**TURKISH JOURNAL OF SCIENCE**
**VOLUME 10, ISSUE 1, 22–34**
**ISSN: 2587–0971**

https://www.tjoscience.com

# Integral Inequalities Involving Functions whose Partial Derivatives are Exponentially Convex on the Co-ordinates

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**Abstract.** In this paper, we establish several novel integral inequalities for functions whose partial derivatives are exponentially convex with respect to each coordinate. To begin with, a new integral identity is derived, which serves as a foundational tool for generating further inequalities. Utilizing well-known classical inequalities, such as Hölder's and Young's inequalities, we derive a range of new upper bounds and estimates. Furthermore, various special cases and corollaries of the main results are discussed to highlight the applicability and generality of the obtained inequalities. These findings contribute to the growing body of research on convexity-based analysis and may have potential implications in related areas of mathematical inequalities and applied analysis.

## 1. Introduction

The concept of convexity plays a fundamental role in the theory of inequalities and has been extensively studied and utilized by numerous researchers, particularly in the development and analysis of various inequality results. Convex functions form the foundation of many classical and modern inequalities due to their rich structural properties and wide applicability in both pure and applied mathematics. A formal definition of convex functions can be found in [1].

**Definition 1.1.** *(See [1]) Let $I$ be an interval in $\mathbb{R}$. Then, $F : I \to \mathbb{R}$ is said to be convex, if*

$$F(\zeta\kappa_1 + (1-\zeta)\kappa_2) \leq \zeta F(\kappa_1) + (1-\zeta) F(\kappa_2)$$

*holds for all $\kappa_1, \kappa_2 \in I$ and $\zeta \in [0,1]$.*

A fundamental motivation underlying the extensive study of generalized convexity concepts lies in the pursuit of refining analytical bounds and extending the applicability of classical inequalities to broader functional frameworks. In this context, a particularly noteworthy and analytically rich subclass of convex functions—termed exponentially convex functions—has emerged, offering enhanced flexibility in various inequality formulations. The formal definition of this class is presented below.

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Received: 21 January 2025; Accepted:8 March 2025; Published: 30 April 2025
*Keywords*. (Co-ordinates, Exponentially convex functions, Hölder inequality.)
2010 *Mathematics Subject Classification*. 26D15, 26A51
*Cited this article as:* Aslan, S. (2025). Integral Inequalities Involving Functions whose Partial Derivatives are Exponentially Convex on the Co-ordinates. Turkish Journal of Science, 10(1), 22-34.

**Definition 1.2.** *(See [2]) A function $F : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be exponentially convex function, if*

$$F\left((1-\zeta)\,\kappa_1+\zeta\kappa_2\right)\leq(1-\zeta)\frac{F(\kappa_1)}{e^{\alpha\kappa_2}}+\zeta\frac{F(\kappa_2)}{e^{\alpha\kappa_2}}$$

*for all $\kappa_1,\kappa_2\in I,\alpha\in\mathbb{R}$ and $\zeta\in[0,1]$.*

As part of their contribution to the theory of generalized convexity, Aslan and Akdemir introduced the notion of exponentially convex functions defined on the co-ordinates. The precise formulation of this class is given below.

**Definition 1.3.** *(See [3]) Let us consider the bidimensional interval $\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$ in $R^2$ with $\kappa_1<\kappa_2$ and $\kappa_3<\kappa_4$. The mapping $F:\Delta\longrightarrow R$ is exponentially convex on the co-ordinates on $\Delta$, if the following inequality holds,*

$$F\left(\zeta\kappa_1+(1-\zeta)\,\kappa_3,\zeta\kappa_2+(1-\zeta)\,\kappa_4\right)\leq\zeta\frac{F(\kappa_1,\kappa_2)}{e^{\alpha(\kappa_1+\kappa_2)}}+(1-\zeta)\frac{F(\kappa_3,\kappa_4)}{e^{\alpha(\kappa_3+\kappa_4)}}$$

*for all $(\kappa_1,\kappa_2),(\kappa_3,\kappa_4)\in\Delta,\alpha\in R$,and $\zeta\in[0,1]$.*

Building upon the previously stated definition of exponentially convex functions, Aslan and Akdemir established an equivalent characterization applicable in the setting of co-ordinated convexity, which is outlined as follows.

**Definition 1.4.** *(see [3]) The mapping $F:\Delta\longrightarrow\mathbb{R}$ is exponentially convex on the co-ordinates on $\Delta$, if the following inequality holds,*

$$\begin{aligned}&F\left(\zeta\kappa_1+(1-\zeta)\,\kappa_2,\xi\kappa_3+(1-\xi)\,\kappa_4\right)\\ \leq\;&\zeta\xi\frac{F(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}+\zeta(1-\xi)\frac{F(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}+(1-\zeta)\xi\frac{F(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}+(1-\zeta)(1-\xi)\frac{F(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}\end{aligned}$$

*for all $(\kappa_1,\kappa_3),(\kappa_1,\kappa_4),(\kappa_2,\kappa_3),(\kappa_2,\kappa_4)\in\Delta,\alpha\in\mathbb{R}$ and $\zeta,\xi\in[0,1]$.*

The formulation of convex functions on the co-ordinates naturally led to the question of whether the classical Hermite–Hadamard inequality could be extended to functions defined on a rectangular domain in $\mathbb{R}^2$. This insightful and motivating question was affirmatively addressed in a seminal work by Dragomir (see [4]), wherein a generalized version of the Hermite–Hadamard inequality for co-ordinated convex functions was established. This result has since become a foundational contribution in the literature, known as the extension of the Hermite–Hadamard inequality from the real line to a rectangular domain in the plane $\mathbb{R}^2$ and is formally stated below.

**Theorem 1.5.** *(see [4]) Suppose that $F:\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]\to\mathbb{R}$ is convex on the co-ordinates on $\Delta$. Then, one has the inequalities;*

$$\begin{aligned}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)&\leq\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(x,y)dxdy\\&\leq\frac{F(\kappa_1,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_2,\kappa_4)}{4}.\end{aligned}$$

*The above inequalities are sharp.*

Aslan and Akdemir have proved the following result for extending Theorem 1.5 to the framework of exponentially convex functions defined on the co-ordinates, thereby enriching the theory of co-ordinated convexity with a broader class of functions.

**Theorem 1.6.** *(See [3]) Let $F:\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]\to\mathbb{R}$ be partial differentiable mapping on $\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$ and $F\in L(\Delta)$, $\alpha\in\mathbb{R}$. If $F$ is exponential convex function on the co-ordinates on $\Delta$, then the following inequality holds;*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(x,y)dxdy\leq\frac{\frac{F(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}+\frac{F(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{F(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{F(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}}{4}.$$

Several recent works have contributed significantly to the growing literature on exponential convex functions and their extensions to the coordinate setting. For detailed discussions on their structural properties, associated inequalities and potential applications, we refer the reader to the works presented in [5]-[17].

## 2. Main Resuls

Throughout this study, $J$ is defined as follows.

$$\begin{aligned} J \;=\; & \frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{72}\left[F(\kappa_1,\kappa_3)+F(\kappa_1,\kappa_4)+F(\kappa_2,\kappa_3)+F(\kappa_2,\kappa_4)\right] \\ & +\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{18}\left[F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right] \\ & -\frac{(\kappa_2-\kappa_1)}{12}\left(\int_{\kappa_3}^{\kappa_4}F(\kappa_1,\tau_2)\,d\tau_2+\int_{\kappa_3}^{\kappa_4}F(\kappa_2,\tau_2)\,d\tau_2\right) \\ & +\frac{(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{18}\left[F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right] \\ & +\frac{2(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}{9}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\frac{(\kappa_2-\kappa_1)}{3}\int_{\kappa_3}^{\kappa_4}F\left(\frac{\kappa_1+\kappa_2}{2},\tau_2\right)d\tau_2 \\ & -\frac{(\kappa_4-\kappa_3)}{12}\left(\int_{\kappa_1}^{\kappa_2}F(\tau_1,\kappa_3)\,d\tau_1+\int_{\kappa_1}^{\kappa_2}F(\tau_1,\kappa_4)\,d\tau_1\right) \\ & -\frac{(\kappa_4-\kappa_3)}{3}\int_{\kappa_1}^{\kappa_2}F\left(\tau_1,\frac{\kappa_3+\kappa_4}{2}\right)d\tau_1+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}F(\tau_1,\tau_2)\,d\tau_1 d\tau_2. \end{aligned}$$

**Lemma 2.1.** *Let $F:\Delta\subseteq R^2\to R$ be a function that is twice partially differentiable on the rectangle $\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$, where $\kappa_1<\kappa_2$ and $\kappa_3<\kappa_4$. If the mixed second-order partial derivative $\frac{\partial^2 F}{\partial\zeta\partial\xi}$ exists and belongs to $L_2(\Delta)$, then the following inequality holds:*

$$\begin{aligned} J= & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32} \\ & \times\left[\int_0^1\int_0^1\left(\zeta-\frac{2}{3}\right)\left(\xi-\frac{2}{3}\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)d\zeta d\xi\right. \\ & +\int_0^1\int_0^1\left(\zeta-\frac{2}{3}\right)\left(\xi-\frac{2}{3}\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)d\zeta d\xi \\ & +\int_0^1\int_0^1\left(\zeta-\frac{2}{3}\right)\left(\frac{2}{3}-\xi\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)d\zeta d\xi \\ & \left.+\int_0^1\int_0^1\left(\frac{2}{3}-\zeta\right)\left(\xi-\frac{2}{3}\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)d\zeta d\xi\right]. \end{aligned}$$

*Proof.* Suppose that

$$\begin{aligned} G_1 &= \int_0^1\int_0^1\left(\zeta-\frac{2}{3}\right)\left(\xi-\frac{2}{3}\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)d\zeta d\xi, \\ G_2 &= \int_0^1\int_0^1\left(\zeta-\frac{2}{3}\right)\left(\xi-\frac{2}{3}\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)d\zeta d\xi, \\ G_3 &= \int_0^1\int_0^1\left(\zeta-\frac{2}{3}\right)\left(\frac{2}{3}-\xi\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)d\zeta d\xi, \\ G_4 &= \int_0^1\int_0^1\left(\frac{2}{3}-\zeta\right)\left(\xi-\frac{2}{3}\right)\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)d\zeta d\xi. \end{aligned}$$

The following equations are obtained using partial integration in double integrals.

$$\begin{aligned}
G_1 &= \frac{4}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F(\kappa_2,\kappa_4)+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\\
&-\frac{8}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}F(\kappa_2,\tau_2)\,d\tau_2+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\\
&+\frac{16}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\frac{16}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}F\left(\frac{\kappa_1+\kappa_2}{2},\tau_2\right)d\tau_2\\
&-\frac{8}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}F(\tau_1,\kappa_4)\,d\tau_1-\frac{16}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}F\left(\tau_1,\frac{\kappa_3+\kappa_4}{2}\right)d\tau_1\\
&+\frac{16}{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}F(\tau_1,\tau_2)\,d\tau_1 d\tau_2,
\end{aligned}$$

$$\begin{aligned}
G_2 &= \frac{4}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F(\kappa_1,\kappa_3)+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\\
&-\frac{8}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}F(\kappa_1,\tau_2)\,d\tau_2+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\\
&+\frac{16}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\frac{16}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}F\left(\frac{\kappa_1+\kappa_2}{2},\tau_2\right)d\tau_2\\
&-\frac{8}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}F(\tau_1,\kappa_3)\,d\tau_1-\frac{16}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}F\left(\tau_1,\frac{\kappa_3+\kappa_4}{2}\right)d\tau_1\\
&+\frac{16}{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}F(\tau_1,\tau_2)\,d\tau_1 d\tau_2,
\end{aligned}$$

$$\begin{aligned}
G_3 &= \frac{4}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F(\kappa_2,\kappa_3)+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\\
&-\frac{8}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}F(\kappa_2,\tau_2)\,d\tau_2+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\\
&+\frac{16}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\frac{16}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}F\left(\frac{\kappa_1+\kappa_2}{2},\tau_2\right)d\tau_2\\
&-\frac{8}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}F(\tau_1,\kappa_3)\,d\tau_1-\frac{16}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}F\left(\tau_1,\frac{\kappa_3+\kappa_4}{2}\right)d\tau_1\\
&+\frac{16}{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}F(\tau_1,\tau_2)\,d\tau_1 d\tau_2,
\end{aligned}$$

$$\begin{aligned}
G_4 = & \frac{4}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F(\kappa_1,\kappa_4)+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\\
& -\frac{8}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}F(\kappa_2,\tau_2)\,d\tau_2+\frac{8}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\\
& +\frac{16}{9(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)}F\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\frac{16}{3(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}F\left(\frac{\kappa_1+\kappa_2}{2},\tau_2\right)d\tau_2\\
& -\frac{8}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}F(\tau_1,\kappa_4)\,d\tau_1-\frac{16}{3(\kappa_4-\kappa_3)(\kappa_2-\kappa_1)^2}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}F\left(\tau_1,\frac{\kappa_3+\kappa_4}{2}\right)d\tau_1\\
& +\frac{16}{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\frac{\kappa_3+\kappa_4}{2}}^{\kappa_4}F(\tau_1,\tau_2)\,d\tau_1 d\tau_2.
\end{aligned}$$

Summing $G_1, G_2, G_3$ and $G_4$ and multiplying both scans by $\frac{(\kappa_2-\kappa_1)^2(\kappa_4-\kappa_3)^2}{32}$ gives the desired result. □

**Theorem 2.2.** *Let $F:\Delta\subseteq R^2\to R$ be partial differentiable mapping on $\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$, where $\kappa_1<\kappa_2$ and $\kappa_3<\kappa_4$, $\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\right|\in L_2(\Delta),\alpha\in R$. If $|\partial F|\in L_2$ is exponentially convex function on the co-ordinates on $\Delta$, then the following inequality holds;*

$$\begin{aligned}
|J| \leq & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2 5^2}{3.2^{12}}\\
& \left(\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right).
\end{aligned}$$

*Proof.* Using the Lemma 2.1 and the absolute value property of integrals, we get;

$$\begin{aligned}
|J| \leq & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32}\\
& \times\left[\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)\right|d\zeta d\xi\right.\\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)\right|d\zeta d\xi\\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\frac{2}{3}-\xi\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)\right|d\zeta d\xi\\
& \left.+\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)\right|d\zeta d\xi\right].
\end{aligned}$$

Using the exponentially convex property of $\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\right|$, then the following inequality is obtained;

$$
\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32} \\
&\times\Bigg[\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\times\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
&+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\Bigg)d\zeta d\xi \\
&+\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\times\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
&+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}\Bigg)d\zeta d\xi \\
&+\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\frac{2}{3}-\xi\right|\times\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
&+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}\Bigg)d\zeta d\xi \\
&+\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|\left|\xi-\frac{2}{3}\right|\times\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
&+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}\Bigg)d\zeta d\xi\Bigg].
\end{aligned}
$$

The necessary algebraic operations are performed and the following result is obtained by integrating separately with respect to $\zeta$ and $\xi$.

$$
\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2 5^2}{3.2^{12}} \\
&\times\left(\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right).
\end{aligned}
$$

□

**Corollary 2.3.** *Under the conditions of Theorem 2.2, if $\alpha = 0$ is taken, the result is the convexity result in the co-ordinates as follows.*

$$
\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2 5^2}{3.2^{12}} \\
&\times\left(\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|+\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|+\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|\right).
\end{aligned}
$$

**Theorem 2.4.** *Let $F : \Delta \subseteq R^2 \to R$ be partial differentiable mapping on $\Delta = [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4]$, where $\kappa_1 < \kappa_2$ and $\kappa_3 < \kappa_4$, $\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\right|^q \in L_2(\Delta), q > 1$ and $\alpha \in R$. If $|\partial F| \in L_2$ is exponentially convex function on the co-ordinates on $\Delta$,* *then the following inequality holds;*

$$
\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{128}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^{\frac{2}{p}}\left(\frac{\left(2^{q+1}-1\right)^{\frac{1}{q}}+1}{(q+1)^{\frac{1}{q}}}\right)^2 \\
& \times\left(\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right),
\end{aligned}
$$

*where* $p^{-1}+q^{-1}=1$.

*Proof.* Using the Lemma 2.1 and the absolute value property of integrals, we get;

$$
\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32} \\
& \times\Bigg[\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)\right|d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)\right|d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\frac{2}{3}-\xi\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)\right|d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)\right|d\zeta d\xi\Bigg].
\end{aligned}
$$

Using the exponentially convex property of $\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\right|^q$, then the following inequality is obtained;

$$
\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32} \\
& \times\Bigg[\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}} \\
& +\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\Bigg)d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}} \\
& +\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}\Bigg)d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\frac{2}{3}-\xi\right|\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}} \\
& +\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}\Bigg)d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|\left|\xi-\frac{2}{3}\right|\Bigg(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}} \\
& +\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}\Bigg)d\zeta d\xi\Bigg].
\end{aligned}
$$

Applying Hölder's inequality to the right-hand side yields the following.

$$
\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32}\times\Bigg[\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

The necessary algebraic operations are performed and the following result is obtained by integrating separately with respect to $\zeta$ and $\xi$.

$$\begin{aligned}|J| \quad &\leq \quad \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32}\times\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^{\frac{2}{p}}\\ &\times\left[\left(\frac{2^{q+1}-1}{2^q(q+1)}\right)^{\frac{2}{q}}\left(\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\right.\\ &+2\left(\frac{2^{q+1}-1}{4^q(q+1)^2}\right)^{\frac{1}{q}}\left(\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\\ &\left.+\left(\frac{1}{2^q(q+1)}\right)^{\frac{2}{q}}\left(\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\right].\end{aligned}$$

By making the necessary adjustments, the following result is obtained.

$$\begin{aligned}|J| \quad &\leq \quad \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{128}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^{\frac{2}{p}}\left(\frac{\left(2^{q+1}-1\right)^{\frac{1}{q}}+1}{(q+1)^{\frac{1}{q}}}\right)^2\\ &\times\left(\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right).\end{aligned}$$

□

**Corollary 2.5.** *Under the conditions of Theorem 2.4, if $\alpha = 0$ is taken, the result is the convexity result in the co-ordinates as follows.*

$$\begin{aligned}|J_2| \quad &\leq \quad \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{128}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^{\frac{2}{p}}\left(\frac{\left(2^{q+1}-1\right)^{\frac{1}{q}}+1}{(q+1)^{\frac{1}{q}}}\right)^2\\ &\times\left(\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|+\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|+\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|\right).\end{aligned}$$

**Theorem 2.6.** *Let $F:\Delta\subseteq R^2\to R$ be partial differentiable mapping on $\Delta=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$, where $\kappa_1<\kappa_2$ and $\kappa_3<\kappa_4$, $\left|\frac{\partial^2F}{\partial\zeta\partial\xi}\right|^q\in L_2(\Delta), q>1$ and $\alpha\in R$. If $|\partial F|\in L_2$ is exponentially convex function on the co-ordinates on $\Delta$, then the following inequality holds;*

$$\begin{aligned}|J| \quad &\leq \quad \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32}\times\left[\frac{16}{p}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^2+\frac{4}{q(q+1)^2}\right.\\ &\left.\times\left(\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\right],\end{aligned}$$

*where $p^{-1}+q^{-1}=1$.*

*Proof.* Using the Lemma 2.1 and the absolute value property of integrals, we get;

$$\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32} \\
& \times\left[\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)\right|d\zeta d\xi\right. \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)\right|d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\frac{2}{3}-\xi\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_2+\frac{1-\zeta}{2}\kappa_1,\frac{1+\xi}{2}\kappa_3+\frac{1-\xi}{2}\kappa_4\right)\right|d\zeta d\xi \\
& \left.+\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|\left|\xi-\frac{2}{3}\right|\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\left(\frac{1+\zeta}{2}\kappa_1+\frac{1-\zeta}{2}\kappa_2,\frac{1+\xi}{2}\kappa_4+\frac{1-\xi}{2}\kappa_3\right)\right|d\zeta d\xi\right].
\end{aligned}$$

Using the exponentially convex property of $\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}\right|^q$, then the following inequality is obtained;

$$\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32} \\
& \times\left[\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}\right.\right. \\
& \left.+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\xi-\frac{2}{3}\right|\left(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}\right. \\
& \left.+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}\right)d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|\left|\frac{2}{3}-\xi\right|\left(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}\right. \\
& \left.+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}\right)d\zeta d\xi \\
& +\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|\left|\xi-\frac{2}{3}\right|\left(\left(\frac{1+\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\left(\frac{1+\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right. \\
& \left.\left.+\left(\frac{1-\zeta}{2}\right)\left(\frac{1+\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\left(\frac{1-\zeta}{2}\right)\left(\frac{1-\xi}{2}\right)\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}\right)d\zeta d\xi\right].
\end{aligned}$$

Applying Young's inequality to the right-hand side yields the following.

$$
\begin{aligned}
|J| \leq\; & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32} \\
& \times\left[\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right)\right. \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\xi-\frac{2}{3}\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\zeta-\frac{2}{3}\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|^q}{e^{q\alpha(\kappa_1+\kappa_4)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1+\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|^q}{e^{q\alpha(\kappa_1+\kappa_3)}}d\zeta d\xi\right) \\
& +\frac{1}{p}\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1+\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|^q}{e^{q\alpha(\kappa_2+\kappa_4)}}d\zeta d\xi\right) \\
& \left.+\frac{1}{p}\left(\int_0^1\int_0^1\left|\frac{2}{3}-\zeta\right|^p\left|\frac{2}{3}-\xi\right|^p d\zeta d\xi\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left(\frac{1-\zeta}{2}\right)^q\left(\frac{1-\xi}{2}\right)^q\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|^q}{e^{q\alpha(\kappa_2+\kappa_3)}}d\zeta d\xi\right)\right].
\end{aligned}
$$

The necessary algebraic operations are performed and the following result is obtained by integrating separately with respect to $\zeta$ and $\xi$.

$$\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32}\left[\frac{16}{p}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^2\right.\\
&+\frac{1}{q}\left(\frac{2^{q+1}-1}{2^q(q+1)}\right)^2\left(\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\\
&+\frac{2}{q}\left(\frac{2^{q+1}-1}{4^q(q+1)^2}\right)\left(\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\\
&\left.+\frac{1}{q}\left(\frac{1}{2^q(q+1)}\right)^2\left(\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\right].
\end{aligned}$$

By making the necessary adjustments, the following result is obtained.

$$\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32}\times\left[\frac{16}{p}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^2+\frac{4}{q(q+1)^2}\right.\\
&\left.\times\left(\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|}{e^{\alpha(\kappa_2+\kappa_4)}}+\frac{\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|}{e^{\alpha(\kappa_1+\kappa_3)}}\right)\right].
\end{aligned}$$

□

**Corollary 2.7.** *Under the conditions of Theorem 2.6, if $\alpha = 0$ is taken, the result is the convexity result in the co-ordinates as follows.*

$$\begin{aligned}
|J| \leq\ & \frac{(\kappa_4-\kappa_3)^2(\kappa_2-\kappa_1)^2}{32}\times\left[\frac{16}{p}\left(\frac{2^{p+1}+1}{3^{p+1}(p+1)}\right)^2+\frac{4}{q(q+1)^2}\right.\\
&\left.\times\left(\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_3)\right|+\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_4)\right|+\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_2,\kappa_4)\right|+\left|\frac{\partial^2 F}{\partial\zeta\partial\xi}(\kappa_1,\kappa_3)\right|\right)\right].
\end{aligned}$$

## 3. Acknowledgement

The principal results presented in this paper are based on the PhD dissertation of the author.

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