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Recent Advances in Integral Inequalities of Hardy-Hilbert Type

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Abstract. Our aim in this study will be to obtain a new kinds of Hardy-Hilbert inequalities in which the weighted function is homogeneous function. Other results are also obtained.

1. Introduction

Hardy-Hilbert's integral inequalities constitute a significant cornerstone in the field of mathematical analysis, offering profound insights into the behavior of integral operators and their associated functions. Named after the eminent mathematicians G.H. Hardy and D. Hilbert, these inequalities have found wide-ranging applications across various branches of mathematics, including functional analysis, partial differential equations, and harmonic analysis.

The well-known Hilbert's inequality and its equivalent form are presented first [2]:

Theorem 1.1. *If $f,g \in L_2([0,\infty))$, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right\}^{\frac{1}{2}} \tag{1}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx \right)^2 dy \le \pi^2 \int_0^\infty f^2(x)dx$$

where π and π^2 are the best possible constants. .

The classical Hilbert's integral inequality (1) had been generalized by Hardy- Riesz (see [1]) in 1925 as the following result. If f,g are nonnegative functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}} \tag{2}$$

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where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible constants. When $p = q = 2$, inequality (2) is reduced to (1). Recently, a number of mathematicians had given lots of generalizations of these inequalities. We mention here some of these contributions in this direction: Li et al. [3] have proved the following Hardy-Hilbert's type inequality using the hypotheses of (1):

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y+\max\{x,y\}}dxdy \le c\left\{\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}}$$

Where the constant factor $c = \sqrt{2}\left(\pi - 2 - \tan^{-1}\sqrt{2}\right)$ is the best possible. Other mathematicians have presented generalizations or new kinds of the above Hardy-Hilbert inequalities, as follows:

Theorem 1.2. *[7] Let $f, g > 0$. If $p > 1,\ q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, are such that*

$$0 < \int_0^\infty t^{p-1-\lambda} f^p(x)\,dx < \infty \text{ and } 0 < \int_0^\infty t^{q-1-\lambda} g^q(y)\,dy < \infty,$$

then one has

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{\max\{x^\lambda, y^\lambda\}}dxdy \le \frac{pq}{\lambda}\left(\int_0^\infty t^{p-1-\lambda} f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty t^{q-1-\lambda} g^q(y)\,dy\right)^{\frac{1}{q}},$$

where the constant factor $\frac{pq}{\lambda}$ is the best possible.

Theorem 1.3. *[11] Let $f, g > 0$. If $p > 1, \lambda > 0,\ \frac{1}{p} + \frac{1}{q} = 1$, and $0 < \lambda = 2 - \frac{1}{p} + \frac{1}{q} \le 1$, then one has*

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}dxdy \le k\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}.$$

Here, k depends on p and q; only if $\frac{1}{p} + \frac{1}{q} = 1,\ \lambda = 2 - \frac{1}{p} + \frac{1}{q} = 1$, k is the best possible.

We also recall that a nonnegative function $f(x, y)$ which is said to be homogeneous function of degree λ if $f(tx, ty) = t^\lambda f(x, y)$ for all $t > 0$. And we say that $h(x, y)$ is increasing if $h(1, u)$and $h(u, 1)$ are increasing functions.

In 2008, Sulaiman [10] gave new integral inequality similar to the Hardy-Hilbert's integral inequality. If $a, b > 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda < \min\left\{(1-b)\frac{p}{q},\ (1-a)\frac{q}{p}\right\}$, $h(x, y)$ is a positive increasing homogeneous function of degree λ, and $f, g \ge 0$ and

$$F(x) = \int_0^x f(t)\,dt, \quad G(x) = \int_0^x g(t)\,dt,$$

then, for all $T > 0$, we have

$$\int_0^T \left(\int_0^T \frac{F(x)\,G(y)\,dx}{h(x,y)}\right)^p dy \tag{3}$$

$$\le T^\alpha \sqrt[p]{pK_2}\sqrt[q]{qK_1}\left(\int_0^T (T-t)\,F^{p-1}(t)\,f(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^T (T-t)\,G^{q-1}(t)\,g(t)\,dt\right)^{\frac{1}{q}}$$

where

$$K_1 = \int_0^1 \frac{u^{b-1}du}{h(u,1)}, \quad K_2 = \int_0^1 \frac{u^{a-1}}{h(1,u)}du.$$

In recent years, some Hilbert-type integral inequalities with homogeneous functions were established [10] by Sulaiman. By means of the technique of real analysis and the weight functions, Sroysang [8] obtained an equivalent statements of a Hilbert-type integral inequality with the generalized homogeneous function. In [9], Sulaiman also presented a new Hardy-Hilbert-type integral inequality with homogeneous functions of order λ. This extension significantly broadened the scope of these inequalities, allowing for their application to a wider range of mathematical contexts. Building upon Sulaiman's groundwork, Wei and Lei further contributed to the advancement of this field in [13] by offering an alternative proof technique for Hardy-Hilbert type inequalities. Despite addressing the same functions and kernels, Wei-Lei's approach introduced novel insights and methodologies, enriching the mathematical discourse surrounding these inequalities. Regarding Hardy-Hilbert integral inequalities regarding different types of functions and approximations see [4], [5], [6], [12], where further references are given.

In this paper, we present a generalization of the integral inequality (3) and its applications. We also aim to extend the existing body of knowledge by introducing a new class of inequalities, drawing upon insights from the aforementioned studies. Our investigation will delve into the intricacies of these inequalities, exploring their implications and potential applications in various mathematical contexts.

2. Main Results

To prove our main results, we require the following lemma:

Lemma 2.1. *Let $k:\mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing, $h:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ be increasing function and $0<\mu+1\le\alpha$. Set for $s\ge 1$,*

$$\Psi(s) = s^{-\alpha}\int_0^s \frac{t^\mu dt}{h(1,t)\max\{k(t^{-1}),k(t)\}}$$

and

$$\Theta(s) = s^{-\alpha}\int_0^s \frac{t^\mu dt}{h(t,1)\max\{k(t^{-1}),k(t)\}}$$

Then

$$\Psi(s)\le\Psi(1) \text{ and } \Theta(s)\le\Theta(1).$$

Proof. With the help of derivative under the integral sign, we get

$$\begin{aligned}
\Psi(s) &= \frac{s^{\mu-\alpha}}{h(1,s)\max\{k(s^{-1}),k(s)\}} - \alpha s^{-\alpha-1}\int_0^s \frac{t^\mu dt}{h(1,t)\max\{k(t^{-1}),k(t)\}} \\
&\le \frac{s^{\mu-\alpha}}{h(1,s)\max\{k(s^{-1}),k(s)\}} - \frac{\alpha s^{-\alpha-1}}{h(1,s)\max\{k(s^{-1}),k(s)\}}\int_0^s t^\mu dt \\
&= \frac{s^{\mu-\alpha}}{h(1,s)\max\{k(s^{-1}),k(s)\}}\left(1-\frac{\alpha}{\mu+1}\right)\le 0.
\end{aligned}$$

This shows that Ψ is nonincreasing and hence $\Psi(s)\le\Psi(1)$. The other part has a similar proof. □

Theorem 2.2. *Assume that $f,h,k>0$, $h,k:\mathbb{R}^{+}\times\mathbb{R}^{+}\to\mathbb{R}^{+}$, h be increasing, homogeneous of degree λ,*

$$0<\lambda<\min\left\{(1-b)\frac{p}{q},\ (1-a)\frac{q}{p}\right\},\ a,b>0,$$

and k is nondecreasing, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. Set

$$F(x)=\int_{0}^{x}f(t)\,dt.$$

Then

i) $k(x)\neq 1$, or in general, $k(x)\neq c$, (c is constant)

$$\int_{0}^{T}\left(\int_{0}^{T}\frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^{p}dy\leq pT^{\alpha p}J_{2}J_{1}^{\frac{p}{q}}\int_{0}^{T}(T-t)\,F^{p-1}(t)\,f(t)\,dt \tag{4}$$

where

$$J_{1}=\int_{0}^{1}\frac{u^{b-1}du}{h(u,1)\max\left\{k(u),k\left(u^{-1}\right)\right\}},\quad J_{2}=\int_{0}^{1}\frac{u^{a-1}du}{h(1,u)\max\left\{k\left(u^{-1}\right),k(u)\right\}}. \tag{5}$$

ii) $k(x)=1$,

$$\int_{0}^{T}\left(\int_{0}^{T}\frac{F(x)}{h(x,y)}dx\right)^{p}dy\leq pT^{\alpha p}K_{2}\left(K_{1}\right)^{\frac{p}{q}}\int_{0}^{T}(T-t)\,F^{p-1}(t)\,f(t)\,dt \tag{6}$$

where

$$K_{1}=\int_{0}^{1}\frac{u^{b-1}du}{h(u,1)},\quad K_{2}=\int_{0}^{1}\frac{u^{a-1}}{h(1,u)}du. \tag{7}$$

Proof. i) From Hölder's inequality, we get

$$\int_{0}^{T}\frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}$$

$$\leq\left(\int_{0}^{T}\frac{y^{a-1}F^{p}(x)\,dx}{x^{(b-1)\frac{p}{q}}h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^{\frac{1}{p}}\left(\int_{0}^{T}\frac{x^{b-1}dx}{y^{(a-1)\frac{q}{p}}h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^{\frac{1}{q}}$$

which yields

$$\left(\int_{0}^{T}\frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^{p} \tag{8}$$

$$\leq\int_{0}^{T}\frac{y^{a-1}F^{p}(x)\,dx}{x^{(b-1)\frac{p}{q}}h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\left(\int_{0}^{T}\frac{x^{b-1}dx}{y^{(a-1)\frac{q}{p}}h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^{\frac{p}{q}}.$$

Change variable in the integral inside with $x = yu$, we have, in view of Lemma 2.1, by writing $\alpha = b + (1-a)\frac{q}{p} - \lambda$

$$
\begin{aligned}
&\int_0^T \frac{x^{b-1}dx}{y^{(a-1)\frac{q}{p}} h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \qquad (9)\\
=\;& y^{(1-a)\frac{q}{p}+b-\lambda}\left(\frac{T}{y}\right)^{\alpha}\left(\frac{T}{y}\right)^{-\alpha}\int_0^{\frac{T}{y}} \frac{u^{b-1}du}{h(u,1)\max\left\{k(u), k\left(u^{-1}\right)\right\}}\\
\leq\;& y^{(1-a)\frac{q}{p}+b-\alpha-\lambda}T^{\alpha}\int_0^1 \frac{u^{b-1}du}{h(u,1)\max\left\{k(u), k\left(u^{-1}\right)\right\}}\\
=\;& y^{(1-a)\frac{q}{p}+b-\alpha-\lambda}T^{\alpha}J_1 = T^{\alpha}J_1.
\end{aligned}
$$

Hence, from (8) and (9), we get

$$
\begin{aligned}
&\left(\int_0^T \frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}}\right)^p\\
\leq\;& T^{\alpha\frac{p}{q}}J_1^{\frac{p}{q}}\int_0^T \frac{y^{a-1}F^p(x)\,dx}{x^{(b-1)\frac{p}{q}}h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}}.
\end{aligned}
$$

Integrating with respect to the y variable , we get

$$
\begin{aligned}
&\int_0^T\left(\int_0^T \frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}}\right)^p dy \qquad (10)\\
\leq\;& T^{\alpha\frac{p}{q}}J_1^{\frac{p}{q}}\int_0^T\int_0^T \frac{y^{a-1}F^p(x)\,dxdy}{x^{(b-1)\frac{p}{q}}h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}}\\
=\;& T^{\alpha\frac{p}{q}}J_1^{\frac{p}{q}}\int_0^T F^p(x)\left(\int_0^T \frac{y^{a-1}dy}{x^{(b-1)\frac{p}{q}}h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}}\right)dx.
\end{aligned}
$$

By calculating the inner integral above, with the same method as above, in view of Lemma 2.1, by writing $\alpha = a + (1-b)\frac{p}{q} - \lambda$,

$$
\begin{aligned}
&\int_0^T \frac{y^{a-1}dy}{x^{(b-1)\frac{p}{q}}h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \qquad (11)\\
=\;& x^{a+(1-b)\frac{p}{q}-\lambda}\left(\frac{T}{x}\right)^{\alpha}\left(\frac{T}{x}\right)^{-\alpha}\int_0^{\frac{T}{x}} \frac{u^{a-1}du}{h(1,u)\max\left\{k\left(u^{-1}\right), k(u)\right\}}\\
\leq\;& x^{a+(1-b)\frac{p}{q}-\alpha-\lambda}T^{\alpha}\int_0^1 \frac{u^{a-1}du}{h(1,u)\max\left\{k\left(u^{-1}\right), k(u)\right\}}
\end{aligned}
$$

$$= \quad x^{a+(1-b)\frac{p}{q}-\alpha-\lambda}T^{\alpha}J_2 = T^{\alpha}J_2.$$

If (11) is substituted into (10) we get

$$\int_0^T\left(\int_0^T \frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^p dy$$

$$\leq \quad T^{\alpha}J_2T^{\alpha\frac{p}{q}}J_1^{\frac{p}{q}}\int_0^T F^p(x)\,dx = T^{\alpha p}J_2J_1^{\frac{p}{q}}\int_0^T F^p(x)\,dx.$$

Since

$$F^p(x) = \int_0^x [F^p(t)]'\,dt = p\int_0^x F^{p-1}(t)\,f(t)\,dt \tag{12}$$

and after interchange of the order of intergration we have

$$\begin{aligned}\int_0^T\left(\int_0^T \frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^p dy &\leq pT^{\alpha p}J_2J_1^{\frac{p}{q}}\int_0^T\int_0^x F^{p-1}(t)\,f(t)\,dtdx \\ &= pT^{\alpha p}J_2J_1^{\frac{p}{q}}\int_0^T (T-t)\,F^{p-1}(t)\,f(t)\,dt.\end{aligned}$$

which shows that the inequality (4) holds.

ii) Similarly, according to Hölder's inequality, we get

$$\int_0^T \frac{F(x)}{h(x,y)}dx = \int_0^T \frac{F(x)\,y^{\frac{a-1}{p}}}{[h(x,y)]^{\frac{1}{p}}\,x^{\frac{b-1}{q}}}\frac{x^{\frac{b-1}{q}}}{[h(x,y)]^{\frac{1}{q}}\,y^{\frac{a-1}{p}}}dx$$

$$\leq \quad \left(\int_0^T \frac{y^{a-1}F^p(x)\,dx}{h(x,y)\,x^{(b-1)\frac{p}{q}}}\right)^{\frac{1}{p}}\times\left(\int_0^T \frac{x^{b-1}dx}{h(x,y)\,y^{(b-1)\frac{q}{p}}}\right)^{\frac{1}{q}}$$

which yields

$$\left(\int_0^T \frac{F(x)}{h(x,y)}dx\right)^p \leq \int_0^T \frac{y^{a-1}F^p(x)}{x^{(b-1)\frac{p}{q}}h(x,y)}dx\left(\int_0^T \frac{x^{b-1}}{y^{(b-1)\frac{q}{p}}h(x,y)}dx\right)^{\frac{p}{q}}. \tag{13}$$

We consider the above last integral in view of Lemma 2.1, by writing $\alpha = a+(1-b)\frac{q}{p}-\lambda$,

$$\begin{aligned}\int_0^T \frac{x^{b-1}}{y^{(b-1)\frac{q}{p}}h(x,y)}dx &= y^{(1-b)\frac{q}{p}-\lambda}\int_0^T \frac{x^{b-1}dx}{h\left(\frac{x}{y},1\right)} \\ &= y^{a+(1-b)\frac{q}{p}-\lambda}\left(\frac{T}{y}\right)^{\alpha}\left(\frac{T}{y}\right)^{-\alpha}\int_0^{\frac{T}{y}} \frac{u^{b-1}du}{h(u,1)}\end{aligned}$$

$$\leq \quad y^{a+(1-b)\frac{q}{p}-\alpha-\lambda}T^{\alpha}K_1 = T^{\alpha}K_1,$$

then we can obtain

$$\left(\int_0^T \frac{F(x)}{h(x,y)}dx\right)^p \leq (T^{\alpha}K_1)^{\frac{p}{q}} \int_0^T \frac{y^{a-1}F^p(x)}{x^{(b-1)\frac{p}{q}}h(x,y)}dx.$$

Integrating with respect to the y variable , we get

$$\int_0^T \left(\int_0^T \frac{F(x)}{h(x,y)}dx\right)^p dy \leq (T^{\alpha}K_1)^{\frac{p}{q}} \int_0^T \int_0^T \frac{y^{a-1}F^p(x)}{x^{(b-1)\frac{p}{q}}h(x,y)}dxdy \tag{14}$$

$$= \quad (T^{\alpha}K_1)^{\frac{p}{q}} \int_0^T F^p(x)\left(\int_0^T \frac{y^{a-1}}{x^{(b-1)\frac{p}{q}}h(x,y)}dy\right)dx.$$

Now, change variable in the integral inside with $y = xu$, we have, in view of Lemma 2.1, by writing $\alpha = a + (1-b)\frac{p}{q} - \lambda$

$$\begin{aligned} \int_0^T \frac{y^{a-1}}{x^{(b-1)\frac{p}{q}}h(x,y)}dy &= x^{a+(1-b)\frac{p}{q}-\lambda}\left(\frac{T}{x}\right)^{\alpha}\left(\frac{T}{x}\right)^{\alpha}\int_0^{\frac{T}{x}} \frac{u^{a-1}}{h(1,u)}du \\ &\leq x^{a+(1-b)\frac{p}{q}-\alpha-\lambda}T^{\alpha}K_2 = T^{\alpha}K_2. \end{aligned}$$

This final result is written instead of (14), it follows that

$$\begin{aligned} \int_0^T \left(\int_0^T \frac{F(x)}{h(x,y)}dx\right)^p dy &\leq T^{\alpha p}K_2(K_1)^{\frac{p}{q}} \int_0^T F^p(x)\,dx \\ &\leq pT^{\alpha p}K_2(K_1)^{\frac{p}{q}} \int_0^T \int_0^x F^{p-1}(t)\,f(t)\,dtdx \\ &= pT^{\alpha p}K_2(K_1)^{\frac{p}{q}} \int_0^T (T-t)\,F^{p-1}(t)\,f(t)\,dt \end{aligned}$$

which shows that the inequality (6) holds. □

Theorem 2.3. *Assume that $f,g,h,k > 0$, $h,k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, h homogeneous of degree λ,*

$$0 < \lambda < \min\left\{(1-b)\frac{p}{q},\ (1-a)\frac{q}{p}\right\},\ a,b > 0,$$

and k is nondecreasing, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Set

$$F(x) = \int_0^x f(t)\,dt, \quad G(x) = \int_0^x g(t)\,dt.$$

Then

i) $k(x) \neq 1$*, or in general,* $k(x) \neq c$*, (c is constant)*

$$\int_0^T \int_0^T \frac{F(x)\, G(y)\, dxdy}{h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \tag{15}$$

$$\leq T^{\alpha} \sqrt[p]{pJ_2} \sqrt[q]{qJ_1} \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{\frac{1}{p}} \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{\frac{1}{q}}$$

where J_1 *and* J_2 *are defined by (5).*

ii) $k(x) = 1$,

$$\int_0^T \int_0^T \frac{F(x)\, G(y)\, dxdy}{h(x,y)} \tag{16}$$

$$\leq T^{\alpha} (pK_2)^{\frac{1}{p}} (qK_1)^{\frac{1}{q}} \left(\int_0^T (T-t) F^{p-1}(t) f(t) dt \right)^{\frac{1}{p}} \left(\int_0^T (T-t) G^{q-1}(t) g(t) dt \right)^{\frac{1}{q}}$$

where K_1 *and* K_2 *are defined by (7).*

Proof. i) By using Hölder's inequality, we have

$$\int_0^T \int_0^T \frac{F(x)\, G(y)\, dxdy}{h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \tag{17}$$

$$= \int_0^T \int_0^T \frac{y^{\frac{a-1}{p}} F(x)}{x^{\frac{b-1}{q}} \left[h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}\right]^{\frac{1}{p}}} \frac{x^{\frac{b-1}{q}} G(y)}{y^{\frac{a-1}{p}} \left[h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}\right]^{\frac{1}{q}}} dxdy$$

$$\leq \left(\int_0^T \int_0^T \frac{y^{a-1} F^p(x)\, dxdy}{x^{(b-1)\frac{p}{q}} h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^T \int_0^T \frac{x^{b-1} G^q(y)\, dxdy}{y^{(a-1)\frac{q}{p}} h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \right)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p}} N^{\frac{1}{q}}.$$

We first consider the following integral:

$$M = \int_0^T \int_0^T \frac{y^{a-1} F^p(x)\, dxdy}{x^{(b-1)\frac{p}{q}} h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \tag{18}$$

$$= \int_0^T F^p(x) \left(\int_0^T \frac{y^{a-1} dy}{x^{(b-1)\frac{p}{q}} h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \right) dx.$$

Change variable in the integral inside with $y = xu$, we have, in view of Lemma 2.1 , by writing $\alpha = a + (1-b)\frac{p}{q} - \lambda$

$$\begin{aligned}
&\int_0^T \frac{y^{a-1}dy}{x^{(b-1)\frac{p}{q}} h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \\
&= x^{a+(1-b)\frac{p}{q}-\lambda} \left(\frac{T}{x}\right)^{\alpha} \left(\frac{T}{x}\right)^{-\alpha} \int_0^{\frac{T}{x}} \frac{u^{a-1}du}{h(1,u)\max\{k(u^{-1}), k(u)\}} \\
&= x^{a+(1-b)\frac{p}{q}-\alpha-\lambda} T^{\alpha} \int_0^{\frac{T}{x}} \frac{u^{a-1}du}{h(1,u)\max\{k(u^{-1}), k(u)\}} \\
&\le x^{a+(1-b)\frac{p}{q}-\alpha-\lambda} T^{\alpha} J_2 = T^{\alpha} J_2.
\end{aligned}$$

If this last result is substituted into (18), we have

$$M = \int_0^T \int_0^T \frac{y^{a-1}F^p(x)\,dxdy}{x^{(b-1)\frac{p}{q}} h(x,y) \max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} = T^{\alpha} J_2 \int_0^T F^p(x)\,dx.$$

Since

$$F^p(x) = \int_0^x [F^p(t)]'\,dt = p\int_0^x F^{p-1}(t) f(t)\,dt \tag{19}$$

and after interchange of the order of intergration we have

$$M \le pT^{\alpha} J_2 \int_0^T \int_0^x F^{p-1}(t) f(t)\,dtdx = pT^{\alpha} J_2 \int_0^T (T-t) F^{p-1}(t) f(t)\,dt.$$

Similarly, the other part follows by using Lemma 2.1, $\alpha = b + (1-a)\frac{q}{p} - \lambda$ to obtain

$$N \le qT^{\alpha} J_1 \int_0^T (T-t) G^{q-1}(t) g(t)\,dt.$$

If M and N are written in (17), we have

$$\begin{aligned}
&\int_0^T \int_0^T \frac{F(x)G(y)\,dxdy}{h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \\
&\le T^{\alpha} \sqrt[p]{pJ_2}\sqrt[q]{qJ_1} \left(\int_0^T (T-t) F^{p-1}(t) f(t)\,dt\right)^{\frac{1}{p}} \left(\int_0^T (T-t) G^{q-1}(t) g(t)\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

which shows that the inequality (15) holds.

ii) Similarly, according to Hölder's inequality, we get

$$\int_0^T \int_0^T \frac{F(x)G(y)\,dxdy}{h(x,y)} \tag{20}$$

$$
\begin{aligned}
&= \int_0^T\int_0^T \frac{F(x)\,y^{\frac{a-1}{p}}}{[h(x,y)]^{\frac{1}{p}}\,x^{\frac{b-1}{q}}}\frac{G(y)\,x^{\frac{b-1}{q}}}{[h(x,y)]^{\frac{1}{q}}\,y^{\frac{a-1}{p}}}dxdy \\
&\le \left(\int_0^T\int_0^T \frac{F^p(x)\,y^{a-1}dxdy}{h(x,y)\,x^{(b-1)\frac{p}{q}}}\right)^{\frac{1}{p}}\times\left(\int_0^T\int_0^T \frac{G^q(y)\,x^{b-1}dxdy}{h(x,y)\,y^{(a-1)\frac{q}{p}}}\right)^{\frac{1}{q}} \\
&= \left(\int_0^T F^p(x)\left(\int_0^T \frac{y^{a-1}dy}{x^{(b-1)\frac{p}{q}}h(x,y)}\right)dx\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^T G^q(y)\left(\int_0^T \frac{x^{b-1}dx}{y^{(a-1)\frac{q}{p}}h(x,y)}\right)dy\right)^{\frac{1}{q}}.
\end{aligned}
$$

We consider the above last integrals with change of variables, by using Lemma 2.1 with $\alpha = a + (1-b)\frac{p}{q} - \lambda$ we have

$$
\begin{aligned}
\int_0^T \frac{y^{a-1}dy}{x^{(b-1)\frac{p}{q}}h(x,y)} &= x^{(1-b)\frac{p}{q}-\lambda}\int_0^T \frac{y^{a-1}dy}{h\left(1,\frac{y}{x}\right)} \\
&= x^{a+(1-b)\frac{p}{q}-\lambda}\left(\frac{T}{x}\right)^{\alpha}\left(\frac{T}{x}\right)^{-\alpha}\int_0^{\frac{T}{x}} \frac{u^{a-1}du}{h(1,u)} \\
&\le x^{a+(1-b)\frac{p}{q}-\alpha-\lambda}T^{\alpha}\int_0^1 \frac{u^{a-1}du}{h(1,u)} = T^{\alpha}K_2
\end{aligned}
\tag{21}
$$

and with $\alpha = b + (1-a)\frac{q}{p} - \lambda$

$$
\begin{aligned}
\int_0^T \frac{x^{b-1}dx}{y^{(a-1)\frac{q}{p}}h(x,y)} &= y^{(1-a)\frac{q}{p}-\lambda}\int_0^T \frac{x^{b-1}dx}{h\left(\frac{x}{y}y,y\right)} \\
&= y^{b+(1-a)\frac{q}{p}-\lambda}\left(\frac{T}{y}\right)^{\alpha}\left(\frac{T}{y}\right)^{-\alpha}\int_0^{\frac{T}{y}} \frac{u^{b-1}du}{h(u,1)} \\
&\le y^{b+(1-a)\frac{q}{p}-\alpha-\lambda}T^{\alpha}\int_0^1 \frac{u^{b-1}du}{h(u,1)} = T^{\alpha}K_1.
\end{aligned}
\tag{22}
$$

Therefore, (21) and (22) are written into (20), it follows that

$$
\int_0^T\int_0^T \frac{F(x)\,G(y)\,dxdy}{h(x,y)} \le (T^{\alpha}K_2)^{\frac{1}{p}}\,(T^{\alpha}K_1)^{\frac{1}{q}}\left(\int_0^T F^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^T G^q(y)\,dy\right)^{\frac{1}{q}}.
$$

By using (19), and after interchange of the order of intergration we have

$$
\int_0^T\int_0^T \frac{F(x)\,G(y)\,dxdy}{h(x,y)}
$$

$$\leq T^{\alpha}(pK_2)^{\frac{1}{p}}(qK_1)^{\frac{1}{q}}\left(\int_0^T\int_0^x F^{p-1}(t)f(t)\,dtdx\right)^{\frac{1}{p}}\left(\int_0^T\int_0^y G^{q-1}(t)g(t)\,dtdy\right)^{\frac{1}{q}}$$

$$= T^{\alpha}(pK_2)^{\frac{1}{p}}(qK_1)^{\frac{1}{q}}\left(\int_0^T (T-t)F^{p-1}(t)f(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^T (T-t)G^{q-1}(t)g(t)\,dt\right)^{\frac{1}{q}}$$

which shows that the inequality (16) holds. □

3. Applications

Corollary 3.1. *Under assumptions of Theorem 2.2 with $a=b=\frac{\lambda}{2}$, then*

i) $k(x)\neq 1$,

$$\int_0^T\left(\int_0^T \frac{F(x)\,dx}{h(x,y)\max\left\{k\left(\frac{x}{y}\right),k\left(\frac{y}{x}\right)\right\}}\right)^p dy \leq pT^{\alpha p}J_3^p\int_0^T (T-t)F^{p-1}(t)f(t)\,dt \tag{23}$$

where

$$J_3=\int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{h(1,u)\max\{k(u),k(u^{-1})\}} \tag{24}$$

ii) $k(x)=1$,

$$\int_0^T\left(\int_0^T \frac{F(x)}{h(x,y)}dx\right)^p dy \leq pT^{\alpha p}K_3^p\int_0^T (T-t)F^{p-1}(t)f(t)\,dt$$

where

$$K_3=\int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{h(1,u)}. \tag{25}$$

Furthermore, when $h(x,y)=(x+y)^{\lambda}$, we have

$$\int_0^T\left(\int_0^T \frac{F(x)}{(x+y)^{\lambda}}dx\right)^p dy \leq pT^{\alpha p}B^p\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^T (T-t)F^{p-1}(t)f(t)\,dt.$$

Proof. For $a=b=\frac{\lambda}{2}$, from result (i) in Theorem 2.2 we get

$$\begin{aligned} J_1 &= \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{h(u,1)\max\{k(u),k(u^{-1})\}}=\int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{h(u,uu^{-1})\max\{k(u),k(u^{-1})\}} \\ &= \int_0^1 \frac{u^{-\frac{\lambda}{2}-1}du}{h(1,u^{-1})\max\{k(u),k(u^{-1})\}}=\int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{h(1,u)\max\{k(u),k(u^{-1})\}}=J_3. \end{aligned}$$

and similarly $J_2=J_3$. Thus, we obtain desired equality (23). From result (ii) in Theorem 2.3, with a similar above method , we get

$$K_1=\int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{h(u,1)}=\int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{h(1,u)}=K_3.$$

On the other hand by putting $h(x,y) = (x+y)^{\lambda}$, we have

$$K_1 = K_2 = K_3 = \int_1^{\infty} \frac{u^{\frac{\lambda}{2}-1}du}{h(1,u)} = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$$

which the desired result can now be obtained. □

Corollary 3.2. *Under assumptions of Theorem 2.3 with $a = b = \frac{\lambda}{2}$, then*
i) $k(x) \neq 1$,

$$\int_0^T \int_0^T \frac{F(x)\,G(y)\,dxdy}{h(x,y)\max\left\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\right\}} \tag{26}$$

$$\leq T^{\alpha} J_3 \sqrt[p]{p}\sqrt[q]{q}\left(\int_0^T (T-t)\,F^{p-1}(t)\,f(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^T (T-t)\,G^{q-1}(t)\,g(t)\,dt\right)^{\frac{1}{q}}$$

where J_3 is defined by (24).
ii) $k(x) = 1$,

$$\int_0^T \int_0^T \frac{F(x)\,G(y)\,dxdy}{h(x,y)}$$

$$\leq T^{\alpha} K_3 \sqrt[p]{p}\sqrt[q]{q}\left(\int_0^T (T-t)\,F^{p-1}(t)\,f(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^T (T-t)\,G^{q-1}(t)\,g(t)\,dt\right)^{\frac{1}{q}}$$

where K_3 is defined by (25).
Furthermore, when $h(x,y) = (x+y)^{\lambda}$, we have

$$\int_0^T \int_0^T \frac{F(x)\,G(y)\,dxdy}{(x+y)^{\lambda}}$$

$$\leq T^{\alpha} K_3 \sqrt[p]{p}\sqrt[q]{q}\left(\int_0^T (T-t)\,F^{p-1}(t)\,f(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^T (T-t)\,G^{q-1}(t)\,g(t)\,dt\right)^{\frac{1}{q}}.$$

Proof. The proof is completed with a method similar to the proof in Corrolary 3.1 □

4. Conclusion

In conclusion, the works of Sulaiman in 2008 and 2010 and Wei-Lei in 2011 represent significant advancements in the theory of Hardy-Hilbert type inequalities. Sulaiman's generalization widened the applicability of these inequalities by accommodating homogeneous kernels of order λ, thereby extending their utility across various mathematical domains. Wei-Lei's alternative proof technique not only provided a fresh perspective but also introduced new methodologies, further enriching the understanding and discourse surrounding Hardy-Hilbert type inequalities. Collectively, these contributions lay a strong foundation for future research endeavors, encouraging continued exploration and innovation in this field.

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