\Psi(\kappa_2) \times \frac{1}{\rho_2^{\beta}\Gamma(\alpha)} e^{a_2(\kappa_2-v)} (\kappa_2-v)^{\alpha-1} \Phi(v) \\
& + {}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\Phi(\kappa_2) \times \frac{1}{\rho_2^{\beta}\Gamma(\alpha)} e^{a_2(\kappa_2-v)} (\kappa_2-v)^{\alpha-1} \Psi(v).
\end{aligned}$$

Integrating both sides of the above equality with respect to v over $[\kappa_2, \kappa_1]$, we get the desired result. □

Remark 2.4. *If we set*

$$\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1} = \frac{1}{\rho_2^{\beta}\Gamma(\alpha)} \sum_{k_2=0}^{\infty} \frac{a_2^{k_2}}{k_2!} \frac{(\kappa_2-\kappa_1)^{\alpha+k_2}}{\beta+k_2},$$

then one can obtain the inequality (9).

Theorem 2.5. *Let $\Psi_i : [0,\infty) \to \mathbb{R}$ be positive increasing and integrable functions on $[0,\infty)$ for $i = 1, 2, ..., n$. For $\alpha > 0$, $0 < \rho_1 \leq 1$, then one has the following inequality:*

$$\left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{n-1} \times \left[{}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\left(\prod_{i=1}^{n}\Psi_i\right)(\kappa_2)\right] \geq \left[\prod_{i=1}^{n}\left({}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\Psi_i(\kappa_2)\right)\right] \tag{14}$$

where $a_1 = \frac{\rho_1-1}{\rho_1}$.

Proof. To prove this inequality, we will use induction on $n \in \mathbb{N}$. For $n = 1$, it is obvious that the inequality (14) holds such as

$${}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\Psi_1(\kappa_2) \geq \left({}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\Psi_1(\kappa_2)\right), \forall\alpha > 0.$$

By using the induction hypothesis, we can assume that

$${}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\left(\prod_{i=1}^{n-1}\Psi_i\right)(\kappa_2) \geq \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]\left[\prod_{i=1}^{n-1}\left({}^{GPF}_{\kappa_1}I^{\alpha,\rho_1}\Psi_i(\kappa_2)\right)\right],$$

where $\forall\alpha, \kappa_2 > 0$.
Since $\Psi_i : [0,\infty) \to \mathbb{R}$ are positive increasing and integrable functions on $[0,\infty)$ for $i = 1, 2, ..., n$, then

$\left(\prod_{i=1}^{n-1}\Psi_i\right)(\kappa_2)$ is an increasing function. Therefore, we can apply inequality (9) for $\prod_{i=1}^{n-1}\Psi_i = \Phi$, $\Psi_n = \Psi$, we get

$$\begin{aligned} & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(\prod_{i=1}^{n-1}\Psi_i\right)(\kappa_2) \geq \left[\prod_{i=1}^{n-1}\left({}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi_i\Psi_n\right)(\kappa_2)\right] \geq {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2) \\ \geq & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1} \times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2) \\ \geq & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1} \\ & \times\left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{2-n}\prod_{i=1}^{n-1}\left({}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi_i(\kappa_2)\right){}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi_n \\ \geq & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{1-n}\prod_{i=1}^{n}\left({}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi_i(\kappa_2)\right). \end{aligned}$$

This completes the proof. □

Theorem 2.6. *Let* $\Psi,\Phi:[0,\infty)\to\mathbb{R}$ *be two integrable functions on* $[0,\infty)$ *such that* Ψ *is increasing and* Φ *is differentiable with* $m=\inf_{t\in[0,\infty)}\Phi'(t)$. *Then one has the following inequality:*

$$\begin{aligned} & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2) \\ \geq & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2) \\ & -\frac{m}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2)+m\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(t\Psi)(\kappa_2) \end{aligned}$$

where $t(x)=x$.

Proof. Suppose that $p(x)=mx$ and $h(x)=\Phi(x)-p(x)$. Note that h is differentiable and increasing on $[0,\infty)$, then we can apply (9) as

$$\begin{aligned} & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi h)(\kappa_2) \\ \geq & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}h(\kappa_2) \\ = & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}h(\kappa_2) \\ & -\left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+m\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}p(\kappa_2). \end{aligned} \tag{15}$$

Since,

$${}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}p(\kappa_2)=m\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2)$$

and

$${}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi p)(\kappa_2)=m\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(t\Psi)(\kappa_2).$$

Then, the inequality (15) implies,

$$
\begin{aligned}
& {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2) \\
= \; & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi h)(\kappa_2) + {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi p)(\kappa_2) \\
\geq \; & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2) \\
& -\left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1} \\
& \times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}p(\kappa_2) + \times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi p)(\kappa_2) \\
\geq \; & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2) \\
& -\frac{m}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2) + m\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(t\Psi)(\kappa_2),
\end{aligned}
$$

which is the desired result. □

Theorem 2.7. *Let $\Psi,\Phi:[0,\infty)\to\mathbb{R}$ be two integrable functions on $[0,\infty)$ such that Ψ and Φ are differentiable with $m_1=\inf_{t\in[0,\infty)}\Psi'(t)$ and $m_2=\inf_{t\in[0,\infty)}\Phi'(t)$. Then one has the following inequality:*

$$
\begin{aligned}
& {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(\Psi\Phi)(\kappa_2) \\
\geq \; & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2) + {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2) \\
& -\frac{m_2}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2) \\
& -\frac{m_1}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2) \\
& +\frac{m_1m_2}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2)\,{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2) \\
& +m_2\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(t\Psi)(\kappa_2) + m_1\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}(t\Phi)(\kappa_2) - m_1m_2\times {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t^2(\kappa_2)
\end{aligned}
$$

where $t(x)=x$.

Proof. Assume that $p_1(x)=m_1x$, $h_1(x)=\Phi(x)-p_1(x)$ and $p_2(x)=m_2x$, $h_2(x)=\Phi(x)-p_2(x)$. Since h_1,h_2 are differentiable and increasing on $[0,\infty)$, then we can apply (9) such that

$$\begin{aligned} & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(h_1h_2\right)(\kappa_2) \qquad (16)\\ \geq\ & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}h_1(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}h_2(\kappa_2)\\ \geq\ & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1}\\ & \times\left({}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}p_1(\kappa_2)\right)\left({}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}p_2(\kappa_2)\right)\\ \geq\ & \left[\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}\right]^{-1}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2){}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2)\\ & -\frac{m_2}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Psi(\kappa_2){}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2)\\ & -\frac{m_1}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\Phi(\kappa_2){}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2)\\ & +\frac{m_1m_2}{\frac{1}{\rho_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(\kappa_2-\kappa_1)^{\alpha+k_1}}{\alpha+k_1}}\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2){}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t(\kappa_2). \end{aligned}$$

Moreover,

$$\begin{aligned} {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(h_1p_2\right)(\kappa_2) &= m_2\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(th_1\right)(\kappa_2)\\ &= m_2\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(t\Psi\right)(\kappa_2)-m_1m_2\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t^2(\kappa_2). \qquad (17) \end{aligned}$$

Similarly, we have

$$\begin{aligned} {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(h_2p_1\right)(\kappa_2) &= m_1\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(th_1\right)(\kappa_2)\\ &= m_1\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(t\Phi\right)(\kappa_2)-m_1m_2\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t^2(\kappa_2) \end{aligned}$$

and

$$ {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(p_1p_2\right)(\kappa_2)=m_1m_2\times{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}t^2(\kappa_2). $$

By using the fact that,

$$\Psi\Phi=\left(h_1+p_1\right)\left(h_2+p_2\right)=h_1h_2+h_1p_2+h_2p_1+p_1p_2.$$

Then, we can obtain

$$\begin{aligned} & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(\Psi\Phi\right)(\kappa_2)\\ =\ & {}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(h_1h_2\right)(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(h_1p_2\right)(\kappa_2)\\ & +{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(h_2p_1\right)(\kappa_2)+{}_{\kappa_1}^{GPF}I^{\alpha,\rho_1}\left(p_1p_2\right)(\kappa_2). \end{aligned}$$

By taking into account this equality together with (16) and (17), we conclude the desired result. □

3. Conclusion

Fractional calculus has evolved from a historical curiosity to a fundamental tool in modern mathematics and science. Its applications across various disciplines emphasize its significance in providing more accurate and comprehensive models for complex systems. As research in this field progresses, fractional calculus is likely to play an increasingly vital role in advancing our understanding of the intricate dynamics inherent in the natural and engineered world. Several researchers have studied on Chebyshev functional in the literature by different motivations. The main purpose of these studies is to obtain optimal bounds and approaches by using concepts of fractional calculus. To provide new and more general bounds and estimations, we have used generalized proportional fractional integral operators for integrable functions. Our findings have been improved by using the expansion of exponential functions in Taylor sense.

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