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Some New Estimates for Hermite-Hadamard-Jensen-Mercer Type Inequalities

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Abstract. The Hermite-Hadamard inequality, which is related to convex functions, is a topic of frequent study in the mathematical community. Similarly, Mercer's inequality, particularly in relation to Hermite-Hadamard and Jensen type inequalities, is also connected to convex functions and has recently been the focus of attention from researchers. In this study, new Hermite-Hadamard-Jensen-Mercer type integral inequalities for convex functions were obtained by employing the Power-mean and Hölder inequalities, as well as their more general forms, namely the Improved Power-mean, Hölder-İşcan, and Young inequalities. Additionally, a comparison was conducted between the obtained results.

1. Introduction

The concept of convex functions has become a significant component of mathematical theory and is frequently employed in a multitude of disciplines. In this context, the historical origins of convex functions can be traced back to the end of the 19th century. In 1893, Hadamard's work makes an indirect reference to the foundational and significant nature of such functions, although this is not explicitly stated. (see [1, 2]) Also, although there are results in the literature indicating the prevalence of convex functions, the first comprehensive study of convex functions was carried out by J.L.W.V. Jensen in 1905 and 1906 [3, 4]. It is widely acknowledged that the theory of convex functions has advanced considerably since Jensen's pioneering work. The concept of convexity, which has become a fundamental building block in many fields, is now often discussed as an inseparable whole within the discipline of mathematics. This distinctive concept has attracted the attention of numerous scientists, including Beckenbach and for many researchers, such as Bellman (1961) and Mitrinovic (1970), it has reached a position that they address in their books. (see [5–7]) By including inequalities for convex functions in their respective books, they have left an important legacy to the mathematical community. In 1987, Pecaric was the first to include only inequalities of convex functions. In addition, many scholars, including Roberts and Varberg (1973) and Niculescu and Persson (2005, 2006), have done extensive research on inequalities for convex functions. Integral inequalities are another area of research within this field. (see [8–12, 49–51])

The concept of convex functions has facilitated the discovery of a multitude of significant and advantageous inequalities, that have proven to be invaluable in various applied sciences. The most notable classical definition of a convex function on convex sets in terms of line segments can be considered as follows:

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$$\mathcal{F}(\psi x + (1-\psi)y) \leq \psi\mathcal{F}(x) + (1-\psi)\mathcal{F}(y)$$

where $\mathcal{F} : I \subset \mathbb{R} \to \mathbb{R}$ is a mapping valid for all $x, y \in I$ and $\psi \in [0,1]$. If $-\mathcal{F}$ is convex, then $\mathcal{F}$ is said to be concave.

In recent decades, convex functions have received a lot of attention, with the basic concept being expanded and generalized in a variety of directions. These functions serve a significant role in many fields of analysis and geometry, and their characteristics have been extensively studied. Readers interested in the aforementioned advancements may reference [12–16], which provides a thorough review of these fields.

The Jensen inequality and its associated inequalities are essential and widely recognized for convex functions. This is due to the fact that the Jensen inequality and the related inequalities have applications in many other domains, such as computer problems, probability theory, optimization, and information theory [17, 18]. In current research, many well-known inequalities for convex functions are often employed.

Let $\mathcal{F}$ be a convex function on the given interval $[\alpha_1, \alpha_2]$ and $0 < x_1 \leq x_2 \leq ... \leq x_n$ and let $\psi = (\psi_1, \psi_2, ..., \psi_n)$ be the non-negative weights such that $\sum_{i=1}^{n} \psi_i = 1$. Then, Jensen's inequality [8] in literature states as follows:

$$\mathcal{F}\left(\sum_{i=1}^{n} \psi_i x_i\right) \leq \psi_i \sum_{i=1}^{n} \mathcal{F}(x_i).$$

Jensen's inequality provides a means of recapturing the concept of a convex function when $n = 2$. Jensen's inequality has a wide range of important applications in areas including statistics, finance, economics, and optimization. On the other hand, information theory uses it especially well to predict estimations of the limits of distance functions (see [19–22]).

Both convex analysis and optimisation employ the Jensen-Mercer inequality extensively, which is a significant mathematical inequality derived from Jensen's inequality. The inequality constrains the convex combination of a function over a set of variables, where the weights of the variables form a probability distribution, by providing an upper bound for the convex combination. The Jensen-Mercer inequality has been applied in a number of fields, including statistics, machine learning, and economics. It is frequently employed to establish critical limits and substantiate significant findings across a range of disciplines.

In 2003, Mercer first proved the following variant of Jensen inequality known as the Jensen-Mercer inequality:

Theorem 1.1. *[23] If $\mathcal{F}$ is a convex function on $[\alpha_1, \alpha_2]$, then*

$$\mathcal{F}\left(\alpha_1 + \alpha_2 - \sum_{i=1}^{n} \psi_i x_i\right) \leq \mathcal{F}(\alpha_1) + \mathcal{F}(\alpha_2) - \sum_{i=1}^{n} \psi_i \mathcal{F}(x_i)$$

for all $x_i \in [\alpha_1, \alpha_2]$ and $\psi_i \in [0,1]$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{n} \psi_i = 1$.

The Jensen-Mercer inequality has been extensively studied by a large number of scholars. There have been several approaches taken: expanding its dimension, deriving it for convex operators with all of its numerous refinements, deriving operator variations for super-quadratic functions, upgrading, and carrying out multiple generalizations with information theory implications (see [24–28]).

The results associated with convex functions are of great significance in the field of inequality theory. One of the most well-known inequalities is the Hermite-Hadamard inequality. The theorem presented below illustrates the Hermite-Hadamard inequality, which occupies a prominent position in the literature.

Theorem 1.2. *$\mathcal{F} : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of real numbers and $\alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$, then:*

$$\mathcal{F}\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{F}(x)dx \leq \frac{\mathcal{F}(\alpha_1) + \mathcal{F}(\alpha_2)}{2}. \tag{1}$$

If $\mathcal{F}$ is a concave function, the inequality direction is reversed. The Hermite-Hadamard inequality provides estimates of both lower and upper bounds for the integral mean of any convex function defined on a bounded and closed interval, including the midpoint and endpoint of the domain of the function. The Hermite-Hadamard inequality has been the subject of extensive research, with numerous generalizations and developments emerging in recent years. The significance of convexity in mathematical analysis has resulted in the inequality receiving considerable attention, with a multitude of generalizations and enhancements being put forth. For further details, please refer to the literature cited in [29–37].

In [38], Kian and Moslehian used the Jensen-Mercer inequality in 2013 to obtain the following Hermite-Hadamard-Mercer type inequalities:

Theorem 1.3. *Let $\mathcal{F}$ be a convex function on $[\alpha_1, \alpha_2]$. Then*

$$\begin{aligned}\mathcal{F}\left(\alpha_1+\alpha_2-\frac{\gamma_1+\gamma_2}{2}\right) &\leq \mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)-\int_0^1 \mathcal{F}(\psi\gamma_1+(1-\psi)\gamma_2)d\psi \\ &\leq \mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)-\mathcal{F}\left(\frac{\gamma_1+\gamma_2}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}\left(\alpha_1+\alpha_2-\frac{\gamma_1+\gamma_2}{2}\right) &\leq \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2} \mathcal{F}(\alpha_1+\alpha_2-\psi)d\psi \\ &\leq \mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)-\frac{\mathcal{F}(\gamma_1)+\mathcal{F}(\gamma_2)}{2}\end{aligned}$$

for all $\gamma_1, \gamma_2 \in [\alpha_1, \alpha_2]$.

Those engaged in research involving diverse integral and convex function applications have a distinctive opportunity to leverage the Hermite-Hadamard-Mercer inequalities. This phenomenon has piqued the interest of experts from a diverse range of academic disciplines. The inequality has been the subject of considerable interest and has inspired a number of new ideas that have been adopted and used in a variety of academic fields, including graph theory, optimisation and economics. The Hermite-Hadamard-Mercer inequality constitutes a significant area of study within the mathematical sciences. (see [39–42])

2. Auxiliary Results

Firstly, it is necessary to establish the following results, which will play an important role in obtaining the main results of the article.

Definition 2.1. *(Beta Function) The Beta function denoted by $\beta(\varkappa_1, \varkappa_2)$ is defined by*

$$\beta(\varkappa_1,\varkappa_2)=\int_0^1 \psi^{\varkappa_1-1}(1-\psi)^{\varkappa_2-1}d\psi, \quad \varkappa_1,\varkappa_2>0.$$

Corollary 2.2. *Beta function provides the following properties:*

1. $\beta(\varkappa_1,\varkappa_2)=\beta(\varkappa_2,\varkappa_1)$
2. $\beta(\varkappa_1,\varkappa_2+1)=\frac{\varkappa_2}{\varkappa_1+\varkappa_2}\beta(\varkappa_1,\varkappa_2)$
3. $\beta(\varkappa_1,\varkappa_2)=\frac{\Gamma(\varkappa_1)\Gamma(\varkappa_2)}{\Gamma(\varkappa_1+\varkappa_2)}$.

Hölder's inequality is a fundamental inequality between integrals and an indispensable tool for the study of L^p spaces. Many new generalizations and refinements have been obtained in the theory of inequalities using different convex functions and this inequality. However, in [43], İşcan proved a new form of the Hölder inequality using a simple method. Using the Hölder-İşcan inequality, better upper bounds are obtained than in previous studies. The Hölder inequality and its new form are as follows:

Theorem 2.3. *(Hölder Inequality for Integrals) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{F}$ and g are real functions defined on $[\alpha_1, \alpha_2]$ such that $|\mathcal{F}|^p$ and $|\mathcal{G}|^q$ are integrable functions on $[\alpha_1, \alpha_2]$, then:*

$$\int_{\alpha_1}^{\alpha_2} |\mathcal{F}(\zeta)\mathcal{G}(\zeta)|\, d\zeta \leq \left(\int_{\alpha_1}^{\alpha_2} |\mathcal{F}(\zeta)|^p\, d\zeta\right)^{\frac{1}{p}} \left(\int_{\alpha_1}^{\alpha_2} |\mathcal{G}(\zeta)|^q\, d\zeta\right)^{\frac{1}{q}}.$$

Theorem 2.4. *(Hölder-İşcan Inequality for Integrals) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{F}$ and g are real functions defined on interval $[\alpha_1, \alpha_2]$ and if $|\mathcal{F}|^p$and $|\mathcal{G}|^q$ are integrable functions on $[\alpha_1, \alpha_2]$ then:*

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} |\mathcal{F}(\zeta)\mathcal{G}(\zeta)|\, d\zeta \quad \leq \quad & \frac{1}{\alpha_2 - \alpha_1}\left\{\left(\int_{\alpha_1}^{\alpha_2} (\alpha_2 - \zeta)\, |\mathcal{F}(\zeta)|^p\, d\zeta\right)^{\frac{1}{p}} \left(\int_{\alpha_1}^{\alpha_2} (\alpha_2 - \zeta)\, |\mathcal{G}(\zeta)|^q\, d\zeta\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_{\alpha_1}^{\alpha_2} (\zeta - \alpha_1)\, |\mathcal{F}(\zeta)|^p\, d\zeta\right)^{\frac{1}{p}} \left(\int_{\alpha_1}^{\alpha_2} (\zeta - \alpha_1)\, |\mathcal{G}(\zeta)|^q\, d\zeta\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

The power-mean integral inequality, which is a different version of the Hölder integral inequality, plays an important role in many branches of mathematical analysis, particularly convex analysis. In [44], Kadakal *et al.* demonstrated and validated an improved power-mean integral inequality, which produces more accurate results than the original inequality. The power-mean inequality and the novel generalised expression are presented as follows:

Theorem 2.5. *(Power-mean Inequality for Integrals) Let $q \geq 1$. If $\mathcal{F}$ and g are real functions defined on $[\alpha_1, \alpha_2]$ such that $|\mathcal{F}|$ and $|\mathcal{G}|^q$ are integrable functions on $[\alpha_1, \alpha_2]$, then:*

$$\int_{\alpha_1}^{\alpha_2} |\mathcal{F}(\zeta)\mathcal{G}(\zeta)|\, d\zeta \leq \left(\int_{\alpha_1}^{\alpha_2} |\mathcal{F}(\zeta)|\, d\zeta\right)^{1-\frac{1}{q}} \left(\int_{\alpha_1}^{\alpha_2} |\mathcal{F}(\zeta)|\, |\mathcal{G}(\zeta)|^q\, d\zeta\right)^{\frac{1}{q}}.$$

Theorem 2.6. *(Improved Power-mean Inequality for Integrals) Let $q \geq 1$. If $\mathcal{F}$ and g are real functions defined on $[\alpha_1, \alpha_2]$ such that $|\mathcal{F}|$ and $|\mathcal{F}||\mathcal{G}|^q$ are integrable functions on $[\alpha_1, \alpha_2]$, then:*

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} |\mathcal{F}(\zeta)\mathcal{G}(\zeta)|\, d\zeta \quad \leq \quad & \frac{1}{\alpha_2 - \alpha_1}\left\{\left(\int_{\alpha_1}^{\alpha_2} (\alpha_2 - \zeta)\, |\mathcal{F}(\zeta)|\, d\zeta\right)^{1-\frac{1}{q}} \left(\int_{\alpha_1}^{\alpha_2} (\alpha_2 - \zeta)\, |\mathcal{F}(\zeta)|\, |\mathcal{G}(\zeta)|^q\, d\zeta\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_{\alpha_1}^{\alpha_2} (\zeta - \alpha_1)\, |\mathcal{F}(\zeta)|\, d\zeta\right)^{1-\frac{1}{q}} \left(\int_{\alpha_1}^{\alpha_2} (\zeta - \alpha_1)\, |\mathcal{F}(\zeta)|\, |\mathcal{G}(\zeta)|^q\, d\zeta\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

The famous Young inequality is defined as follows:

Theorem 2.7. *[45] Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\alpha_1\alpha_2 \leq \frac{1}{p}\alpha_1^p + \frac{1}{q}\alpha_2^q \tag{2}$$

where α_1 and α_2 are nonnegative reel numbers. The reversed version of inequality (2) reads

$$\alpha_1\alpha_2 \geq \frac{1}{p}\alpha_1^p + \frac{1}{q}\alpha_2^q, \quad \alpha_1, \alpha_2 > 0, \quad 0 < p < 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

The celebrated Hölder inequality, regarded as one of the most pivotal inequalities in analysis, was exemplified in this manner through the utilization of inequality (2). It constitutes a pivotal contribution to numerous domains of both applied and pure mathematics and is indispensable in addressing numerous issues in the social, cultural, and natural sciences.

The most prevalent formulation of Young's inequality, which is frequently employed to illustrate the celebrated inequality for L_p functions, is as follows:

$$\alpha_1^{\psi}\alpha_2^{1-\psi} \leq \psi\alpha_1 + (1-\psi)\alpha_2,$$

where $\alpha_1, \alpha_2 > 0$ and $0 \leq \psi \leq 1$.

The Hermite-Hadamard inequality and the Mercer inequality, which pertain to convex functions, continue to be a topic of significant interest within the mathematical community, having recently attracted considerable research attention. The aim of this study was to derive novel Hermite-Hadamard-Jensen-Mercer type integral inequalities for convex functions by employing the Power-mean, Hölder and and Young inequalities, as well as their more extended variants, namely the Improved Power-mean, Hölder-İşcan. Additionally, a comparison was conducted between the obtained results. In conclusion, the methodology described in this article is expected to stimulate further research in this area.

3. Main Results

We start by establishing novel auxiliary identity for differentiable functions, which can be used to derive future advancements.

Lemma 3.1. *Let $\mathcal{F} : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a differentiable mapping on (α_1, α_2) with $\alpha_1 \leq \alpha_2$. If $\mathcal{F}'' \in L[\alpha_1, \alpha_2]$, then the following equality for integrals holds:*

$$\begin{aligned} & \frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1} \mathcal{F}(u)du \\ = & \frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1 \psi(1-\psi)\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))d\psi \end{aligned} \tag{3}$$

for all $\gamma_1, \gamma_2 \in [\alpha_1, \alpha_2]$.

Proof. In order to demonstrate equality (3), it is necessary to apply twice the partial integration method to the right-hand side of the equation. Namely,

$$\begin{aligned} & \quad (4) \\ = & \int_0^1 \psi(1-\psi)\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))d\psi \\ = & \psi(1-\psi)\frac{\mathcal{F}'(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))}{\gamma_1-\gamma_2}\Bigg|_0^1 \\ & -\int_0^1 (1-2\psi)\frac{\mathcal{F}'(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))}{\gamma_1-\gamma_2}d\psi \\ = & \frac{1}{\gamma_2-\gamma_1}\left[(1-2\psi)\frac{\mathcal{F}(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))}{\gamma_1-\gamma_2}\Bigg|_0^1\right. \\ & \left. -\frac{2}{\gamma_2-\gamma_1}\int_0^1 \mathcal{F}(\alpha_1+\alpha_2-(\psi\gamma_1+(1-\psi)\gamma_2))d\psi\right] \\ = & \frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{(\gamma_2-\gamma_1)^2} \\ & -\frac{2}{(\gamma_2-\gamma_1)^2}\int_0^1 \mathcal{F}(\alpha_1+\alpha_2-(\psi\gamma_1+(1-\psi)\gamma_2))d\psi. \end{aligned}$$

Using the change of the variable $u = \alpha_1 + \alpha_2 - (\psi\gamma_1 + (1-\psi)\gamma_2)$ for $\psi \in [0,1]$ and multiplying the both sides (4) by $\frac{\gamma_2-\gamma_1}{2}$, we obtain

$$\begin{aligned} & \frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du \\ = & \frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1 \psi(1-\psi)\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1)d\psi. \end{aligned}$$

Thus, we get the required identity. □

Remark 3.2. *If we take $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$ in Lemma 3.1, then the equality (3) reduces to the equality*

$$\frac{\mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)}{2} - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{F}(u)du = \frac{(\alpha_2-\alpha_1)^2}{2}\int_0^1 \psi(1-\psi)\mathcal{F}''(\psi\alpha_1+(1-\psi)\alpha_2)d\psi$$

which is proved by Alomari and Darus in [46].

Theorem 3.3. *Let $\mathcal{F} : [\alpha_1,\alpha_2] \to \mathbb{R}$ be a differentiable mapping on (α_1,α_2) with $\alpha_1 \leq \alpha_2$. If $|\mathcal{F}''|$ is a convex function on $[\alpha_1,\alpha_2]$, $\gamma_1,\gamma_2 \in [\alpha_1,\alpha_2]$ and $\psi \in [0,1]$, then the following inequality for integrals holds:*

$$\begin{aligned} & \left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\ \leq & \frac{(\gamma_2-\gamma_1)^2}{12}\left(|\mathcal{F}''(\alpha_1)|+|\mathcal{F}''(\alpha_2)| - \frac{\left|\mathcal{F}''(\gamma_1)\right|+\left|\mathcal{F}''(\gamma_2)\right|}{2}\right). \end{aligned} \tag{5}$$

Proof. Taking absolute values on both sides of Lemma 3.1 and using Jensen-Mercer inequality, we have

$$\begin{aligned} & \left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\ \leq & \frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1 \left|\psi(1-\psi)\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1)\right|d\psi \\ \leq & \frac{\gamma_2-\gamma_1}{2}\int_0^1 \left|\psi(1-\psi)\right|\left[|\mathcal{F}''(\alpha_1)|+|\mathcal{F}''(\alpha_2)| - \left(\psi\left|\mathcal{F}''(\gamma_1)\right| + (1-\psi)\left|\mathcal{F}''(\gamma_2)\right|\right)\right]d\psi \\ = & \frac{(\gamma_2-\gamma_1)^2}{12}\left(|\mathcal{F}''(\alpha_1)|+|\mathcal{F}''(\alpha_2)| - \frac{\left|\mathcal{F}''(\gamma_1)\right|+\left|\mathcal{F}''(\gamma_2)\right|}{2}\right). \end{aligned}$$

This completes the proof. □

Remark 3.4. *If we choose $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$ in Theorem 3.3, then inequality (5) reduces to Proposition 2 proved by Sarıkaya and Aktan in [47].*

Theorem 3.5. *Let $\mathcal{F} : [\alpha_1,\alpha_2] \to \mathbb{R}$ be a differentiable mapping on (α_1,α_2) such $\mathcal{F} \in L[\alpha_1,\alpha_2]$ with $\alpha_1 < \alpha_2$. If $|\mathcal{F}''|^q$ is a convex function on $[\alpha_1,\alpha_2]$, for $p > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned} & \left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\ \leq & \frac{(\gamma_2-\gamma_1)^2}{2}\left[\beta(p+1,p+1)\right]^{\frac{1}{p}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q - \frac{\left|\mathcal{F}''(\gamma_1)\right|^q+\left|\mathcal{F}''(\gamma_2)\right|^q}{2}\right)^{\frac{1}{q}}. \end{aligned} \tag{6}$$

$\beta(.,.)$ is the Beta function.

Proof. From Lemma 3.1 and using Hölder inequality, we get

$$\begin{aligned} &\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\ \leq\ & \frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1\left|\psi(1-\psi)\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|d\psi \\ \leq\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left(\int_0^1\left|\psi(1-\psi)\right|^p d\psi\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|^q\right)^{\frac{1}{q}}d\psi \end{aligned}$$

Utilizing Jensen-Mercer inequality because of the convexity of $|\mathcal{F}''|^q$ on $[\alpha_1,\alpha_2]$, we obtain

$$\begin{aligned} &\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\ \leq\ & \frac{(\gamma_2-\gamma_1)^2}{2} \\ &\times\left\{\left(\int_0^1\left|\psi(1-\psi)\right|^p d\psi\right)^{\frac{1}{p}}\left(\int_0^1\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\left(\psi\left|\mathcal{F}''(\gamma_1)\right|^q+(1-\psi)\left|\mathcal{F}''(\gamma_2)\right|^q\right)\right)d\psi\right)^{\frac{1}{q}}\right\} \\ =\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q+\left|\mathcal{F}''(\gamma_2)\right|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

and

$$\int_0^1\left|\psi(1-\psi)\right|^p d\psi=\int_0^1\psi^p\left(1-\psi\right)^p d\psi=\beta(p+1,p+1),\quad \psi\in[0,1]$$

the proof is completed by simple integral calculation as above. □

Remark 3.6. *If we choose $\gamma_1=\alpha_1$ and $\gamma_2=\alpha_2$ in Theorem 3.5, then inequality (6) reduces to Corollary 2 proved by Özdemir et al. in [48]*

$$\begin{aligned} &\left|\frac{\mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{F}(u)du\right| \\ \leq\ & \frac{(\alpha_2-\alpha_1)^2}{2}\left[\beta(p+1,p+1)\right]^{\frac{1}{p}}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

with

$$\begin{aligned} \beta(p+1,p+1) &= 2^{1-2(p+1)}\beta\left(\frac{1}{2},p+1\right) \\ &= 2^{1-2(p+1)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)} \end{aligned}$$

and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ *where* $\sqrt{\pi}<2$.

Theorem 3.7. *Let $\mathcal{F} : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a differentiable mapping on (α_1, α_2) such $\mathcal{F}'' \in L[\alpha_1, \alpha_2]$ with $\alpha_1 < \alpha_2$. If $|\mathcal{F}''|^q$ is a convex function on $[\alpha_1, \alpha_2]$, for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \tag{7}$$
$$\leq \frac{(\gamma_2-\gamma_1)^2}{2}\left(\beta(p+1,p+2)\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2} - \frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{6} - \frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2} - \frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{3} - \frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{6}\right)^{\frac{1}{q}}\right\}.$$

Proof. From Lemma 3.1 and using Hölder-İşcan inequality, we can write

$$\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right|$$
$$\leq \frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1 \left|\psi(1-\psi)\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1)\right|d\psi$$
$$\leq \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\left(\int_0^1(1-\psi)\left|\psi(1-\psi)\right|^p d\psi\right)^{\frac{1}{p}}\left(\int_0^1(1-\psi)\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\psi\left|\psi(1-\psi)\right|^p d\psi\right)^{\frac{1}{p}}\left(\int_0^1\psi\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|^q\right)^{\frac{1}{q}}\right\}.$$

From the Jensen-Mercer inequality, we have

$$\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2} - \frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right|$$
$$\leq \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\left(\int_0^1(1-\psi)\left|\psi(1-\psi)\right|^p d\psi\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^1(1-\psi)\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\left(\psi\left|\mathcal{F}''(\gamma_2)\right|^q+(1-\psi)\left|\mathcal{F}''(\gamma_1)\right|^q\right)\right)d\psi\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\psi\left|\psi(1-\psi)\right|^p d\psi\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_0^1\psi\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-(\psi\left|\mathcal{F}''(\gamma_2)\right|^q+(1-\psi)\left|\mathcal{F}''(\gamma_1)\right|^q)\right)d\psi\right)^{\frac{1}{q}}\right\}$$
$$= \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\left(\beta(p+1,p+2)\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2} - \frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{6} - \frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\beta(p+2,p+1)\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2} - \frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{3} - \frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{6}\right)^{\frac{1}{q}}\right\}.$$

From the property of the beta function, $\beta(\gamma_1,\gamma_2)=\beta(\gamma_2,\gamma_1)$,

$$\beta(p+1,p+2)=\beta(p+2,p+1),$$

Thus, The proof is completed. □

Remark 3.8. *If we choose $\gamma_1=\alpha_1$ and $\gamma_2=\alpha_2$ in Theorem 3.7, then inequality (7) reduces to the following the inequality;*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{F}(u)du\right| \\
\leq\ & \frac{(\alpha_2-\alpha_1)^2}{2}\left[\beta(p+1,p+2)\right]^{\frac{1}{p}} \\
&\times\left[\left(\frac{|\mathcal{F}''(\alpha_1)|^q+2|\mathcal{F}''(\alpha_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 3.9. *Inequality (7) is better than inequality (6). Indeed , since the function $g:[0,\infty)\to\mathbb{R}$, $g(\varepsilon)=\epsilon^r$, $r\in[0,1]$ is concave, we can write*

$$\frac{\omega_1^r+\omega_2^r}{2}=\frac{g(\omega_1)+g(\omega_2)}{2}\leq g\left(\frac{\omega_1+\omega_2}{2}\right)=\frac{\omega_1^r+\omega_2^r}{2} \tag{8}$$

for all $\omega_1,\omega_2\geq 0$. In inequality (8), if we choose

$$\begin{aligned}
\omega_1 &= \frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}-\frac{|\mathcal{F}''(\gamma_2)|^q}{6}-\frac{|\mathcal{F}''(\gamma_1)|^q}{3}, \\
\omega_2 &= \frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}-\frac{|\mathcal{F}''(\gamma_2)|^q}{3}-\frac{|\mathcal{F}''(\gamma_1)|^q}{6}
\end{aligned}$$

and $r=\frac{1}{q}$, then we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}-\frac{|\mathcal{F}''(\gamma_2)|^q}{6}-\frac{|\mathcal{F}''(\gamma_1)|^q}{3}\right)^{\frac{1}{q}} \\
&+\frac{1}{2}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}-\frac{|\mathcal{F}''(\gamma_2)|^q}{3}-\frac{|\mathcal{F}''(\gamma_1)|^q}{6}\right)^{\frac{1}{q}} \\
\leq\ & \left(\frac{1}{2}\right)^{\frac{1}{q}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{|\mathcal{F}''(\gamma_2)|^q+|\mathcal{F}''(\gamma_1)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, using the property of Beta functions, the most regular form of the Theorem 3.7 is as follows.

$$
\begin{aligned}
&\frac{(\gamma_2-\gamma_1)^2}{2}\left(\beta(p+1,p+2)\right)^{\frac{1}{p}}\\
&\times\left\{\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{3}\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{3}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{6}\right)^{\frac{1}{q}}\right\}\\
\leq\ &(\gamma_2-\gamma_1)^2\left(\beta(p+1,p+2)\right)^{\frac{1}{p}}\left\{\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{2}\right)^{\frac{1}{q}}\right\}\\
=\ &\frac{(\gamma_2-\gamma_1)^2}{2^{\frac{1}{q}}}\left(\beta(p+1,p+2)\right)^{\frac{1}{p}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Also, from the following property, we can write

$$\beta(p+1,p+2)=\frac{\beta(p+1,p+1)}{2}$$

and we obtain

$$
\begin{aligned}
&\frac{(\gamma_2-\gamma_1)^2}{2^{\frac{1}{q}}}\left(\beta(p+1,p+2)\right)^{\frac{1}{p}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{2}\right)^{\frac{1}{q}}\\
=\ &\frac{(\gamma_2-\gamma_1)^2}{2}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Therefore, the right-hand side of the inequality (6) is established. This completes the proof.

Theorem 3.10. *Let $\mathcal{F}:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a positive mapping on (α_1,α_2) such $\mathcal{F}''\in L[\alpha_1,\alpha_2]$. If $|\mathcal{F}''|^q$ is a convex function on $[\alpha_1,\alpha_2]$, for $q\geq 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \qquad (9)\\
\leq\ &\frac{(\gamma_2-\gamma_1)^2}{12}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q+\left|\mathcal{F}''(\gamma_2)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Suppose that $q\geq 1$. From Lemma 3.1 and using the power-mean inequality, we have

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right|\\
\leq\ &\frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1\left|\psi(1-\psi)\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|d\psi\\
\leq\ &\frac{(\gamma_2-\gamma_1)^2}{2}\left(\int_0^1\left|\psi(1-\psi)\right|d\psi\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi(1-\psi)\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|^q\right)^{\frac{1}{q}}d\psi.
\end{aligned}
$$

Using the Jensen-Mercer inequality because of the convexity of $|\mathcal{F}''|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\
\leq\;& \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\left(\int_0^1 \left|\psi(1-\psi)\right| d\psi\right)^{\frac{1}{p}}\right. \\
&\left.\times\left(\int_0^1 \left|\psi(1-\psi)\right|\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\left(\psi\left|\mathcal{F}''(\gamma_1)\right|^q+(1-\psi)\left|\mathcal{F}''(\gamma_2)\right|^q\right)\right)d\psi\right)^{\frac{1}{q}}\right\} \\
=\;& \frac{(\gamma_2-\gamma_1)^2}{12}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q+\left|\mathcal{F}''(\gamma_2)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Remark 3.11. *If we choose $\gamma_1=\alpha_1$ and $\gamma_2=\alpha_2$ in Theorem 3.10, then we obtain*

$$\left|\frac{\mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{F}(u)du\right|\leq\frac{(\alpha_2-\alpha_1)^2}{12}\left[\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}\right]^{\frac{1}{q}}.$$

Theorem 3.12. *Let $\mathcal{F}:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable function on (α_1,α_2) with $\alpha_1<\alpha_2$ and $\mathcal{F}''\in L[\alpha_1,\alpha_2]$ .If $|\mathcal{F}''|^q$ is a convex function on $[\alpha_1,\alpha_2]$, for $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \qquad (10)\\
\leq\;& \frac{(\gamma_2-\gamma_1)^2}{4}\left(\frac{1}{6}\right)^{\frac{1}{p}}\left\{\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{15}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{10}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{10}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{15}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Suppose that $q\geq 1$. From Lemma 3.1 and using the improved power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\
\leq\;& \frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1\left|\psi(1-\psi)\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1)\right|d\psi \\
\leq\;& \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\left(\int_0^1(1-\psi)\left|\psi-\psi^2\right|d\psi\right)^{\frac{1}{p}}\right. \\
&\times\left(\int_0^1(1-\psi)\left|\psi-\psi^2\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|^q d\psi\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1\psi\left|\psi-\psi^2\right|d\psi\right)^{\frac{1}{p}}\left(\int_0^1\psi\left|\psi-\psi^2\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_2+(1-\psi)\gamma_1))\right|^q d\psi\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Similarly, utilizing Jensen-Mercer inequality, we have

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\
\leq\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\left(\int_0^1(1-\psi)\left|\psi-\psi^2\right|d\psi\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\int_0^1(1-\psi)\left|\psi-\psi^2\right|\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\left(\psi\left|\mathcal{F}''(\gamma_1)\right|^q+(1-\psi)\left|\mathcal{F}''(\gamma_2)\right|^q\right)\right)d\psi\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\psi\left|\psi-\psi^2\right|d\psi\right)^{\frac{1}{p}} \\
&\quad\left.\times\left(\int_0^1\psi\left|\psi-\psi^2\right|\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\left(\psi\left|\mathcal{F}''(\gamma_1)\right|^q+(1-\psi)\left|\mathcal{F}''(\gamma_2)\right|^q\right)\right)d\psi\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\left(\frac{1}{12}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{12}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{30}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{20}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{12}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{12}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{20}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{30}\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(\gamma_2-\gamma_1)^2}{4}\left(\frac{1}{6}\right)^{\frac{1}{p}}\left\{\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{15}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{10}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{10}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{15}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

Remark 3.13. *Inequality (10) is better than inequality (9). Indeed, using the inequlity (8) in Remark 3.9, we have*

$$\begin{aligned}
\omega_1 &= \frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{15}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{10}, \\
\omega_2 &= \frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{10}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{15}
\end{aligned}$$

and for $r=\frac{1}{q}$, then we can write

$$\begin{aligned} &\frac{1}{2}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{15}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{10}\right)^{\frac{1}{q}} \\ &+\frac{1}{2}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{10}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{15}\right)^{\frac{1}{q}} \\ \leq\ & \left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{12}\right)^{\frac{1}{q}} \\ =\ & \left(\frac{1}{6}\right)^{\frac{1}{q}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{2}\right)^{\frac{1}{q}}. \end{aligned}$$

Thus, we obtain the following inequality:

$$\begin{aligned} &\frac{(\gamma_2-\gamma_1)^2}{4}\left(\frac{1}{6}\right)^{\frac{1}{p}}\left\{\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{15}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{10}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{6}-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q}{10}-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q}{15}\right)^{\frac{1}{q}}\right\} \\ \leq\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left(\frac{1}{6}\right)^{\frac{1}{p}}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{2}\right)^{\frac{1}{q}} \\ =\ & \frac{(\gamma_2-\gamma_1)^2}{12}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_2)\right|^q+\left|\mathcal{F}''(\gamma_1)\right|^q}{2}\right)^{\frac{1}{q}}. \end{aligned}$$

Therefore, the right-hand side of the inequality (9) is established.

Theorem 3.14. *Let $\mathcal{F}:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a differentiable mapping on (α_1,α_2) such that $\mathcal{F}\in L[\alpha_1,\alpha_2]$ and $\frac{1}{p}+\frac{1}{q}=1$. If $|\mathcal{F}''|^q$ is a convex function, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\ \leq\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\frac{\beta(p+1,p+1)}{p}+\frac{1}{q}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q+\left|\mathcal{F}''(\gamma_2)\right|^q}{2}\right)\right\}. \end{aligned} \tag{11}$$

Proof. From Lemma 3.1 and using Young inequality, we get

$$\begin{aligned} &\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\ \leq\ & \frac{(\gamma_2-\gamma_1)^2}{2}\int_0^1\left|\psi(1-\psi)\right|\left|\mathcal{F}''(\alpha_1+\alpha_2-(\psi\gamma_1+(1-\psi)\gamma_2)\right|d\psi \\ \leq\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left(\int_0^1\frac{1}{p}\left|\psi(1-\psi)\right|^p d\psi+\int_0^1\frac{1}{q}\left|\mathcal{F}''\left(\alpha_1+\alpha_2-(\psi\gamma_1+(1-\psi)\gamma_2)\right|^q d\psi\right). \end{aligned}$$

Using the Jensen-Mercer inequality because of the convexity of $|\mathcal{F}''|^q$ on $[\alpha_1, \alpha_2]$, we get

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1+\alpha_2-\gamma_1)+\mathcal{F}(\alpha_1+\alpha_2-\gamma_2)}{2}-\frac{1}{\gamma_2-\gamma_1}\int_{\alpha_1+\alpha_2-\gamma_2}^{\alpha_1+\alpha_2-\gamma_1}\mathcal{F}(u)du\right| \\
\leq\ & \frac{(\gamma_2-\gamma_1)^2}{2} \\
&\times\left(\int_0^1 \frac{1}{p}\left|\psi(1-\psi)\right|^p d\psi+\int_0^1 \frac{1}{q}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\left(\psi\left|\mathcal{F}''(\gamma_1)\right|^q+(1-\psi)\left|\mathcal{F}''(\gamma_2)\right|^q\right)\right)d\psi\right) \\
=\ & \frac{(\gamma_2-\gamma_1)^2}{2}\left\{\frac{\beta(p+1,p+1)}{p}+\frac{1}{q}\left(|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q-\frac{\left|\mathcal{F}''(\gamma_1)\right|^q+\left|\mathcal{F}''(\gamma_2)\right|^q}{2}\right)\right\}.
\end{aligned}$$

This completes the proof. □

Corollary 3.15. *If we choose $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$ in Theorem 3.14, then we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\alpha_1)+\mathcal{F}(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{F}(u)du\right| \\
\leq\ & \frac{(\alpha_2-\alpha_1)^2}{2}\left\{\frac{\beta(p+1,p+1)}{p}+\frac{1}{q}\left(\frac{|\mathcal{F}''(\alpha_1)|^q+|\mathcal{F}''(\alpha_2)|^q}{2}\right)\right\}.
\end{aligned}$$

4. Conclusions and Future Research Work

The concept of a convex function is defined in terms of an inequality, and as a result, it is related to a number of significant inequalities. Of these, the Hermite-Hadamard and Mercer inequalities are particularly noteworthy, having been the subject of extensive study within the academic literature. This work derived novel Hermite-Hadamard-Jensen-Mercer type integral inequalities for convex functions utilizing the Power-mean and Hölder inequalities, along with their more generalized variants, namely the Improved Power-mean, Hölder-İşcan, and Young inequalities. Additionally, comparisons were conducted between the acquired results. In these comparisons, Theorem 3.7 yields a better result compared to Theorem 3.5. Similarly, Theorem 3.12 yields a better result compared to Theorem 3.10. Furthermore, researchers can obtain a new lemma for second-sense differentiable functions and discover new results for other convex functions. It is our hope that this article will inspire new and interesting research in this field.

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