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On the Fixed Point Property for Nonexpansive Mappings on Large Classes in Köthe-Toeplitz Duals of Certain Difference Sequence Spaces

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Abstract. In 2000, Et and Esi introduced new type of generalized difference sequences by using the structure of Çolak's work from 1989 where he defined new types of sequence spaces while Çolak was also inspired by Kızmaz's idea about the difference operator he studied in 1981. In this study, we consider Et and Esi's work and study Köthe-Toeplitz duals of their generalized difference sequence spaces. We take their study in terms of fixed point theory and find large classes of closed, bounded and convex subsets in those duals with fixed point property for nonexpansive mappings.

1. Introduction and preliminaries

Researches have shown that the fixed point exists for some function classes defined on certain classes of sets in some spaces, while it cannot be found at all in others. Fixed point theory has examined how this happens or does not happen. Researchers have made classifications and characterizations. In 1965, Browder [4] proved that every Hilbert space has a property satisfying that every nonexpansive mapping defined on any closed, bounded, and convex (cbc) nonempty subset domain with the same range has a fixed point. Since that time, spaces with this property have been considered to have the fixed point property for nonexpansive mappings (fppne). Then, researchers considered looking for the spaces with the property and if the property still exists when larger classes of mappings are taken. Then also they have seen spaces failing the properties. For example, in 1965, Browder [5] and Göhde [17] with independent studies, they saw that uniform convex Banach spaces have the fppne. Then, Kirk [20] generalized the result for the reflexive Banach spaces with normal structure. In fact, Goebel and Kirk [14] noticed that Kirk's result was able to extend for uniformly Lipschitz mappings and some researchers have studied estimating the Lipschitz coefficient satisfying the property for uniform Lipschitz mappings on different Banach spaces. For example, Goebel and Kirk [15] showed that for Hilbert spaces, the best Lipschitz coefficient would be a scalar less than a number in the interval $\left[\sqrt{2}\,,\ \frac{\pi}{2}\right]$, and Goebel and Kirk [14] and Lim [21] showed independently that for a Lebesgue space L^p when $2 < p < \infty$, the coefficient is smaller by a scalar larger than or equal to $\left(1 + \frac{1}{2^p}\right)^{\frac{1}{p}}$ while Alspach [1] showed that when $p = 2$, there exists a fixed point free Lipschitz mapping with Lispchitz coefficient $\sqrt{2}$ defined on a cbc subset. In fact, $\sqrt{2}$ is the smallest Lipschitz coefficient for Alspach's mapping. We need to note that, similar to the definition of the Banach spaces satisfying the fppne,

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if a Banach space has a property that every uniformly Lipschitz mapping defined on any cbc nonempty subset domain with the same range has a fixed point, then that Banach space has the fixed point property for uniformly Lischitz mapping (fppul). In terms of fixed point property for uniformly Lipschitz mappings, Dowling, Lennard, and Turett [8] showed that if a Banach space contains an isomorphic copy of ℓ^1, then it fails the fppul. It is a well-known fact by researchers that c_0 or ℓ^1 is almost isometrically embedded in every non-reflexive Banach space with an unconditional basis (see [24]). For this reason, classical non-reflexive Banach spaces fail the fixed point property for non-expansive mappings, that is, in these spaces, there can be a closed, convex and bounded subset and a non-expansive invariant T mapping defined on that set such that T has no fixed point. This result is based on well-known theorems in the literature (see for example Theorem 1.c.12 in [24] and Theorem 1.c.5 in [25]). These theorems state that for a Banach lattice or Banach space with an unconditional basis to be reflexive, it is necessary and sufficient if it does not contain any isomorphic copies of c_0 or ℓ^1. Therefore, this close relation to the reflexivity or nonreflexivity of Banach space, researchers have worked for years and questioned whether c_0 or ℓ^1 can be renormed to have a fixed point for nonexpansive mappings. Lin [22] showed in his study that what was thought was not true and that at least ℓ^1 could be renormed to have the fixed point property for nonexpansive mappings. Then, the remaining question was if the same could have been done for c_0, but the answer still remains open. Since the researchers have considered trying to obtain the analogous results for well-known other classical nonreflexive Banach spaces, another experiment was done for Lebesgue integrable functions space $L_1[0,1]$ by Hernandes-Lineares and Maria [23] but they were able to obtain the positive answer when they restricted the nonexpansive mappings by assuming they were affine as well. One can say that there is no doubt most research has been inspired by the ideas of the study [16] where Goebel and Kuczumow proved that while ℓ^1 fails the fixed point property since one can easily find a cbc nonweakly compact subset there and a fixed point free invariant nonexpansive map, it is possible to find a very large class subsets in target such that invariant nonexpansive mappings defined on the members of the class have fixed points. In fact, it is easy to notice the traces of those ideas in [22] work. Even Goebel and Kuczumow's work has inspired many other researchers to investigate if there exist more example of nonreflexive Banach spaces with large classes satisfying fixed point property. For example, in 2004, Kaczor and Prus [18] wanted to generalize Goebel and Kuczumow's findings and they proved that affine asymptotically nonexpansive invariant mappings defined on a large class of cbc subsets in ℓ^1 can have fixed points. Moreover, in [12], Kaczor and Prus' results were extended by having been found larger classes satisfying the fixed point property for affine asymptotically nonexpansive mappings. Thus, affinity condition became a tool for their works. In fact, another well-known nonreflexive Banach space, Lebesgue space $L_1[0,1]$, was studied in [23] and in their study they obtained an analogous result to [22] as they showed that $L_1[0,1]$ can be renormed to have the fixed point property for affine nonexpansive mappings. In this study, we will investigate some Banach spaces analogous to ℓ^1. In the present work, we study Goebel-Kuczumow analogy for Köthe-Toeplitz duals of certain generalized difference sequence spaces investigated by Et and Esi [11]. We prove that a very large class of closed, bounded and convex subsets in Köthe-Toeplitz duals of the difference sequence spaces generalized by Et and Esi has the fixed point property for nonexpansive mappings. Therefore, firstly we would like to give the definition of Cesàro sequence spaces which was defined by Shiue [29] in 1970, and next we present Kızmaz's difference sequence space definition in [19] by noting that we work on a space which is derived from his ideas' generalizations such that many researchers (see for example [6, 9–11, 27, 30]) have generalized his work as well.

In fact, we need to note that Et and Esi's work [11] and Et and Çolak's work [10] used a common difference sequence definition from Çolak's work [6].

Now, first we recall that Shiue [29], in 1970, introduced the Cesàro sequence spaces written as

$$\mathrm{ces}_p = \left\{ (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

such that $\ell^p \subset \mathrm{ces}_p$ and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

such that $\ell^\infty \subset ces_\infty$ where $1 \leq p < \infty$. Then, from the definition of Cesàro sequence spaces, Kızmaz [19], defined difference sequence spaces for ℓ^∞, c, and c_0 and symbolized them by $\ell^\infty(\triangle)$, $c(\triangle)$, and $c_0(\triangle)$, respectively. In his introduction, he defined the difference operator $\triangle$ applied to the sequence $x = (x_n)_n$ using the formula $\triangle x = (x_k - x_{k+1})_k$. In fact, he investigated Köthe-Toeplitz duals and their topological properties.

As one of the researchers generalizing his ideas, Çolak [6] in 1989, introduced firstly a generalized difference sequence space by taking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ and then denoting a new difference operator by $\triangle_v$ such that for any sequence $x = (x_n)_n$, he defined the difference sequence of that $\triangle_v x = (v_k x_k - v_{k+1} x_{k+1})_k$. Then, Et and Esi [11] in 2000, generalized Çolak's difference sequence space by defining

$$\triangle_v(\ell^\infty) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle_v x \in \ell^\infty \right\},$$

$$\triangle_v(c) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle_v x \in c \right\},$$

$$\triangle_v(c_0) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \triangle_v x \in c_0 \right\}.$$

Furthermore, their m^{th} order generalized difference sequence space is given for any $m \in \mathbb{N}$ by $\triangle_v^0 x = (v_k x_k)_k$, $\triangle_v^m x = (\triangle_v^m x_k)_k = \left(\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1}\right)_k$ with $\triangle_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for each $k \in \mathbb{N}$.

Next, in 2004, Bektaş, Et and Çolak [2] obtained the Köthe-Toeplitz duals for the generalized difference sequence space of Et and Esi's. We may recall here that their m^{th} order difference sequence space has the following norm for any $m \in \mathbb{N}$:

$$\|x\|_v^{(m)} = \sum_{k=1}^{m} |v_k x_k| + \left\|\triangle_v^m x\right\|_\infty$$

Then, the corresponding Köthe-Toeplitz dual was obtained as in [2] and [11] such that it is written as below:

$$\begin{aligned} D_1^m &:= \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; \left(n^m {v_n}^{-1} a_n\right)_n \in \ell^1 \right\} \\ &= \left\{ a = (a_n)_n \subset \mathbb{R} \colon \; \|a\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|} < \infty \right\}. \end{aligned}$$

Note that $D_1^m \subset \ell^1$ if $k^m \left|{v_k}^{-1}\right| > 1$ for each $k, m \in \mathbb{N}$ and $\ell^1 \subset D_1^m$ if $k^m \left|{v_k}^{-1}\right| < 1$ for each $k, m \in \mathbb{N}$.

Now, we will need the following well-known preliminaries before giving our main results. [15] may be suggested as a good reference for these fundamentals.

Definition 1.1. *Consider that $(X, \|\cdot\|)$ is a Banach space and let C be a non-empty cbc subset. Let : $C \to C$ be a mapping. We say that*
1. T is an affine mapping if for every $t \in [0,1]$ and $a, b \in C$, $T((1-t)a + tb) = (1-t)T(a) + t\,T(b)$.
2. T is a nonexpansive mapping if for every $a, b \in C$, $\| T(a) - T(b) \| \leq \| a - b \|$.

Then, we will easily obtain an anologous key lemma from the below lemma in the work [16].

Lemma 1.2. *Let $\{u_n\}$ be a sequence in ℓ^1 converging to u in weak-star topology. Then, for every $w \in \ell^1$,*

$$r(w) = r(u) + \|w - u\|_1$$

where

$$r(w) = \limsup_n \|u_n - w\|_1.$$

Note that our scalar field in this study will be real numbers although Çolak [6] considers complex values of $v = (v_n)_n$ while introducing his structer of the difference sequence which is taken as the fundamental concept in this study.

As a brief summary just before starting the section for main results of our paper to remind the reader important facts for the study, we may note the following. This study explores the fixed point property (FPP) for nonexpansive mappings within the context of Köthe-Toeplitz duals of generalized difference sequence spaces. Inspired by foundational work on absolutely summable scalar sequences, we generalize prior results to spaces with richer geometric structures. In 2024, as an alternative proof of García-Falset et al [13], Dalby [7] demonstrated that uniformly nonsquare Banach spaces possess the FPP, emphasizing the role of geometric conditions. Berinde and Päcurar [3] introduced saturated classes of contractive mappings, broadening fixed point applicability through enriched contractions. In 2022, Oppenheim [26] linked Banach property (T) with FPP in higher-rank simple Lie groups, revealing algebraic influences on fixed point existence. By avoiding reliance on the affinity hypothesis, our approach identifies larger classes of closed, bounded, and convex subsets that exhibit the FPP, incorporating these recent insights to strengthen the theoretical framework.

2. Main results

In this section, we will present our results. As mentioned in the first section, we investigate Goebel and Kuzmunow analogy for the space D_1^m for each $m \in \mathbb{N}$. We aim to show that there is a large class of cbc subsets in D_1^m such that every nonexpansive invariant mapping defined on the subsets in the class taken has a fixed point. Recall that the invariant mappings have the same domain and the range.

First, due to isometric isomorphism, using Lemma 1.2, we will provide the straight analogous result as a lemma below which will be a key step as in the works such as [16], and [12] and in fact the methods in the study [12] will be our lead in this work.

Lemma 2.1. *Let $m \in \mathbb{N}$ and $\{u_n\}$ be a sequence in the Banach space D_1^m and assume $\{u_n\}$ converges to u in weak-star topology. Then, for every $w \in D_1^m$,*

$$r(w) = r(u) + \|w - u\|^{(m)}$$

where

$$r(w) = \limsup_n \|u_n - w\|^{(m)} .$$

Then, we obtain our results by the following theorems.

Theorem 2.2. *Fix $m \in \mathbb{N}$ and $t \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := t\, v_1\, e_1$, $f_2 := \frac{t\, v_2}{2^m}\, e_2$, and $f_n := \frac{v_n}{n^m} e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E^{(m)} = E_t{}^{(m)}$ of D_1^m by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \alpha_n f_n : \ \forall n \in \mathbb{N}, \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} .$$

Then, $E^{(m)}$ has the fixed point property for $\|\,.\,\|^{(m)}$-nonexpansive mappings.

Proof. Fix $m \in \mathbb{N}$ and $t \in (0,1)$. Let $T{:}E^{(m)} \to E^{(m)}$ be a $\|\,.\,\|^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(m)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|^{(m)} \underset{n}{\to} 0$. Due to isometric isomorphism, D_1^m shares common geometric properties with ℓ^1 and so both D_1^m and its predual have similar fixed point theory properties to ℓ^1 and c_0, respectively. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in D_1^m, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(m)}$ by

$$C^{(m)}:=\overline{E^{(m)}}^{w^*}=\left\{\sum_{n=1}^{\infty}\alpha_n f_n\text{: } each\ \alpha_n\geq 0\ and\ \sum_{n=1}^{\infty}\alpha_n\leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u\in C^{(m)}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r:D_1^m\to[0,\infty)$ defined by

$$r(w)=\limsup_n \left\|u^{(n)}-w\right\|^{(m)},\ \forall w\in D_1^m$$

such that for every $w\in D_1^m$,

$$r(w)=r(u)+\|u-w\|^{(m)}.$$

Case 1. $u\in E^{(m)}$.

Then, $r(Tu)=r(u)+\|Tu-u\|^{(m)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu-u^{(n)}\right\|^{(m)} \\ &\leq \limsup_n \left\|Tu-T\left(u^{(n)}\right)\right\|^{(m)}+\limsup_n \left\|u^{(n)}-T\left(u^{(n)}\right)\right\|^{(m)} \\ &\leq \limsup_n \left\|u-u^{(n)}\right\|^{(m)}+0 \\ &= r(u). \end{aligned} \tag{1}$$

Thus, $r(Tu)=r(u)+\|Tu-u\|^{(m)}\leq r(u)$ and so $\|Tu-u\|^{(m)}=0$. Therefore, $Tu=u$.

Case 2. $u\in C^{(m)}\backslash E^{(m)}$.

Then, we may find scalars satisfying $u=\sum_{n=1}^{\infty}\delta_n f_n$ such that $\sum_{n=1}^{\infty}\delta_n<1$ *and* $\delta_n\geq 0$, $\forall n\in\mathbb{N}$.

Define $\xi:=1-\sum_{n=1}^{\infty}\delta_n$ and for $\beta\in\left[\frac{-\delta_1}{\xi},\frac{\delta_2}{\xi}+1\right]$ define

$$h_\beta:=(\delta_1+\beta\xi)f_1+(\delta_2+(1-\beta)\xi)f_2+\sum_{n=3}^{\infty}\delta_n f_n.$$

Then,

$$\left\|h_\beta-u\right\|^{(m)}=\left\|\beta t\xi v_1 e_1+(1-\beta)\,\xi\frac{t\,v_2 e_2}{2^m}\right\|^{(m)}=t\left|\beta\right|\xi+t\left|1-\beta\right|\xi.$$

$\left\|h_\beta-u\right\|^{(m)}$ is minimized for $\beta\in[0,1]$ and its minimum value would be $t\xi$.

Now fix $w\in E^{(m)}$. Then, we may find scalars satisfying $w=\sum_{n=1}^{\infty}\alpha_n f_n$ such that $\sum_{n=1}^{\infty}\alpha_n=1$ with $\alpha_n\geq 0$, $\forall n\in\mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k\in\mathbb{N}$ where $\gamma_1:=t\,v_1$, $\gamma_2:=\frac{t\,v_2}{2^m}$, and $\gamma_n:=\frac{v_n}{n^m}$ for all integers $n\geq 3$ such that for each $n\in\mathbb{N}$, $f_n=\gamma_n e_n$.

Then,

$$\begin{aligned}\|w-u\|^{(m)} &= \left\|\sum_{k=1}^{\infty}\alpha_k f_k - \sum_{k=1}^{\infty}\delta_k f_k\right\|^{(m)}\\ &= \left\|\sum_{k=1}^{\infty}(\alpha_k-\delta_k) f_k\right\|^{(m)}\\ &= \sum_{k=1}^{\infty}\left|(\alpha_k-\delta_k)\frac{k^m\gamma_k}{v_k}\right|.\end{aligned}$$

Hence,

$$\begin{aligned}\|w-u\|^{(m)} &\geq \sum_{k=1}^{\infty} t\ |\alpha_k-\delta_k|\\ &\geq t\left|\sum_{k=1}^{\infty}(\alpha_k-\delta_k)\right|\\ &= t\left|1-\sum_{k=1}^{\infty}\delta_k\right|\\ &= t\xi.\end{aligned}$$

Hence,

$$\|w-u\|^{(m)} \geq t\xi = \left\|h_\beta-u\right\|^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=3}^{\infty}|\alpha_k-\delta_k| = 0$; that is, we have $\|w-u\|^{(m)}=t\xi$ if and only if $\alpha_k=\delta_k$ for every $k \geq 3$; or say, $\|w-u\|^{(m)}=t\xi$ if and only if $w=h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function $\rho : [0,1] \to E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda := \rho([0,1])$ is a compact convex subset and so $\|w-u\|^{(m)}$ achieves its minimum value at $w=h_\beta$ and for any $h_\beta \in \Lambda$, we get

$$\begin{aligned}r\left(h_\beta\right) &= r(u) + \left\|h_\beta-u\right\|^{(m)}\\ &\leq r(u) + \left\|Th_\beta-u\right\|^{(m)}\\ &= r\left(Th_\beta\right) = \limsup_n \left\|Th_\beta-u^{(n)}\right\|^{(m)}\end{aligned}$$

then, like the inequality 1, we get

$$\begin{aligned}r\left(h_\beta\right) &\leq \limsup_n \left\|Th_\beta-T\left(u^{(n)}\right)\right\|^{(m)} + \limsup_n \left\|u^{(n)}-T\left(u^{(n)}\right)\right\|^{(m)}\\ &\leq \limsup_n \left\|h_\beta-u^{(n)}\right\|^{(m)} + \limsup_n \left\|u^{(n)}-T\left(u^{(n)}\right)\right\|^{(m)}\\ &\leq \limsup_n \left\|h_\beta-u^{(n)}\right\|^{(m)}+0\\ &= r\left(h_\beta\right).\end{aligned}$$

Hence, $r(h_\beta) \leq r\left(Th_\beta\right) \leq r(h_\beta)$ and so $r\left(Th_\beta\right) = r\left(h_\beta\right)$.

Therefore,

$$r(u) + \left\|Th_\beta-u\right\|^{(m)} = r(u) + \left\|h_\beta-u\right\|^{(m)}.$$

Thus, $\left\|Th_\beta - u\right\|^{(m)} = \left\|h_\beta - u\right\|^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's fixed point theorem [28] easily we get the result T has a fixed point since T is continuous; thus, h_β is the unique minimizer of $\|w - u\|^{(m)}$ $: w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings. □

Theorem 2.3. *Fix $m \in \mathbb{N}$ and $t \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := t\, v_1\, e_1$, $f_2 := \frac{t\, v_2}{2^m}\, e_2$, $f_3 := \frac{t\, v_3}{3^m}\, e_3$, and $f_n := \frac{v_n}{n^m} e_n$ for all integers $n \geq 4$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E^{(m)} = E_t{}^{(m)}$ of D_1^m by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} \alpha_n f_n : \ \forall n \in \mathbb{N}, \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} .$$

Then, $E^{(m)}$ has the fixed point property for $\| \, . \, \|^{(m)}$-nonexpansive mappings.

Proof. Fix $m \in \mathbb{N}$ and $t \in (0,1)$. Let $T : E^{(m)} \to E^{(m)}$ be a $\| \, . \, \|^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(m)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|^{(m)} \underset{n}{\to} 0$. Due to isometric isomorphism, D_1^m shares common geometric properties with ℓ^1 and so both D_1^m and its predual have similar fixed point theory properties to ℓ^1 and c_0, respectively. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology and c_0 is separable, in D_1^m, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(m)}$ by

$$C^{(m)} := \overline{E^{(m)}}^{w^*} = \left\{ \sum_{n=1}^{\infty} \alpha_n f_n \text{: each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n \leq 1 \right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u \in C^{(m)}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r : D_1^m \to [0, \infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|^{(m)}, \ \forall w \in D_1^m$$

such that for every $w \in D_1^m$,

$$r(w) = r(u) + \|u - w\|^{(m)}.$$

Case 1. $u \in E^{(m)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|^{(m)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|^{(m)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|^{(m)} + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(m)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|^{(m)} + 0 \\ &= r(u) . \end{aligned} \tag{2}$$

Thus, $r(Tu) = r(u) + \|Tu - u\|^{(m)} \leq r(u)$ and so $\|Tu - u\|^{(m)} = 0$. Therefore, $Tu = u$.

Case 2. $u \in C^{(m)} \setminus E^{(m)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\delta_n \geq 0$, $\forall n \in \mathbb{N}$.

Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\beta \in \left[\frac{-\delta_1}{\xi}, \frac{\delta_2}{\xi} + 1\right]$, define

$$h_\beta := \left(\delta_1 + \frac{\beta}{2}\xi\right) f_1 + \left(\delta_2 + \frac{\beta}{2}\xi\right) f_2 + (\delta_3 + (1-\beta)\,\xi)\, f_3 + \sum_{n=4}^{\infty} \delta_n f_n.$$

Then,

$$\left\|h_\beta - u\right\|^{(m)} = \left\|\frac{\beta}{2} t\xi v_1 e_1 + \frac{\beta}{2} t\xi \frac{v_2}{2^m} e_2 + (1-\beta)\,\xi \frac{t\, v_3 e_3}{3^m}\right\|^{(m)} = t\left|\frac{\beta}{2}\right|\xi + t\left|\frac{\beta}{2}\right|\xi + t\left|1-\beta\right|\xi.$$

$\left\|h_\beta - u\right\|^{(m)}$ is minimized for $\beta \in [0,1]$ and its minimum value would be $t\xi$.

Now fix $w \in E^{(m)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ with $\alpha_n \geq 0$, $\forall n \in \mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k \in \mathbb{N}$ where $\gamma_1 := t\, v_1$, $\gamma_2 := \frac{t\, v_2}{2^m}$, $\gamma_3 := \frac{t\, v_3}{3^m}$, and $\gamma_n := \frac{v_n}{n^m}$ for all integers $n \geq 4$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\begin{aligned}
\|w - u\|^{(m)} &= \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(m)} \\
&= \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\, f_k\right\|^{(m)} \\
&= \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k)\, \frac{k^m \gamma_k}{v_k}\right| \\
&\geq \sum_{k=1}^{\infty} t\, |\alpha_k - \delta_k| \\
&\geq t\left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right| \\
&= t\left|1 - \sum_{k=1}^{\infty} \delta_k\right| \\
&= t\xi.
\end{aligned}$$

Hence,

$$\|w - u\|^{(m)} \geq t\xi = \left\|h_\beta - u\right\|^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=4}^{\infty} |\alpha_k - \delta_k| = 0$; that is, we have $\|w-u\|^{(m)} = t\xi$ if and only if $\alpha_k = \delta_k$ for every $k \geq 4$; or say, $\|w-u\|^{(m)} = t\xi$ if and only if $w = h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function $\rho : [0,1] \to E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda := \rho([0,1])$ is a compact convex subset and so $\|w-u\|^{(m)}$ achieves its minimum value at $w = h_\beta$ and for any $h_\beta \in \Lambda$, we get

$$\begin{aligned}
r\left(h_\beta\right) &= r(u) + \left\|h_\beta - u\right\|^{(m)} \\
&\leq r(u) + \left\|Th_\beta - u\right\|^{(m)} \\
&= r\left(Th_\beta\right) = \limsup_n \left\|Th_\beta - u^{(n)}\right\|^{(m)}
\end{aligned}$$

then same as the inequality 2, we get

$$\begin{aligned} r\left(h_\beta\right) &\leq \limsup_n \left\| Th_\beta - T\left(u^{(n)}\right)\right\|^{(m)} + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right)\right\|^{(m)} \\ &\leq \limsup_n \left\| h_\beta - u^{(n)}\right\|^{(m)} + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right)\right\|^{(m)} \\ &\leq \limsup_n \left\| h_\beta - u^{(n)}\right\|^{(m)} + 0 \\ &= r\left(h_\beta\right). \end{aligned}$$

Hence, $r(h_\beta) \leq r\left(Th_\beta\right) \leq r(h_\beta)$ and so $r\left(Th_\beta\right) = r\left(h_\beta\right)$.
Therefore,

$$r(u) + \left\| Th_\beta - u\right\|^{(m)} = r(u) + \left\| h_\beta - u\right\|^{(m)}.$$

Thus, $\left\| Th_\beta - u\right\|^{(m)} = \left\| h_\beta - u\right\|^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's fixed point theorem [28] we can easily we get the result T has a fixed point since T is continuous. Thus, h_β is the unique minimizer of $\|w - u\|^{(m)} \ :w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings.

□

3. Conclusion

In this paper, we extended the fixed point property for nonexpansive mappings to large classes of subsets in Köthe-Toeplitz duals of generalized difference sequence spaces. Building on the ideas of Goebel and Kuczumow [16], our work eliminates the affinity condition required by other approaches, such as those of Kaczor and Prus [18]. In 2024, Dalby [7] and previously in 2006 García-Falset et al [13] highlighted geometric conditions like uniform nonsquareness as sufficient for FPP, while Berinde and Păcurar [3] expanded fixed point results to enriched contractions, offering broader applications. In 2022, Oppenheim [26] demonstrated the interplay between Banach property (T) and fixed points in higher-rank simple Lie groups, providing a novel perspective. By leveraging these insights, we demonstrate the broad applicability of our methods. Future research could explore analogous properties in other Banach spaces or further generalize the concepts introduced here.

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