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Fekete-Szegö Inequalities for Sakaguchi Type Bi-univalent Functions Defined by Gegenbauer Polynomials

Murat ÇAĞLAR[a], Mucahit BUYANKARA[b], Ayşegül KARAMAN[c]

[a]*Department of Mathematics, Faculty of Science Erzurum Technical University, Erzurum, Türkiye*
[b]*Vocational School of Social Sciences, Bingöl University, Bingöl, Türkiye*
[c]*Department of Mathematics, Faculty of Science Erzurum Technical University, Erzurum, Türkiye*

Abstract. We introduce and investigate in this paper new subclasses of bi-univalent functions associated with the Gegenbauer polynomials in the open unit disc which satisfy subordination conditions defined in a symmetric domain. For these new subclasses, we obtain estimates for the Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $\left|a_3 - \mu a_2^2\right|$.

1. Introduction

Let A represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \Delta), \tag{1}$$

which are analytic in $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$.

A subclass of A with members that are univalent in Δ is indicated by the symbol S. The Koebe one-quarter theorem [9] guarantees that a disk with a radius of $1/4$ exists in the image of Δ for every univalent function $f \in A$. As a result, each univalent function f has a satisfied inverse function f^{-1}

$$f^{-1}(f(z)) = z, (z \in \Delta) \text{ and } f\left(f^{-1}(\omega)\right) = \omega, (|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

If f and f^{-1} are univalent in Δ, then we say that $f \in A$ is bi-univalent in Δ. The class of bi-univalent functions defined on the unit disk Δ is denoted by Σ. Due to the fact that $f \in \Sigma$ has the Maclaurin series described by (1), a calculation reveals that $g = f^{-1}$ has the expansion

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2{}^2 - a_3\right)\omega^3 + \ldots. \tag{2}$$

Corresponding author: MÇ mail address: murat.caglar@erzurum.edu.tr ORCID:0000-0001-8147-0343, MB ORCID:0000-0003-2683-9229, AK ORCID:0000-0002-2893-9461

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We know that the class Σ is not empty. For example, the functions

$$f_1(z) = \frac{z}{z-1},\ f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z},\ f_3(z) = -\log(1-z)$$

with their respective inverses

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega'},\ \ f_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1},\ \ f_3^{-1}(\omega) = \frac{e^{\omega}-1}{e^{\omega}}$$

belong to Σ.

Also, the Koebe function does not belong to Σ.

The research of analytical and bi-univalent functions is reintroduced in the publication [25]; previous studies include those of [6],[7],[12],[17],[18],[20],[21]. Several authors have introduced new subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [6]-[8],[17],[22],[24],[25]).

Let f and g be the analytic functions in Δ. We say that f is subordinate to g and denoted by

$$f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwarz function w, which is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$ such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

If g is a univalent function in Δ, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

In [17], by means of Loewner's method, the Fekete-Szegö inequality for the coefficients of $f \in S$ is that

$$\left|a_3 - \mu a_2^2\right| \le 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) \text{ for } 0 \le \mu < 1.$$

As $\mu \to 1^-$, the elementary inequality $\left|a_3 - a_2^2\right| \le 1$ is obtained. Moreover, the coefficient functional

$$F_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions, f in the open unit disk Δ plays an important role in geometric function theory. The problem of maximizing the absolute value of the functional $F_\mu(f)$ is called the Fekete-Szegö problem.

The Fekete–Szegö inequalities introduced in 1933, see [11], preoccupied researchers regarding different classes of univalent functions [10],[14],[19],[26]; hence, it is obvious that such inequalities were obtained regarding bi-univalent functions too and very recently published papers can be cited to support the assertion that the topic still provides interesting results [1],[3],[28].

Orthogonal polynomials have been extensively explored in recent years from a variety of angles because of their relevance in mathematical statistics, engineering, mathematical physics and probability theory. Classical orthogonal polynomials are the most typically encountered orthogonal polynomials in applications (Hermite polynomials, Laguerre polynomials, and Jacobi polynomials). The general subclass of Jacobi polynomials is the set of Gegenbauer polynomials, this class includes Legendre polynomials and Chebyshev polynomials as subclasses. For a recent connection between the classical orthogonal polynomials and geometric function theory, we mention [1]-[4],[13],[15],[27].

The Gegenbauer polynomials [16] are defined in terms of the Jacobi polynomials $P_n^{(u,v)}$, with $v = u = \lambda - \frac{1}{2}$, $(\lambda > -\frac{1}{2},\ \lambda \neq 0)$, which are described by

$$\mathcal{B}_n^\lambda(l) = \frac{\Gamma(n+2)\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda)\,\Gamma\left(n+\lambda+\frac{1}{2}\right)} P_n^{\left(\lambda+\frac{1}{2},\lambda-\frac{1}{2}\right)}(l)$$

$$= \binom{n-1+2\lambda}{n} \sum_{k=0}^{n} \frac{\binom{n}{k}(2\lambda+n)_k}{(\lambda+\frac{1}{2})_k} \left(\frac{l-1}{2}\right)^k. \tag{3}$$

From (3), it follows that $\mathcal{B}_n^{\lambda}(l)$ is a polynomial of degree n with real coefficients and $\mathcal{B}_n^{\lambda}(1) = \binom{n-1+2\lambda}{n}$, while the leading coefficient of $\mathcal{B}_n^{\lambda}(l)$ is $2^n\binom{n+\lambda-1}{n}$. According to Jacobi polynomial theory, for $\mu = v = \lambda - \frac{1}{2}$, with $\lambda > -\frac{1}{2}$, and $\lambda \neq 0$, we have

$$\mathcal{B}_n^{\lambda}(-l) = (-1)^n \mathcal{B}_n^{\lambda}(l).$$

In [16] and [23], Gegenbauer polynomials' generating function is provided by

$$\frac{2^{\lambda-\frac{1}{2}}}{(1-2lz+z^2)^{\frac{1}{2}}\left(1-lz+\sqrt{1-2lz+z^2}\right)^{\lambda-\frac{1}{2}}} = \frac{(\lambda-\frac{1}{2})_n}{(2\lambda)_n}\mathcal{B}_n^{\lambda}(l)\,t^n, \tag{4}$$

and this equivalence may be deduced from the Jacobi polynomial generating function.

From (4), we obtain

$$\phi_l^{\lambda}(z) = \frac{1}{(1-2lz+z^2)^{\lambda}} = \sum_{n=0}^{\infty} \mathcal{B}_n^{\lambda}(l)\,z^n,\ z \in \Delta,\ l \in [-1,1],\ \lambda \in \left(\frac{-1}{2}, +\infty\right) \backslash \{0\}, \tag{5}$$

and the proof is given in the papers [15]-[17].

When $\lambda = 1$, the relation (5) yields the ordinary generating function for the Chebyshev polynomials, and when $\lambda = \frac{1}{2}$, we get the ordinary generating function for the Legendre polynomials (see [5]).

The Taylor-Maclaurin series expansion for the function $\phi_l^{\lambda}(z)$ is as follows:

$$\phi_l^{\lambda}(z) = z + \mathcal{B}_1^{\lambda}(l)\,z^2 + \mathcal{B}_2^{\lambda}(l)\,z^3 + \mathcal{B}_3^{\lambda}(l)\,z^4 + \cdots + \mathcal{B}_{n-1}^{\lambda}(l)\,z^2(l)\,z^n + \ldots, \tag{6}$$

where

$$\mathcal{B}_0^{\lambda}(l) = 1, \mathcal{B}_1^{\lambda}(l) = 2\lambda l, \mathcal{B}_2^{\lambda}(l) = 2\lambda(\lambda+1)\,l^2 - \lambda = 2(\lambda)_2 l^2 - \lambda. \tag{7}$$

and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & n \in \mathbb{N}. \end{cases}$$

Many researchers have recently explored bi-univalent functions associated with Gegenbauer polynomials, refs. [2]-[4],[13],[27].

In this paper, we introduce and investigate novel subclasses of bi-univalent functions that are associated with the Gegenbauer polynomials, defined within the open unit disc. These functions are subject to subordination conditions that are established in a symmetric domain. By considering these specific subclasses, we provide detailed estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, which are crucial in understanding the behavior and properties of these functions. Additionally, we delve into the analysis of the Fekete-Szegö inequality, focusing on the expression $\left|a_3 - \mu a_2^2\right|$, where μ is a constant, to investigate the extremal properties of these coefficients. The results obtained in this work not only provide new insights into the structure of bi-univalent functions associated with the Gegenbauer polynomials but also contribute to the broader theory of coefficient estimates and inequalities in complex analysis, particularly in the context of subclasses of univalent functions.

2. Main Results

First, we define new subclasses of bi-univalent functions, associated with Gegenbauer polynomials.

Definition 2.1. *We say that f the form* (1) *is in the class $G_\Sigma\left(\phi_l^\lambda\right)$ $((l \in 0,1])$ $\left(l \neq \frac{1}{\sqrt{2}}\right)$, if the following subordinations hold:*

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \phi_l^\lambda(z) \tag{8}$$

and

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} \prec \phi_l^\lambda(\omega), \tag{9}$$

$z, \omega \in \Delta$, ϕ_l^λ is given by (6), *and $g = f^{-1}$ is given by* (2).

Definition 2.2. *We say that f the form* (1) *is in the class $K_\Sigma\left(\phi_l^\lambda\right)$ $((l \in 0,1])$, if the following subordinations hold:*

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} \prec \phi_l^\lambda(z) \tag{10}$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'} \prec \phi_l^\lambda(\omega), \tag{11}$$

$z, \omega \in \Delta$, ϕ_l^λ is given by (7), *and $g = f^{-1}$ is given by* (2).

Lemma 2.3. *[21, p.172]. Suppose $w(z) = \sum_{n=1}^{\infty} w_n z^n,\ z \in \Delta$, is an analytic function in Δ such that $|w(z)| < 1, z \in \Delta$. Then,*

$$|w_1| \leq 1, |w_n| \leq 1 - |w_1|^2, n = 2, 3, \ldots.$$

3. Initial Taylor Coefficients Estimates for the Class $G_\Sigma\left(\phi_l^\lambda\right)$

For the functions belonging to a class $G_\Sigma\left(\phi_l^\lambda\right)$, we will obtain upper bounds for the modulus of coefficients a_2 and a_3.

Theorem 3.1. *If the class $G_\Sigma\left(\phi_l^\lambda\right)$ contains all the functions f given by* (1), *then*

$$|a_2| \leq \frac{\lambda l\sqrt{2\lambda l}}{\sqrt{\left|\lambda(1-2l^2)\right|}}, \tag{12}$$

and

$$|a_3| \leq \lambda l(1 + \lambda l). \tag{13}$$

Proof. Let $f \in G_\Sigma\left(\phi_l^\lambda\right)$ and $g = f^{-1}$. We have the following from the definition in formulas (8) and (9)

$$\frac{2zf'(z)}{f(z)-f(-z)} = \Phi_l^\lambda(\upsilon(z)) \tag{14}$$

and

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} = \Phi_l^\lambda(\nu(\omega)) \tag{15}$$

where the analytical υ and ν functions have the form

$$\upsilon(z) = c_1 z + c_2 z^2 + \ldots, \tag{16}$$

$$\nu(\omega) = d_1\omega + d_2\omega^2 + \ldots, \tag{17}$$

and $\upsilon(0) = 0 = \nu(0)$, $|\upsilon(z)| < 1$, $|\nu(\omega)| < 1$, $z, \omega \in \Delta$.

It follows that, from Lemma 2.3, that

$$\left|c_j\right| \leq 1, \left|d_j\right| \leq 1, \text{ where } j \in \mathbb{N}. \tag{18}$$

If we replace (16) and (17) in (14) and (15), respectively, we obtain

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + \mathcal{B}_1^\lambda(l)\,\upsilon(z) + \mathcal{B}_2^\lambda(l)\,\upsilon^2(z) + \ldots, \tag{19}$$

and

$$\frac{2\omega f'(\omega)}{f(\omega) - f(-\omega)} = 1 + \mathcal{B}_1^\lambda(l)\,\nu(\omega) + \mathcal{B}_2^\lambda(l)\,\nu^2(\omega) + \ldots. \tag{20}$$

In view of (1) and (2), from (19) and (20), we obtain

$$\begin{aligned} & 1 + 2a_2 z + 2a_3 z^2 + \ldots \\ = \; & 1 + \mathcal{B}_1^\lambda(l)\,c_1 z + \left[\mathcal{B}_1^\lambda(l)\,c_2 + \mathcal{B}_2^\lambda(l)\,{c_1}^2\right] z^2 + \ldots \end{aligned}$$

and

$$\begin{aligned} & 1 - 2a_2\omega + \left(4a_2^2 - 2a_3\right)\omega^2 + \ldots \\ = \; & 1 + \mathcal{B}_1^\lambda(l)\,d_1\omega + \left[\mathcal{B}_1^\lambda(l)\,d_2 + \mathcal{B}_2^\lambda(l)\,d_1^2\right]\omega^2 + \ldots. \end{aligned}$$

It gives rise to the following relationships:

$$2a_2 = \mathcal{B}_1^\lambda(l)c_1, \tag{21}$$

$$2a_3 = \mathcal{B}_1^\lambda(l)\,c_2 + \mathcal{B}_2^\lambda(l)\,c_1^2, \tag{22}$$

and

$$-2a_2 = \mathcal{B}_1^\lambda(l)d_1, \tag{23}$$

$$4{a_2}^2 - 2a_3 = \mathcal{B}_1^\lambda(l)\,d_2 + \mathcal{B}_2^\lambda(l)\,d_1^2. \tag{24}$$

From (21) and (23), it follows that

$$c_1 = -d_1, \tag{25}$$

and

$$8a_2^2 = \left[\mathcal{B}_1^\lambda(l)\right]^2\left(c_1^2 + d_1^2\right)$$

$$a_2^2 = \frac{\left[\mathcal{B}_1^\lambda(l)\right]^2\left(c_1^2 + d_1^2\right)}{8}. \tag{26}$$

Adding (22) and (24), using (26), we obtain

$$a_2^2 = \frac{\left[\mathcal{B}_1^\lambda(l)\right]^3(c_2 + d_2)}{4\left[\mathcal{B}_1^\lambda(l)\right]^2 - 8\mathcal{B}_2^\lambda(l)}. \tag{27}$$

Using the relation (7), from (18) for c_2 and d_2 we get (12).

Using (25) and (26), by subtracting (24) from the relation (22), we get

$$\begin{aligned} a_3 &= \frac{\mathcal{B}_2^\lambda(l)\left(c_1^2 - d_1^2\right) + \mathcal{B}_1^\lambda(l)\,(c_2 - d_2)}{4} + a_2^2 \\ &= \frac{\mathcal{B}_1^\lambda(l)\,(c_2 - d_2) + \mathcal{B}_2^\lambda(l)(c_1^2 - d_1^2)}{4} + \frac{\left[\mathcal{B}_1^\lambda(l)\right]^2\left(c_1^2 + d_1^2\right)}{8} \end{aligned} \tag{28}$$

Once again applying (18) and using (7), for the coefficients c_1, d_1, c_2, d_2, we deduce (13). □

Corollary 3.2. *If the class* $G_\Sigma\left(\phi_1^1\right)$ *contains all the functions* f *given by* (1)*, then*

$$|a_2| \le \sqrt{2},$$

and

$$|a_3| \le 2.$$

Corollary 3.3. *If the class* $G_\Sigma\left(\phi_{\frac{1}{2}}^1\right)$ *contains all the functions* f *given by* (1)*, then*

$$|a_2| \le \frac{\sqrt{2}}{2},$$

and

$$|a_3| \le \frac{3}{4}.$$

Corollary 3.4. *If the class* $G_\Sigma\left(\phi_{\frac{1}{2}}^{\frac{1}{2}}\right)$ *contains all the functions* f *given by* (1)*, then*

$$|a_2| \le \frac{1}{2\sqrt{2}},$$

and

$$|a_3| \le \frac{5}{16}.$$

4. The Fekete-Szegö problem for the Function Class $G_\Sigma\left(\phi_l^\lambda\right)$

Due to the Zaprawa result, which is discussed in [28], we will obtain the Fekete-Szegö inequality for the class $G_\Sigma\left(\phi_l^\lambda\right)$.

Theorem 4.1. *If* f *given by* (1) *is in the class* $G_\Sigma\left(\phi_l^\lambda\right)$ *where* $\mu \in \mathbb{R}$*, then, we have*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \lambda l, & \text{if } \left|h(\mu)\right| \le \frac{1}{4}, \\ 4\lambda l\left|h(\mu)\right|, & \text{if } \left|h(\mu)\right| \ge \frac{1}{4}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\lambda l^2}{4\lambda l^2 - 2(2(\lambda+1)l^2 - 1)}.$$

Proof. If $f \in G_\Sigma\left(\phi_l^\lambda\right)$ is given by (1), from (27) and (28), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\mathcal{B}_1^\lambda(l)(c_2 - d_2)}{4} + (1-\mu)a_2^2 \\ &= \frac{\mathcal{B}_1^\lambda(l)(c_2 - d_2)}{4} + \frac{(1-\mu)\left[\mathcal{B}_1^\lambda(l)\right]^3(c_2 + d_2)}{4\left[\mathcal{B}_1^\lambda(l)\right]^2 - 8\mathcal{B}_2^\lambda(l)} \\ &= \mathcal{B}_1^\lambda(l)\left[\frac{c_2}{4} - \frac{d_2}{4} + \frac{(1-\mu)\left[\mathcal{B}_1^\lambda(l)\right]^2 c_2}{4\left[\mathcal{B}_1^\lambda(l)\right]^2 - 8\mathcal{B}_2^\lambda(l)} + \frac{(1-\mu)\left[\mathcal{B}_1^\lambda(l)\right]^2 d_2}{4\left[\mathcal{B}_1^\lambda(l)\right]^2 - 8\mathcal{B}_2^\lambda(l)}\right] \\ &= \mathcal{B}_1^\lambda(l)\left[\left(h(\mu) + \frac{1}{4}\right)c_2 + \left(h(\mu) - \frac{1}{4}\right)d_2\right], \end{aligned}$$

where

$$h(\mu)=\frac{(1-\mu)\left[\mathcal{B}_1^{\lambda}(l)\right]^2}{4\left[\mathcal{B}_1^{\lambda}(l)\right]^2-8\mathcal{B}_2^{\lambda}(l)}.$$

Now, by using (7)

$$a_3-\mu a_2^2=2\lambda l\left[\left(h(\mu)+\frac{1}{4}\right)c_2+\left(h(\mu)-\frac{1}{4}\right)d_2\right],$$

where

$$h(\mu)=\frac{(1-\mu)\lambda l^2}{4\lambda l^2-2(2(\lambda+1)l^2-1)}.$$

Therefore, given (7) and (18), we conclude that the required inequality holds. □

Corollary 4.2. *If f given by* (1) *is in the class $G_{\Sigma}\left(\phi_1^1\right)$ where $\mu\in\mathbb{R}$, then, we have*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}1, & if\quad \left|\mu-1\right|\le\frac{1}{2},\\ 2\left|1-\mu\right|, & if\quad \left|\mu-1\right|\ge\frac{1}{2}.\end{cases}$$

Corollary 4.3. *If f given by* (1) *is in the class $G_{\Sigma}\left(\phi_{\frac{1}{2}}^1\right)$ where $\mu\in\mathbb{R}$, then, we have*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{1}{2}, & if\quad \left|1-\mu\right|\le 1,\\ \left|\frac{1-\mu}{2}\right|, & if\quad \left|1-\mu\right|\ge 1.\end{cases}$$

Corollary 4.4. *If f given by* (1) *is in the class $G_{\Sigma}\left(\phi_{\frac{1}{2}}^{\frac{1}{2}}\right)$ where $\mu\in\mathbb{R}$, then, we have*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{1}{4}, & if\quad \left|1-\mu\right|\le 2,\\ \left|\frac{1-\mu}{8}\right|, & if\quad \left|1-\mu\right|\ge 2.\end{cases}$$

5. Coefficients Estimates for the Class $K_{\Sigma}\left(\phi_l^{\lambda}\right)$

We will obtain upper bounds of $|a_2|$ and $|a_3|$ for the functions belonging to a class $K_{\Sigma}\left(\phi_l^{\lambda}\right)$.

Theorem 5.1. *If the class $K_{\Sigma}\left(\phi_l^{\lambda}\right)$ contains all the functions f given by* (1), *then*

$$|a_2|\le\frac{\lambda l\sqrt{2\lambda l}}{\sqrt{\left|\lambda(2-4l^2-\lambda l^2)\right|}},\tag{29}$$

and

$$|a_3|\le\frac{\lambda l}{3}+\frac{\lambda^2 l^2}{4}.\tag{30}$$

Proof. Let $f\in K_{\Sigma}\left(\phi_l^{\lambda}\right)$ and $g=f^{-1}$. We have the following from the definition in formulas (10) and (11)

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'}=\Phi_l^{\lambda}(\upsilon(z))\tag{31}$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'}=\Phi_l^{\lambda}(\nu(\omega))\tag{32}$$

where the analytical υ and ν functions have the form

$$\upsilon(z) = c_1 z + c_2 z^2 + \ldots, \tag{33}$$

$$\nu(\omega) = d_1\omega + d_2\omega^2 + \ldots, \tag{34}$$

and $\upsilon(0) = 0 = \nu(0)$, $|\upsilon(z)| < 1$, $|\nu(\omega)| < 1$, $z, \omega \in \Delta$.

It follows that, from Lemma 2.3, that

$$\left|c_j\right| \le 1, \left|d_j\right| \le 1, \text{ where } j \in \mathbb{N}. \tag{35}$$

If we replace (33) and (34) in (31) and (32), respectively, we obtain

$$\frac{2[zf'(z)]'}{[f(z) - f(-z)]'} = 1 + \mathcal{B}_1^\lambda(l)\,\upsilon(z) + \mathcal{B}_2^\lambda(l)\,\upsilon^2(z) + \ldots, \tag{36}$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega) - f(-\omega)]'} = 1 + \mathcal{B}_1^\lambda(l)\,\nu(\omega) + \mathcal{B}_2^\lambda(l)\,\nu^2(\omega) + \ldots. \tag{37}$$

In view of (1) and (2), from (36) and (37), we obtain

$$\begin{aligned} & 1 + 4a_2 z + 6a_3 z^2 + \ldots \\ = \; & 1 + \mathcal{B}_1^\lambda(l)\,c_1 z + \left[\mathcal{B}_1^\lambda(l)\,c_2 + \mathcal{B}_2^\lambda(l)\,c_1^2\right] z^2 + \ldots \end{aligned}$$

and

$$\begin{aligned} & 1 - 4a_2\omega + \left(12{a_2}^2 - 6a_3\right)\omega^2 + \ldots \\ = \; & 1 + \mathcal{B}_1^\lambda(l)\,d_1\omega + \left[\mathcal{B}_1^\lambda(l)\,d_2 + \mathcal{B}_2^\lambda(l)\,d_1^2\right]\omega^2 + \ldots. \end{aligned}$$

It gives rise to the following relationships:

$$4a_2 = \mathcal{B}_1^\lambda(l)c_1, \tag{38}$$

$$6a_3 = \mathcal{B}_1^\lambda(l)\,c_2 + \mathcal{B}_2^\lambda(l)\,c_1^2, \tag{39}$$

and

$$-4a_2 = \mathcal{B}_1^\lambda(l)d_1, \tag{40}$$

$$12{a_2}^2 - 6a_3 = \mathcal{B}_1^\lambda(l)\,d_2 + \mathcal{B}_2^\lambda(l)\,d_1^2. \tag{41}$$

From (38) and (40), it follows that

$$c_1 = -d_1, \tag{42}$$

and

$$32a_2^2 = \left[\mathcal{B}_1^\lambda(l)\right]^2\left(c_1^2 + d_1^2\right),$$

$$a_2^2 = \frac{\left[\mathcal{B}_1^\lambda(l)\right]^2\left(c_1^2 + d_1^2\right)}{32}. \tag{43}$$

Adding (39) and (41), using (43), we obtain

$$a_2^2 = \frac{\left[\mathcal{B}_1^\lambda(l)\right]^3(c_2 + d_2)}{4(3\left[\mathcal{B}_1^\lambda(l)\right]^2 - 8\mathcal{B}_2^\lambda(l))}. \tag{44}$$

Using the relation (7), from (35) for c_2 and d_2 we get (29).Using (42) and (43), by subtracting (41) from the relation (39), we get

$$\begin{aligned} a_3 &= \frac{\mathcal{B}_2^{\lambda}(l)\left(c_1^2-d_1^2\right)+\mathcal{B}_1^{\lambda}(l)(c_2-d_2)}{12}+a_2^2 \\ &= \frac{\mathcal{B}_1^{\lambda}(l)(c_2-d_2)+\mathcal{B}_2^{\lambda}(l)(c_1^2-d_1^2)}{12}+\frac{\left[\mathcal{B}_1^{\lambda}(l)\right]^2\left(c_1^2+d_1^2\right)}{32} \end{aligned} \tag{45}$$

Once again applying (42) and using (7), for the coefficients c_1, d_1, c_2, d_2, we deduce (30). □

Corollary 5.2. *If the class $K_\Sigma\left(\phi_1^1\right)$ contains all the functions f given by* (1), *then*

$$|a_2| \le \frac{\sqrt{6}}{3},$$

and

$$|a_3| \le \frac{7}{12}.$$

Corollary 5.3. *If the class $K_\Sigma\left(\phi_{\frac{1}{2}}^1\right)$ contains all the functions f given by* (1), *then*

$$|a_2| \le \frac{\sqrt{3}}{3},$$

and

$$|a_3| \le \frac{11}{48}.$$

Corollary 5.4. *If the class $K_\Sigma\left(\phi_{\frac{1}{2}}^{\frac{1}{2}}\right)$ contains all the functions f given by* (1), *then*

$$|a_2| \le \frac{1}{\sqrt{14}},$$

and

$$|a_3| \le \frac{19}{192}.$$

6. The Fekete-Szegö problem for the Function Class $K_\Sigma\left(\phi_l^\lambda\right)$

We obtain the Fekete-Szegö inequality fort he class $K_\Sigma\left(\phi_l^\lambda\right)$ due to the result of Zaprawa; see [28].

Theorem 6.1. *If f given by* (1) *is in the class $K_\Sigma\left(\phi_l^\lambda\right)$ where $\mu \in \mathbb{R}$, then, we have*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{\lambda l}{3}, & \text{if } \left|h(\mu)\right| \le \frac{1}{12}, \\ 4\lambda l\left|h(\mu)\right|, & \text{if } \left|h(\mu)\right| \ge \frac{1}{12}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\lambda l^2}{12\lambda l^2-2(2(\lambda+1)l^2-1)}.$$

Proof. If $f \in K_\Sigma\left(\phi_l^\lambda\right)$ is given by (1), from (44) and (45), we have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{\mathcal{B}_1^\lambda(l)(c_2-d_2)}{12}+(1-\mu)a_2^2 \\ &= \frac{\mathcal{B}_1^\lambda(l)(c_2-d_2)}{12}+\frac{(1-\mu)\left[\mathcal{B}_1^\lambda(l)\right]^3(c_2+d_2)}{4(3\left[\mathcal{B}_1^\lambda(l)\right]^2-8\mathcal{B}_2^\lambda(l))} \\ &= \mathcal{B}_1^\lambda(l)\left[\frac{c_2}{12}-\frac{d_2}{12}+\frac{(1-\mu)\left[\mathcal{B}_1^\lambda(l)\right]^2c_2}{4(3\left[\mathcal{B}_1^\lambda(l)\right]^2-8\mathcal{B}_2^\lambda(l))}+\frac{(1-\mu)\left[\mathcal{B}_1^\lambda(l)\right]^2d_2}{4(3\left[\mathcal{B}_1^\lambda(l)\right]^2-8\mathcal{B}_2^\lambda(l))}\right] \\ &= \mathcal{B}_1^\lambda(l)\left[\left(h(\mu)+\frac{1}{12}\right)c_2+\left(h(\mu)-\frac{1}{12}\right)d_2\right], \end{aligned}$$

where

$$h(\mu)=\frac{(1-\mu)\left[\mathcal{B}_1^\lambda(l)\right]^2}{4(3\left[\mathcal{B}_1^\lambda(l)\right]^2-8\mathcal{B}_2^\lambda(l))}$$

Now, by using (7)

$$a_3-\mu a_2^2=2\lambda l\left[\left(h(\mu)+\frac{1}{12}\right)c_2+\left(h(\mu)-\frac{1}{12}\right)d_2\right],$$

where

$$h(\mu)=\frac{(1-\mu)\lambda l^2}{12\lambda l^2-2(2(\lambda+1)l^2-1)}.$$

Therefore, given (7) and (35), we conclude that the required inequality holds. □

Corollary 6.2. *If f given by* (1) *is in the class $K_\Sigma\left(\phi_1^1\right)$ where $\mu\in\mathbb{R}$, then, we have*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{1}{3}, & if\ \left|1-\mu\right|\le\frac{1}{2},\\ \frac{2}{3}\left|1-\mu\right|, & if\ \left|1-\mu\right|\ge\frac{1}{2}.\end{cases}$$

Corollary 6.3. *If f given by* (1) *is in the class $K_\Sigma\left(\phi_{\frac{1}{2}}^1\right)$ where $\mu\in\mathbb{R}$, then, we have*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{1}{6}, & if\ \left|1-\mu\right|\le 1,\\ \left|\frac{1-\mu}{6}\right|, & if\ \left|1-\mu\right|\ge 1.\end{cases}$$

Corollary 6.4. *If f given by* (1) *is in the class $K_\Sigma\left(\phi_{\frac{1}{2}}^{\frac{1}{2}}\right)$ where $\mu\in\mathbb{R}$, then, we have*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{1}{12}, & if\ \left|1-\mu\right|\le\frac{4}{3},\\ \left|\frac{1-\mu}{16}\right|, & if\ \left|1-\mu\right|\ge\frac{4}{3}.\end{cases}$$

7. Conclusions

In this paper, we introduced and investigated a new subclass of bi-univalent functions in the open unit disk defined by Gegenbauer polynomials and satisfies subordination conditions. Furthermore, we obtain upper bounds for $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $\left|a_3-\mu a_2{}^2\right|$ for functions in this subclass. Also, the approach presented here has been extended to establish new subfamilies of bi-univalent functions with the other special functions. The related outcomes may be left to the researchers for practice.

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