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Some Notes on Diagonal Lifts in the Semi-Cotangent Bundle

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Abstract. The main purpose of the present paper is to study diagonal lift tensor fields of type (1,1) from tangent bundle $T(M_n)$ to semi-cotangent (pull-back) bundle $(t^*(M_n), \pi_2)$.

1. Lifts of Vector Fields on a Cross-Section in the Semi-Cotangent Bundle

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $T(M_n)$ the tangent bundle determined by a natural projection (submersion) $\pi_1 : T(M_n) \to M_n$. We use the notation $(x^i) = (x^{\overline{\alpha}}, x^\alpha)$, where the indices $i, j, ...$ run from 1 to $2n$, the indices $\alpha, \beta, ...$ from 1 to n and the indices $\overline{\alpha}, \overline{\beta}, ...$ from $n+1$ to $2n$, x^α are coordinates in M_n, $x^{\overline{\alpha}} = y^\alpha$ are fibre coordinates of the tangent bundle $T(M_n)$. If $(x^{i'}) = (x^{\overline{\alpha}'}, x^{\alpha'})$ is another system of local adapted coordinates in the tangent bundle $T(M_n)$, then we have

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1}$$

The Jacobian of (1) has components

$$(A_j^{i'}) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_\beta^{\alpha'} & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$, $A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}$. Let $T_x^*(M_n)(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^{\overline{\alpha}}, x^\alpha) \in T(M_n))$ be the cotangent space at a point x of M_n. If p_α are components of $p \in T_x^*(M_n)$ with respect to the natural coframe $\{dx^\alpha\}$, i.e. $p = p_i\, dx^i$, then by definition the set $t^*(M_n)$ of all points $\left(x^I\right) = (x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$, $x^{\overline{\overline{\alpha}}} = p_\alpha$; $I, J, ... = 1, ..., 3n$ with projection $\pi_2 : t^*(M_n) \to T(M_n)$ (i.e. $\pi_2 : (x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}}) \to (x^{\overline{\alpha}}, x^\alpha)$) is a semi-cotangent (pull-back [11]) bundle of the cotangent bundle by submersion $\pi_1 : T(M_n) \to M_n$ (For definition of the pull-back bundle, see for example [1], [3], [4], [5],[6],[10],[12]). It is remarkable fact that the semi-cotangent (pull-back) bundle has a degenerate symplectic structure [11]

$$\omega : (\omega_{AB}) = dp = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\delta_\beta^\alpha \\ 0 & \delta_\alpha^\beta & 0 \end{pmatrix}.$$

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It is clear that the pull-back bundle $t^*(M_n)$ of the cotangent bundle $T^*(M_n)$ also has the natural bundle structure over M_n, its bundle projection $\pi : t^*(M_n) \to M_n$ being defined by $\pi : (x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}}) \to (x^{\alpha})$, and hence $\pi = \pi_1 \circ \pi_2$. Thus $(t^*(M_n), \pi_1 \circ \pi_2)$ is the composite bundle [[13], p.9] or step-like bundle [14].

We analyze some properties of diagonal lift of tensor fields of type (1,1) in semi-cotangent bundles with the help of adapted frames.

We denote by $\Im^p_q(T(M_n))$ and $\Im^p_q(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of type (p, q) on $T(M_n)$ and M_n respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞−functions on $T(M_n)$ and M_n, respectively.

To a transformation (1) of local coordinates of $T(M_n)$, there corresponds on $t^*(M_n)$ the coordinate transformation [8], [9]:

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} y^{\beta}, \\ x^{\alpha'} = x^{\alpha'}\left(x^{\beta}\right), \\ x^{\overline{\overline{\alpha}}'} = \frac{\partial x^{\beta}}{\partial x^{\alpha'}} p_{\beta}. \end{cases} \tag{2}$$

The Jacobian of (2) has components [8], [9]:

$$\overline{A} : (A^{I'}_{J}) = \begin{pmatrix} A^{\alpha'}_{\beta} & A^{\alpha'}_{\beta\varepsilon} y^{\varepsilon} & 0 \\ 0 & A^{\alpha'}_{\beta} & 0 \\ 0 & p_{\sigma} A^{\beta'}_{\beta} A^{\sigma}_{\beta'\alpha'} & A^{\beta}_{\alpha'} \end{pmatrix}, \tag{3}$$

where

$$A^{\alpha'}_{\beta\varepsilon} = \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta} \partial x^{\varepsilon}}, \quad A^{\alpha}_{\beta'\alpha'} = \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'} \partial x^{\alpha'}}.$$

We denote by $\Im^p_q(T(M_n))$ and $\Im^p_q(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of type (p, q) on $T(M_n)$ and M_n, respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞ −functions on $T(M_n)$ and M_n, respectively.

Let θ be a covector field on $T(M_n)$. Then the transformation $p \to \theta_p$, θ_p being the value of θ at $p \in T(M_n)$, determines a cross-section β_θ of semi-cotangent bundle. Thus if $\sigma : M_n \to T^*(M_n)$ is a cross-section of $(T^*(M_n), \widetilde{\pi}, M_n)$, such that $\widetilde{\pi} \circ \sigma = I_{(M_n)}$, an associated cross-section $\beta_\theta : T(M_n) \to t^*(M_n)$ of semi-cotangent (pull-back) bundle $(t^*(M_n), \pi_2, T(M_n))$ of cotangent bundle by using projection (submersion) of the tangent bundle $T(M_n)$ defined by [[2], p. 217-218], [[7], p. 301]:

$$\beta_\theta\left(x^{\overline{\alpha}}, x^{\alpha}\right) = \left(x^{\overline{\alpha}}, x^{\alpha}, \sigma \circ \pi_1\left(x^{\overline{\alpha}}, x^{\alpha}\right)\right) = \left(x^{\overline{\alpha}}, x^{\alpha}, \sigma\left(x^{\alpha}\right)\right) = \left(x^{\overline{\alpha}}, x^{\alpha}, \theta_{\alpha}\left(x^{\beta}\right)\right).$$

If the covector field θ has the local components $\theta_\alpha\left(x^{\beta}\right)$, the cross-section $\beta_\theta\left(T(M_n)\right)$ of $t^*(M_n)$ is locally expressed by

$$x^{\overline{\alpha}} = y^{\alpha} = V^{\alpha}\left(x^{\beta}\right), \quad x^{\alpha} = x^{\alpha}, \quad x^{\overline{\overline{\alpha}}} = p_{\alpha} = \theta_{\alpha}\left(x^{\beta}\right) \tag{4}$$

with respect to the coordinates $x^A = (x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$. $x^{\overline{\alpha}} = y^{\alpha}$ being considered as parameters. Differentiating (4) by $x^{\overline{\alpha}} = y^{\alpha}$, we have vector fields $B_{(\overline{\beta})}$ $(\overline{\beta} = 1, ..., n)$ with components

$$B_{(\overline{\beta})} = \frac{\partial x^A}{\partial x^{\overline{\beta}}} = \partial_{\overline{\beta}} x^A = \begin{pmatrix} \partial_{\overline{\beta}} V^{\alpha} \\ \partial_{\overline{\beta}} x^{\alpha} \\ \partial_{\overline{\beta}} \theta_{\alpha} \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta\left(T(M_n)\right)$ [8], [9].

Thus $B_{(\overline{\beta})}$ have components

$$B_{(\overline{\beta})} : \left(B^A_{(\overline{\beta})}\right) = \begin{pmatrix} \delta^{\alpha}_{\beta} \\ 0 \\ 0 \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, where

$$\delta^{\alpha}_{\overline{\beta}} = A^{\alpha}_{\overline{\beta}} = \frac{\partial x^{\alpha}}{\partial x^{\overline{\beta}}}.$$

Let $X \in \Im^1_0(T(M_n))$, i.e. $X = X^{\alpha}\partial_{\alpha}$. We denote by BX the vector field with local components

$$BX : \left(B^A_{(\overline{\beta})} X^{\overline{\beta}}\right) = \begin{pmatrix} \delta^{\alpha}_{\overline{\beta}} X^{\overline{\beta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\alpha}_{\overline{\beta}} X^{\overline{\beta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X^{\alpha} \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, which is defined globally along $\beta_\theta(T(M_n))$. Then a mapping

$$B : \Im^1_0(T(M_n)) \to \Im^1_0(\beta_\theta(T(M_n)))$$

is defined by (5). The mapping B is the differential of $\beta_\theta : T(M_n) \to t^*(M_n)$ and so an isomorphism of $\Im^1_0(T(M_n))$ onto $\Im^1_0(\beta_\theta(T(M_n)))$ [8], [9].

Since a cross-section is locally expressed by $x^{\overline{\alpha}} = y^{\alpha} = const.$, $x^{\overline{\overline{\alpha}}} = p_{\alpha} = const.$, $x^{\alpha} = x^{\alpha}$, x^{α} being considered as parameters. Differentiating (4) by x^{α}, we have vector fields $C_{(\beta)}$ $(\beta = n+1, ..., 2n)$ with components

$$C_{(\beta)} = \frac{\partial x^A}{\partial x^{\beta}} = \partial_{\beta} x^A = \begin{pmatrix} \partial_{\beta} V^{\alpha} \\ \partial_{\beta} x^{\alpha} \\ \partial_{\beta} \theta_{\alpha} \end{pmatrix},$$

which are tangent to the cross-section $\beta_\theta(T(M_n))$.

Thus $C_{(\beta)}$ have components

$$C_{(\beta)} : \left(C^A_{(\beta)}\right) = \begin{pmatrix} \partial_{\beta} V^{\alpha} \\ \delta^{\alpha}_{\beta} \\ \partial_{\beta} \theta_{\alpha} \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, where

$$\delta^{\alpha}_{\beta} = A^{\alpha}_{\beta} = \frac{\partial x^{\alpha}}{\partial x^{\beta}}.$$

Let $X \in \Im^1_0(T(M_n))$. Then we denote by CX the vector field with local components

$$CX : \left(C^A_{(\beta)} X^{\beta}\right) = \begin{pmatrix} X^{\beta}\partial_{\beta} V^{\alpha} \\ X^{\alpha} \\ X^{\beta}\partial_{\beta} \theta_{\alpha} \end{pmatrix} \tag{6}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ in $t^*(M_n)$, which is defined globally along $\beta_\theta(T(M_n))$. Then a mapping

$$C : \Im^1_0(T(M_n)) \to \Im^1_0(\beta_\theta(T(M_n)))$$

is defined by (6). The mapping C is the differential of $\beta_\theta : T(M_n) \to t^*(M_n)$ and so an isomorphism of $\Im^1_0(T(M_n))$ onto $\Im^1_0(\beta_\theta(T(M_n)))$ [8], [9].

Now, consider $\omega \in \Im^0_1(M_n)$ and vector field $X \in \Im^1_0(T(M_n))$, then ${}^{vv}\omega$ (vertical lift), ${}^{cc}X$ (complete lift) and ${}^{HH}X$ (horizontal lift) have respectively, components on the semi-cotangent bundle $t^*(M_n)$ [8], [9]:

$${}^{vv}\omega : \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix}, \quad {}^{cc}X : \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon} X^{\alpha} \\ X^{\alpha} \\ -p_{\sigma}(\partial_{\alpha} X^{\sigma}) \end{pmatrix}, \quad {}^{HH}X : \begin{pmatrix} -\Gamma^{\alpha}_{\beta} X^{\beta} \\ X^{\alpha} \\ X^{\beta}\Gamma_{\beta\alpha} \end{pmatrix} \tag{7}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$, where

$$\Gamma^{\alpha}_{\beta} = V^{\varepsilon}\Gamma^{\alpha}_{\varepsilon\beta}, \quad \Gamma_{\beta\alpha} = \theta_{\varepsilon}\Gamma^{\varepsilon}_{\beta\alpha}.$$

On the other hand, the fibre is locally represented by

$$x^{\overline{\alpha}} = y^{\alpha} = const., \quad x^{\alpha} = const., \quad x^{\overline{\alpha}} = p_{\alpha} = p_{\alpha},$$

p_{α} being considered as parameters. Thus, on differentiating with respect to p_{α}, we easily see that the vector fields $E_{\left(\overline{\overline{\beta}}\right)} =^{vv} \left(dx^{\beta}\right)$ $(\overline{\overline{\beta}} = 2n+1, ..., 3n)$ with components

$$E_{\left(\overline{\overline{\beta}}\right)} : \left(E^{A}_{\left(\overline{\overline{\beta}}\right)}\right) = \partial_{\left(\overline{\overline{\beta}}\right)} x^{A} = \begin{pmatrix} \partial_{\overline{\overline{\beta}}} y^{\alpha} \\ \partial_{\overline{\overline{\beta}}} x^{\alpha} \\ \partial_{\overline{\overline{\beta}}} p_{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta^{\beta}_{\alpha} \end{pmatrix}$$

is tangent to the fibre, where

$$\delta^{\beta}_{\alpha} = A^{\beta}_{\alpha} = \frac{\partial x^{\beta}}{\partial x^{\alpha}}.$$

Let ω be an 1-form with local components ω_{α} on M_n, so that ω is a 1-form with local expression $\omega = \omega_{\alpha} dx^{\alpha}$. We denote by $E\omega$ the vector field with local components

$$E\omega : \left(E^{A}_{\left(\overline{\overline{\beta}}\right)} \omega_{\beta}\right) = \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix}, \tag{8}$$

which is tangent to the fibre. Then a mapping

$$E : \Im^{0}_{1}(M_n) \to \Im^{1}_{0}(t^{*}(M_n))$$

is defined by (8) and so an isomorphism of $\Im^{0}_{1}(M_n)$ in to $\Im^{1}_{0}(t^{*}(M_n))$ [8], [9].

We consider in $\pi^{-1}(U)$ $3n$ local vector fields $B_{(\overline{\beta})}, C_{(\beta)}$ and $E_{\left(\overline{\overline{\beta}}\right)}$ along $\beta_{\theta}\left(T(M_n)\right)$, which are respectively represented by

$$B_{(\overline{\beta})} = B\frac{\partial}{\partial x^{\overline{\beta}}}, \quad C_{(\beta)} = C\frac{\partial}{\partial x^{\beta}}, \quad E_{\left(\overline{\overline{\beta}}\right)} = E dx^{\beta}.$$

Theorem 1.1. *Let X be a vector field on $T(M_n)$. We have along $\beta_{\theta}\left(T(M_n)\right)$ the formula*

$$^{cc}X = CX + B\left(L_V X\right) + E\left(-L_X \theta\right),$$

where $L_V X$ denotes the Lie derivative of X with respect to V, and $L_X\theta$ denotes the Lie derivative of θ with respect to X [8], [9].

On the other hand, on putting $C_{\left(\overline{\overline{\beta}}\right)} = E_{\left(\overline{\overline{\beta}}\right)}$, we write the adapted frame of $\beta_{\theta}\left(T(M_n)\right)$ as $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$. The adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of $\beta_{\theta}\left(T(M_n)\right)$ is given by the matrix

$$\widetilde{A} = \left(\widetilde{A}^{A}_{B}\right) = \begin{pmatrix} \delta^{\alpha}_{\beta} & \partial_{\beta} V^{\alpha} & 0 \\ 0 & \delta^{\alpha}_{\beta} & 0 \\ 0 & \partial_{\beta}\theta_{\alpha} & \delta^{\beta}_{\alpha} \end{pmatrix}. \tag{9}$$

Since the matrix $\widetilde{A}$ in (9) is non-singular, it has the inverse. Denoting this inverse by $\left(\widetilde{A}\right)^{-1}$, we have

$$\left(\widetilde{A}\right)^{-1} = \left(\widetilde{A}_C^B\right)^{-1} = \begin{pmatrix} \delta_\theta^\beta & -\partial_\theta V^\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\partial_\theta \theta_\beta & \delta_\beta^\theta \end{pmatrix}, \tag{10}$$

where $\widetilde{A}\left(\widetilde{A}\right)^{-1} = (\widetilde{A}_B^A)\left(\widetilde{A}_C^B\right)^{-1} = \delta_C^A = \widetilde{I}$, where $A = \left(\overline{\alpha}, \alpha, \overline{\overline{\alpha}}\right)$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = \left(\overline{\theta}, \theta, \overline{\overline{\theta}}\right)$.

Then we see from Theorem 1.1 that the complete lift ${}^{cc}X$ of a vector field $X \in \Im_0^1(T(M_n))$ has along $\beta_\theta\,(T(M_n))$ components of the form

$${}^{cc}X : \begin{pmatrix} L_V X^\alpha \\ X^\alpha \\ -L_X \theta_\alpha \end{pmatrix}$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ [8], [9].

Theorem 1.2. *The complete lift ${}^{cc}X$ of a vector field X in M_n to $t^*(M_n)$ is tangent to the cross-section $\beta_\theta\,(T(M_n))$ determined by a $1-$form θ and vector field V in M_n if and only if*

$$L_X\theta = 0, L_V X = 0,$$

where $L_V X$ denotes the Lie derivative of X with respect to V, and $L_X\theta$ denotes the Lie derivative of θ with respect to X.

BX, CX and $E\omega$ also have components:

$$BX : \begin{pmatrix} X^\alpha \\ 0 \\ 0 \end{pmatrix}, \quad CX : \begin{pmatrix} 0 \\ X^\alpha \\ 0 \end{pmatrix}, \quad E\omega : \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix} \tag{11}$$

respectively, with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of the cross-section $\beta_\theta\,(T(M_n))$ determined by a 1-form θ on $T(M_n)$ [8], [9].

2. Complete Lift of Tensor Fields of Type (1,1) on a Cross-Section in Semi-Cotangent Bundle

Suppose now that $F \in \Im_1^1(T(M_n))$ and F has local components F_β^α in a neighborhood U of M_n, $F = F_\beta^\alpha \partial_\alpha \otimes dx^\beta$. Then the semi-cotangent (pull-back) bundle $t^*(M_n)$ of cotangent bundle $T^*(M_n)$ by using projection of the tangent bundle $T(M_n)$ admits the complete lift ${}^{cc}F$ of F with components [8], [9]:

$${}^{cc}F : ({}^{cc}F_J^I) = \begin{pmatrix} F_\beta^\alpha & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix}, \tag{12}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ on $t^*(M_n)$. Then ${}^{cc}F$ has components F_B^A given by

$${}^{cc}F = ({}^{cc}F_B^A) = \begin{pmatrix} F_\beta^\alpha & L_V F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \varphi_F\theta & F_\alpha^\beta \end{pmatrix} \tag{13}$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of the cross-section $\beta_\theta\left(T(M_n)\right)$ determined by a 1-form θ in $T(M_n)$, where $A = \left(\overline{\alpha}, \alpha, \overline{\overline{\alpha}}\right)$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$ [8], [9]. Also, the component ${}^{cc}F_{\beta}^{\overline{\overline{\alpha}}}$ of ${}^{cc}F_B^A$ is defined as Tachibana operator $\phi_F\theta$ of F, i.e.,

$$ {}^{cc}F_{\beta}^{\overline{\overline{\alpha}}} = \phi_F\theta = (\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma)\theta_\sigma - F_\beta^\gamma \partial_\gamma \theta_\alpha + F_\alpha^\gamma \partial_\beta \theta_\gamma, $$

and $L_V F_\beta^\alpha$ *denotes the Lie derivative of* F_β^α *with respect to* V, i.e.,

$$ L_V F_\beta^\alpha = V^\gamma \partial_\gamma F_\beta^\alpha + F_\gamma^\alpha \partial_\beta V^\gamma - F_\beta^\gamma \partial_\gamma V^\alpha. $$

3. Adapted Frames and Diagonal Lifts of Affinor Fields

Let ∇ be a symmetric affine connection in M_n. In each coordinate neighborhood $\{U, x^\alpha\}$ of M_n, we put

$$ X_{(\alpha)} = \frac{\partial}{\partial x^\alpha}, \quad \theta^{(\alpha)} = dx^\alpha. $$

Then $3n$ local vector fields $Y_{(\alpha)}$, ${}^{HH}X_{(\alpha)}$ and ${}^{vv}\theta^{(\alpha)}$ have respectively components of the form

$$ Y_{(\alpha)} : \begin{pmatrix} \delta_\alpha^\beta \\ 0 \\ 0 \end{pmatrix}, \quad {}^{HH}X_{(\alpha)} : \begin{pmatrix} -\Gamma_\beta^\alpha \\ \delta_\alpha^\beta \\ \Gamma_{\beta\alpha} \end{pmatrix}, \quad {}^{vv}\theta^{(\alpha)} : \begin{pmatrix} 0 \\ 0 \\ \delta_\beta^\alpha \end{pmatrix} \tag{14} $$

with respect to the induced coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$ in $\pi^{-1}(U)$, where we have used (7). We call the set $\left\{Y_{(\alpha)}, {}^{HH}X_{(\alpha)}, {}^{vv}\theta^{(\alpha)}\right\}$ the frame adapted to the symmetric affine connection ∇ in $\pi^{-1}(U)$. On putting

$$ \widehat{e_{(\overline{\alpha})}} = Y_{(\alpha)}, \quad \widehat{e_{(\alpha)}} = {}^{HH}X_{(\alpha)}, \quad \widehat{e_{\left(\overline{\overline{\alpha}}\right)}} = {}^{vv}\theta^{(\alpha)} \tag{15} $$

we write the adapted frame as

$$ \left\{\widehat{e_{(B)}}\right\} = \left\{\widehat{e_{(\overline{\alpha})}}, \widehat{e_{(\alpha)}}, \widehat{e_{\left(\overline{\overline{\alpha}}\right)}}\right\}. \tag{16} $$

The adapted frame $\left\{\widehat{e_{(B)}}\right\} = \left\{\widehat{e_{(\overline{\alpha})}}, \widehat{e_{(\alpha)}}, \widehat{e_{\left(\overline{\overline{\alpha}}\right)}}\right\}$ is given by the matrix

$$ \widehat{A} : \left(\widehat{A_B^A}\right) = \begin{pmatrix} \delta_\beta^\alpha & -\Gamma_\beta^\alpha & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & \Gamma_{\beta\alpha} & \delta_\alpha^\beta \end{pmatrix}. \tag{17} $$

Since the matrix $\widehat{A}$ in (17) is non-singular, it has the inverse. Denoting this inverse by $\left(\widehat{A}\right)^{-1}$, we have

$$ \left(\widehat{A}\right)^{-1} : \left(\widehat{A_C^B}\right)^{-1} = \begin{pmatrix} \delta_\theta^\beta & \Gamma_\theta^\beta & 0 \\ 0 & \delta_\theta^\beta & 0 \\ 0 & -\Gamma_{\theta\beta} & \delta_\beta^\theta \end{pmatrix}, \tag{18} $$

where $\widehat{A}\left(\widehat{A}\right)^{-1} = (\widehat{A_B^A})\left(\widehat{A_C^B}\right)^{-1} = \delta_C^A = \widetilde{I}$, where $A = \left(\overline{\alpha}, \alpha, \overline{\overline{\alpha}}\right)$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = \left(\overline{\theta}, \theta, \overline{\overline{\theta}}\right)$.

If we take account of (16), we see that the diagonal lift ${}^{DD}F$ of $F \in \Im_1^1(T(M_n))$ has components [8], [9]:

$$ {}^{DD}F : ({}^{DD}F_J^I) = \begin{pmatrix} -F_\beta^\alpha & -\Gamma_\varepsilon^\alpha F_\beta^\varepsilon - \Gamma_\beta^\varepsilon F_\varepsilon^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & \Gamma_{\beta\sigma} F_\alpha^\sigma + \Gamma_{\alpha\sigma} F_\beta^\sigma & -F_\alpha^\beta \end{pmatrix}, \tag{19} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ on $t^{*}(M_n)$, where

$$\Gamma^{\alpha}_{\varepsilon} = y^{\gamma}\Gamma^{\alpha}_{\gamma\varepsilon}, \quad \Gamma_{\alpha\sigma} = p_{\gamma}\Gamma^{\gamma}_{\alpha\sigma}.$$

which proves (19).

We now see, from (16), that the diagonal lift ${}^{DD}F$ of $F \in \Im^1_1(T(M_n))$ has components of the form

$${}^{DD}F : ({}^{DD}F^{A}_{B}) = \begin{pmatrix} -F^{\alpha}_{\beta} & 0 & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & 0 & -F^{\beta}_{\alpha} \end{pmatrix}$$

with respect to the adapted frame $\left\{\widehat{e_{(B)}}\right\}$ in $t^{*}(M_n)$.

We now obtain from (19) that the diagonal lift ${}^{DD}F$ of an affinor field $F \in \Im^1_1(T(M_n))$ has along $\beta_\theta\,(T(M_n))$ components of the form [8], [9]:

$${}^{DD}F : \begin{pmatrix} -F^{\alpha}_{\beta} & -(\nabla_{\varepsilon}V^{\alpha})F^{\varepsilon}_{\beta} - \left(\nabla_{\beta}V^{\varepsilon}\right)F^{\alpha}_{\varepsilon} & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & -\left(\nabla_{\beta}\theta_{\sigma}\right)F^{\sigma}_{\alpha} - (\nabla_{\alpha}\theta_{\sigma})F^{\sigma}_{\beta} & -F^{\beta}_{\alpha} \end{pmatrix}, \tag{20}$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$.

Then we see from (7) that the horizontal lift ${}^{HH}X$ of a vector field $X \in \Im^1_0(T(M_n))$ has along $\beta_\theta\,(T(M_n))$ components of the form

$${}^{HH}X : \begin{pmatrix} -X^{\beta}\left(\nabla_{\beta}V^{\alpha}\right) \\ X^{\alpha} \\ -\left(\nabla_{\beta}\theta_{\alpha}\right)X^{\beta} \end{pmatrix} \tag{21}$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ [8], [9].

Using (7), (20) and (21), we have along $\beta_\theta\,(T(M_n))$:

Theorem 3.1. *If F and X are affinor and vector fields on $T(M_n)$, and $\omega \in \Im^0_1(M_n)$, then with respect to a symetric affine connection ∇ in M_n, we have [8], [9]:*

(*i*) ${}^{DD}F\left({}^{HH}X\right) = {}^{HH}(FX)$,

(*ii*) ${}^{DD}F\left({}^{vv}\omega\right) = -{}^{vv}(\omega \circ F)$.

Theorem 3.2. *If $F, G \in \Im^1_1(M_n)$, then with respect to a symetric affine connection ∇ in M_n, we have [9]*:

$${}^{DD}F\,{}^{DD}G + {}^{DD}G\,{}^{DD}F = {}^{HH}(FG + GF).$$

Theorem 3.3. *If $F, G \in \Im^1_1(M_n)$, then with respect to a symetric affine connection ∇ in M_n, we have [9]*:

$${}^{DD}F\,{}^{HH}G + {}^{DD}G\,{}^{HH}F = {}^{HH}F\,{}^{DD}G + {}^{HH}G\,{}^{DD}F = {}^{DD}(FG + GF).$$

Putting $F = G$ in Theorem 3.2 and Theorem 3.3, we have

$$\begin{aligned} {}^{HH}F\,{}^{DD}F &= {}^{DD}F\,{}^{HH}F = {}^{DD}(F^2) \\ ({}^{DD}F)^{2p} &= {}^{HH}(F^{2p}),\ ({}^{DD}F)^{2p+1} = {}^{DD}(F^{2p+1}), (p = 1, 2, ...) \end{aligned}$$

for any $F \in \Im^1_1(T(M_n))$.

Theorem 3.4. *The diagonal lift $\widehat{J}$ of the identity tensor field I of type* $(1,1)$ *has the components [9]:*

$$\widehat{J}:\begin{pmatrix} -\delta^{\alpha}_{\beta} & 2\Gamma^{\alpha}_{\beta} & 0 \\ 0 & \delta^{\alpha}_{\beta} & 0 \\ 0 & 2\Gamma_{\beta\alpha} & -\delta^{\beta}_{\alpha} \end{pmatrix}. \tag{22}$$

From Theorem 3.4, we have

Theorem 3.5. *The diagonal lift $\widehat{J}$ of the identity tensor filed I of type* $(1,1)$ *satisfies* $\widehat{J^2} = I$.

Proof. In fact, from (22), we easily see that

$$\begin{aligned} \widehat{J^2} &= \widehat{J}\big(\widehat{J}\big) = (\widehat{J^A_B})\big(\widehat{J^B_C}\big) \\ &= \begin{pmatrix} -\delta^{\alpha}_{\beta} & 2\Gamma^{\alpha}_{\beta} & 0 \\ 0 & \delta^{\alpha}_{\beta} & 0 \\ 0 & 2\Gamma_{\beta\alpha} & -\delta^{\beta}_{\alpha} \end{pmatrix}\begin{pmatrix} -\delta^{\beta}_{\theta} & 2\Gamma^{\beta}_{\theta} & 0 \\ 0 & \delta^{\beta}_{\theta} & 0 \\ 0 & 2\Gamma_{\theta\beta} & -\delta^{\theta}_{\beta} \end{pmatrix} \\ &= \begin{pmatrix} \delta^{\alpha}_{\theta} & 2\Gamma^{\alpha}_{\theta} - 2\Gamma^{\alpha}_{\theta} & 0 \\ 0 & \delta^{\alpha}_{\theta} & 0 \\ 0 & 2\Gamma_{\theta\alpha} - 2\Gamma_{\theta\alpha} & \delta^{\theta}_{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} \delta^{\alpha}_{\theta} & 0 & 0 \\ 0 & \delta^{\alpha}_{\theta} & 0 \\ 0 & 0 & \delta^{\theta}_{\alpha} \end{pmatrix} \\ &= \delta^{A}_{C} \\ &= \widehat{I}. \end{aligned}$$

□

Theorem 3.6. *The lifts ${}^{HH}X$ of $X \in \Im^1_0(T(M_n))$ and ${}^{vv}\omega$ of $\omega \in \Im^0_1(M_n)$ have respectively components*

$$(i)\ {}^{HH}X : \begin{pmatrix} 0 \\ X^{\alpha} \\ 0 \end{pmatrix},\ (ii)\ {}^{vv}\omega : \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix}$$

with respect to the adapted frame $\left\{\widehat{e_{(B)}}\right\} = \left\{\widehat{e_{(\overline{\alpha})}}, \widehat{e_{(\alpha)}}, \widehat{e_{(\overline{\overline{\alpha}})}}\right\}$, *$X^{\alpha}$ and ω_{α} being local components of X and ω respectively.*

Proof. (i) If $X \in \Im^1_0(T(M_n))$, from (7) and from (17), then we have

$$\begin{aligned} \widehat{A}^{HH}X &= \begin{pmatrix} \delta^{\alpha}_{\beta} & -\Gamma^{\alpha}_{\beta} & 0 \\ 0 & \delta^{\alpha}_{\beta} & 0 \\ 0 & \Gamma_{\beta\alpha} & \delta^{\beta}_{\alpha} \end{pmatrix}\begin{pmatrix} -\Gamma^{\beta}_{\theta}X^{\theta} \\ X^{\beta} \\ X^{\theta}\Gamma_{\beta\theta} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ X^{\alpha} \\ 0 \end{pmatrix}. \end{aligned}$$

(ii) If $\omega \in \Im^0_1(M_n)$, from (7) and from (17), then we have

$$\widehat{A}^{vv}\omega = \begin{pmatrix} \delta^{\alpha}_{\beta} & -\Gamma^{\alpha}_{\beta} & 0 \\ 0 & \delta^{\alpha}_{\beta} & 0 \\ 0 & \Gamma_{\beta\alpha} & \delta^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix}.$$

□

Using Theorem 3.1, we have

Theorem 3.7. *$F, G \in \Im^{1}_{1}(M_n)$, then*

$$\left[{}^{DD}F, {}^{DD}G\right] = {}^{DD}[F, G].$$

Proof. If X is an arbitrary vector field in $T(M_n)$, then

$$\begin{aligned} \left[{}^{DD}F, {}^{DD}G\right]^{HH} X &= {}^{DD}F^{DD}G^{HH}X - {}^{DD}G^{DD}F^{HH}X \\ &= {}^{DD}F^{HH}(GX) - {}^{DD}G^{HH}(FX) \\ &= {}^{HH}(FGX - GFX) \\ &= {}^{HH}([F, G]X) \\ &= {}^{DD}[F, G]^{HH}X \end{aligned}$$

by virtue of Theorem 3.1. Thus we have $\left[{}^{DD}F, {}^{DD}G\right] = {}^{DD}[F, G]$. □

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