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Integral Inequalities for n-Fractional Polynomial Convex Functions via AB-Fractional Integral Operators

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Abstract. In recent years, many new equations involving fractional integral operators have been obtained and numerous inequalities have been proven using these equations. In this paper, we obtained some integral inequalities via Atangana-Baleanu fractional integral operators for n-fractional polynomial convex functions using the identity by proved Set et al. [10]. Some of the inequalities proved are reduced to existing inequalities in the literature for some special values of the parameters. And also, the inequalities obtained produce new results for some special values of the parameters.

1. Introduction

Convex functions, defined through an inequality condition, have significantly contributed to the development of inequality theory.The definition of this function is as follows.

Definition 1.1. *The function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \tag{1}$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The Hermite-Hadamard inequality is one of the important results for convex functions, and it has played a significant role in helping researchers derive numerous new findings in the field of inequality theory. This important inequality is given follow.

Assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

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holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

We recall the definitions of n-fractional polynomial convex function introduce by İşcan in [12].

Definition 1.2. *Let $n \in \mathbb{N}$. A non-nagative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial convex function if the inequality*

$$f(tu + (1-t)v) \leq \frac{1}{n}\sum_{i=1}^{n} t^{\frac{1}{i}} f(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{\frac{1}{i}} f(v) \tag{3}$$

hold for every $u, v \in I$ and $t \in [0,1]$.

The class of all n-fractional polynomial convex function is denoted by $FPC(I)$.

The utilization of fractional operators in the field of inequality has introduced a novel perspective to the field. Numerous new results have been derived by researchers through the use of these operators. One of the recently introduced operators is the Atangana-Baleanu fractional integral operator, which was introduced by Atangana and Baleanu using the Laplace transform and the convolution theorem as follow.

Definition 1.3. [2] *The fractional integral associate to the new fractional derivative with non-local kernel of a function $f \in H^1(a,b)$ as defined:*

$${}^{AB}_{a}I^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy$$

where $b > a, \alpha \in [0,1]$.

In [1], the right-hand side of the integral operator was introduced by Abdeljawad and Baleanu as follows: The right fractional new integral with Mittag-Leffler kernel of order $\alpha \in [0,1]$ is defined by

$${}^{AB}I^{\alpha}_{b}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}dy.$$

For more results related to different kinds of fractional operators, we suggest to the interested readers the papers [3–9].

In this paper, we will denote normalization function as $B(\alpha)$ with $B(0) = B(1) = 1$.

The Gamma function $\Gamma(z)$ developed by Euler is usually defined as follow.

Definition 1.4. [11] *Assume that $\Re(z) > 0$, the Gamma function is denoted by $\Gamma(z)$ and defined as follow.*

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt.$$

The definition of Beta function is as follow.

Definition 1.5. [11] *Assume that $\Re(\eta) > 0$ and $\Re(\rho) > 0$, the Beta function is denoted by $\beta(\eta,\rho)$ and defined as*

$$\beta(\eta,\rho) = \int_0^1 t^{\eta-1}(1-t)^{\rho-1}dt.$$

In [10], Set et al. proved identity that we using to obtained our main results via Atangana-Baleanu fractional integral operators as following.

Lemma 1.6. *$f : [a, b] \to \mathbb{R}$ be differentiable function on (a, b) with $a < b$. Then we have the following identity for Atangana-Baleanu fractional integral operators*

$$
\begin{aligned}
& {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a) + (b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \\
= \; & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha} f'(kt + (1-k)a)\, dk - \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha} f'(kb + (1-k)t)\, dk
\end{aligned}
$$

where $\alpha \in (0, 1], t \in [a, b]$, $k \in [0, 1]$, $B(\alpha) > 0$ is normalization function and $\Gamma(.)$ is Gamma function.

The main purpose of this article is to obtain some integral inequalities that includes the Atangana-Baleanu fractional integral operators for n-fractional polynomial convex function with the help of the identity by proved Set et al. in [10].

2. Main Results

In this part, we obtained some fractional integral inequalities for $n-$fractional polynomial convex functions with help of identity by proved Set et al. in [10] as following:

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable function on (a, b) with $a < b$ and $f' \in L[a, b]$. If $|f'|$ is a n-fractional polynomial convex function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$
\begin{aligned}
& \left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a) + (b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \qquad (4) \\
\leq \; & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)n} \left[\left|f'(t)\right| \sum_{r=1}^{n} B\left(\frac{1}{r} + 1, \alpha + 1\right) + \left|f'(a)\right| \sum_{r=1}^{n} \frac{r}{\alpha r + r + 1} \right] \\
& + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)n} \left[\left|f'(b)\right| \sum_{r=1}^{n} \frac{r}{\alpha r + r + 1} + \left|f'(t)\right| \sum_{r=1}^{n} B\left(\alpha + 1, \frac{1}{r} + 1\right) \right]
\end{aligned}
$$

where $t \in [a, b]$, $\alpha \in (0, 1]$, $B(\alpha) > 0$ is normalization function and $\Gamma(.)$ is Gamma function.

Proof. *By using the identity that is given in Lemma 1.6 and properties of absolute value, we can write*

$$
\begin{aligned}
& \left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a) + (b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
= \; & \left| \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha} f'(kt + (1-k)a)\, dk - \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha} f'(kb + (1-k)t)\, dk \right| \\
\leq \; & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha} \left|f'(kt + (1-k)a)\right| dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha} \left|f'(kb + (1-k)t)\right| dk.
\end{aligned}
$$

By using n-fractional polynomial convexity of $|f'|$, we get

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}_{t}\{f(t)\} + {}^{AB}_{b}I^{\alpha}_{t}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha} \left[\frac{1}{n}\sum_{r=1}^{n} k^{\frac{1}{r}} \left|f'(t)\right| + \frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}} \left|f'(a)\right| \right] dk \\
&+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha} \left[\frac{1}{n}\sum_{r=1}^{n} k^{\frac{1}{r}} \left|f'(b)\right| + \frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}} \left|f'(t)\right| \right] dk \\
=\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)n} \left(\left|f'(t)\right| \sum_{r=1}^{n} \int_0^1 (1-k)^{\alpha} k^{\frac{1}{r}} dk + \left|f'(a)\right| \sum_{r=1}^{n} \int_0^1 (1-k)^{\alpha+\frac{1}{r}} dk \right) \\
&+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)n} \left(\left|f'(b)\right| \sum_{r=1}^{n} \int_0^1 k^{\alpha+\frac{1}{r}} dk + \left|f'(t)\right| \sum_{r=1}^{n} \int_0^1 k^{\alpha}(1-k)^{\frac{1}{r}} dk \right) \\
=\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)n} \left[\left|f'(t)\right| \sum_{r=1}^{n} B\left(\frac{1}{r}+1, \alpha+1\right) + \left|f'(a)\right| \sum_{r=1}^{n} \frac{r}{\alpha r + r + 1} \right] \\
&+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)n} \left[\left|f'(b)\right| \sum_{r=1}^{n} \frac{r}{\alpha r + r + 1} + \left|f'(t)\right| \sum_{r=1}^{n} B\left(\alpha+1, \frac{1}{r}+1\right) \right]
\end{aligned}$$

and the proof is completed. □

Corollary 2.2. *In Theorem 2.1, if we choose $t = \frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{AB}I^{\alpha}_{b} f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[f(a)+f(b)\right] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)} \right| \\
\leq\ & \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)n} \left[2\left|f'\left(\frac{a+b}{2}\right)\right| \sum_{r=1}^{n} B\left(\frac{1}{r}+1, \alpha+1\right) + \left[\left|f'(a)\right| + \left|f'(b)\right|\right] \sum_{r=1}^{n} \frac{r}{\alpha r + r + 1} \right].
\end{aligned}$$

Remark 2.3. *In Theorem 2.1, if we choose $n = 1$, the inequality* (4) *reduces to the inequality in Theorem 2.2 in [10].*

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is a n-fractional polynomial convex function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \qquad (5)\\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \left(\frac{\left[\left|f'(t)\right|^q + \left|f'(a)\right|^q\right]}{n} \sum_{r=1}^{n} \frac{r}{r+1} \right)^{\frac{1}{q}} \\
&+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \left(\frac{\left[\left|f'(b)\right|^q + \left|f'(t)\right|^q\right]}{n} \sum_{r=1}^{n} \frac{r}{r+1} \right)^{\frac{1}{q}}
\end{aligned}$$

where $t \in [a,b]$, $\alpha \in (0,1]$, $p^{-1} + q^{-1} = 1$, $q > 1$, $B(\alpha) > 0$ is normalization function and $\Gamma(.)$ is Gamma function.

Proof. By using Lemma 1.6 and properties of absolute value, we have

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha} \left|f'(kt+(1-k)a)\right| dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha} \left|f'(kb+(1-k)t)\right| dk.
\end{aligned}$$

By applying Hölder inequality, we have

$$\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(kt+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(kb+(1-k)t)\right|^q dk\right)^{\frac{1}{q}}\right].$$

By using n-fractional polynomial convexity of $|f'|^q$*, we obtain*

$$\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1 (1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{1}{n}\sum_{r=1}^{n}k^{\frac{1}{r}}\left|f'(t)\right|^q+\frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}}\left|f'(a)\right|^q\right]dk\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{1}{n}\sum_{r=1}^{n}k^{\frac{1}{r}}\left|f'(b)\right|^q+\frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}}\left|f'(t)\right|^q\right]dk\right)^{\frac{1}{q}}$$

$$= \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1 (1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\frac{\left|f'(t)\right|^q}{n}\sum_{r=1}^{n}\int_0^1 k^{\frac{1}{r}}dk+\frac{\left|f'(a)\right|^q}{n}\sum_{r=1}^{n}\int_0^1(1-k)^{\frac{1}{r}}dk\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\frac{\left|f'(b)\right|^q}{n}\sum_{r=1}^{n}\int_0^1 k^{\frac{1}{r}}dk+\frac{\left|f'(t)\right|^q}{n}\sum_{r=1}^{n}\int_0^1(1-k)^{\frac{1}{r}}dk\right)^{\frac{1}{q}}$$

$$= \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left[\left|f'(t)\right|^q+\left|f'(a)\right|^q\right]}{n}\sum_{r=1}^{n}\frac{r}{r+1}\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left[\left|f'(b)\right|^q+\left|f'(t)\right|^q\right]}{n}\sum_{r=1}^{n}\frac{r}{r+1}\right)^{\frac{1}{q}}.$$

So, the proof is completed. □

Corollary 2.5. *In Theorem 2.4, if we choose* $t=\frac{a+b}{2}$*, we obtain*

$$\left| {}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right) + {}^{AB}I^{\alpha}_{b}f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[f(a)+f(b)\right] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)} \right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left[\left|f'(\frac{a+b}{2})\right|^q+\left|f'(a)\right|^q\right]}{n}\sum_{r=1}^{n}\frac{r}{r+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left[\left|f'(b)\right|^q+\left|f'(\frac{a+b}{2})\right|^q\right]}{n}\sum_{r=1}^{n}\frac{r}{r+1}\right)^{\frac{1}{q}}\right].$$

Remark 2.6. *In Theorem 2.4, if we choose* $n=1$*, the inequality* (5) *reduces to the inequality in Theorem 2.5 in [10].*

Theorem 2.7. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is a n-fractional polynomial convex function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right| \qquad (6)\\ \leq\;& \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(t)\right|^{q}}{n}\sum_{r=1}^{n}B(\frac{1}{r}+1,\alpha+1)+\frac{\left|f'(a)\right|^{q}}{n}\sum_{r=1}^{n}\frac{r}{\alpha r+r+1}\right)^{\frac{1}{q}}\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^{q}}{n}\sum_{r=1}^{n}\frac{r}{\alpha r+r+1}+\frac{\left|f'(t)\right|^{q}}{n}\sum_{r=1}^{n}B(\alpha+1,\frac{1}{r}+1)\right)^{\frac{1}{q}}\end{aligned}$$

where $t\in[a,b]$, $\alpha\in(0,1]$, $q\geq 1$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is Gamma function.

Proof. *By using the identity that is given in Lemma 1.6 and properties of absolute value, we can write*

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \leq\;& \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.\end{aligned}$$

By applying power mean inequality, we get

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \leq\;& \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1(1-k)^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|^{q}dk\right)^{\frac{1}{q}}\right]\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1k^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1k^{\alpha}\left|f'(kb+(1-k)t)\right|^{q}dk\right)^{\frac{1}{q}}\right].\end{aligned}$$

By using n-fractional polynomial convexity of $|f'|^q$, we obtain

$$\begin{aligned}&\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\ \leq\;& \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1(1-k)^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1(1-k)^{\alpha}\left[\frac{1}{n}\sum_{r=1}^{n}k^{\frac{1}{r}}\left|f'(t)\right|^{q}\right.\right.\right.\\ &+\left.\left.\left.\frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}}\left|f'(a)\right|^{q}\right]dk\right)^{\frac{1}{q}}\right]\\ &+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1k^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1k^{\alpha}\left[\frac{1}{n}\sum_{r=1}^{n}k^{\frac{1}{r}}\left|f'(b)\right|^{q}+\frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}}\left|f'(t)\right|^{q}\right]dk\right)^{\frac{1}{q}}\right]\end{aligned}$$

$$= \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^\alpha dk\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(t)\right|^q}{n}\sum_{r=1}^{n}\int_0^1 (1-k)^\alpha k^{\frac{1}{r}} dk\right.\right.$$
$$\left.\left. + \frac{\left|f'(a)\right|^q}{n}\sum_{r=1}^{n}\int_0^1 (1-k)^{\alpha+\frac{1}{r}} dk\right)^{\frac{1}{q}}\right]$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^\alpha dk\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^q}{n}\sum_{r=1}^{n}\int_0^1 k^{\alpha+\frac{1}{r}} dk + \frac{\left|f'(t)\right|^q}{n}\sum_{r=1}^{n}\int_0^1 k^\alpha (1-k)^{\frac{1}{r}} dk\right)^{\frac{1}{q}}\right]$$
$$= \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(t)\right|^q}{n}\sum_{r=1}^{n} B(\frac{1}{r}+1,\alpha+1) + \frac{\left|f'(a)\right|^q}{n}\sum_{r=1}^{n}\frac{r}{\alpha r+r+1}\right)^{\frac{1}{q}}$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^q}{n}\sum_{r=1}^{n}\frac{r}{\alpha r+r+1} + \frac{\left|f'(t)\right|^q}{n}\sum_{r=1}^{n} B(\alpha+1,\frac{1}{r}+1)\right)^{\frac{1}{q}}$$

and the proof is completed. □

Corollary 2.8. *In Theorem 2.7, if we choose $t = \frac{a+b}{2}$, we obtain*

$$\left|{}^{AB}_{\ a}I^\alpha f\left(\frac{a+b}{2}\right) +^{AB} I_b^\alpha f\left(\frac{a+b}{2}\right) - \frac{(b-a)^\alpha}{2^\alpha B(\alpha)\Gamma(\alpha)}\big[f(a)+f(b)\big] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|$$
$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|f'(\frac{a+b}{2})\right|^q}{n}\sum_{r=1}^{n} B(\frac{1}{r}+1,\alpha+1) + \frac{\left|f'(a)\right|^q}{n}\sum_{r=1}^{n}\frac{r}{\alpha r+r+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|f'(b)\right|^q}{n}\sum_{r=1}^{n}\frac{r}{\alpha r+r+1} + \frac{\left|f'(\frac{a+b}{2})\right|^q}{n}\sum_{r=1}^{n} B(\alpha+1,\frac{1}{r}+1)\right)^{\frac{1}{q}}\right].$$

Remark 2.9. *In Theorem 2.7, if we choose $n = 1$, the inequality* (6) *reduces to the inequality in Theorem 2.10 in [10].*

Theorem 2.10. *Let $f : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is a n-fractional polynomial convex function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left|{}^{AB}_{\ a}I^\alpha\{f(t)\} +^{AB} I_b^\alpha\{f(t)\} - \frac{(t-a)^\alpha f(a)+(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right| \tag{7}$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)} + \left[\frac{\left[\left|f'(a)\right|^q + \left|f'(t)\right|^q\right]}{qn}\right]\sum_{r=1}^{n}\frac{r}{r+1}\right)$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)} + \left[\frac{\left[\left|f'(b)\right|^q + \left|f'(t)\right|^q\right]}{qn}\right]\sum_{r=1}^{n}\frac{r}{r+1}\right)$$

where $p^{-1}+q^{-1} = 1$, $t \in [a,b]$, $\alpha \in (0,1]$, $q > 1$, $B(\alpha) > 0$ is normalization function and $\Gamma(.)$ is Gamma function.

Proof. *By using Lemma 1.6 and properties of absolute value, we have*

$$\left|{}^{AB}_{\ a}I^\alpha\{f(t)\} +^{AB} I_b^\alpha\{f(t)\} - \frac{(t-a)^\alpha f(a)+(b-t)^\alpha f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^\alpha \left|f'(kt+(1-k)a)\right| dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^\alpha \left|f'(kb+(1-k)t)\right| dk.$$

By applying the Young inequality, we get

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_{b}^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 (1-k)^{\alpha p}dk + \frac{1}{q}\int_0^1 \left|f'(kt+(1-k)a)\right|^q dk\right] \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{\alpha p}dk + \frac{1}{q}\int_0^1 \left|f'(kb+(1-k)t)\right|^q dk\right].
\end{aligned}$$

By using n-fractional polynomial convexity of $|f'|^q$*, we get*

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_{b}^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \\
\leq\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 (1-k)^{\alpha p}dk + \frac{1}{q}\int_0^1 \left[\frac{1}{n}\sum_{r=1}^{n} k^{\frac{1}{r}}\left|f'(t)\right|^q + \frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}}\left|f'(a)\right|^q\right]dk\right] \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{\alpha p}dk + \frac{1}{q}\int_0^1 \left[\frac{1}{n}\sum_{r=1}^{n} k^{\frac{1}{r}}\left|f'(b)\right|^q + \frac{1}{n}\sum_{r=1}^{n}(1-k)^{\frac{1}{r}}\left|f'(t)\right|^q\right]dk\right] \\
=\ & \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)} + \left[\frac{\left[\left|f'(a)\right|^q + \left|f'(t)\right|^q\right]}{qn}\right]\left(\sum_{r=1}^{n}\frac{r}{r+1}\right)\right) \\
&+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)} + \left[\frac{\left[\left|f'(b)\right|^q + \left|f'(t)\right|^q\right]}{qn}\right]\left(\sum_{r=1}^{n}\frac{r}{r+1}\right)\right)
\end{aligned}$$

and the proof is completed. □

Corollary 2.11. *In Theorem 2.10, if we choose* $t = \frac{a+b}{2}$*, we obtain*

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}f\left(\frac{a+b}{2}\right) + {}^{AB}I_{b}^{\alpha}f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[f(a)+f(b)\right] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)} \right| \\
\leq\ & \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{2}{p(\alpha p+1)} + \left[\frac{2\left|f'(\frac{a+b}{2})\right|^q + \left|f'(a)\right|^q + \left|f'(b)\right|^q}{qn}\right]\left(\sum_{r=1}^{n}\frac{r}{r+1}\right)\right).
\end{aligned}$$

Remark 2.12. *In Theorem 2.10, if we choose* $n = 1$*, the inequality* (7) *reduces to the inequality in Theorem 2.8 in [10].*

3. Conclusion

In recent years, a variety of new equations involving fractional integral operators have been introduced and numerous inequalities have been established based on these equations. In the first section of this study, some definitions, theorems, and equations are presented to obtain the main results. In the second section, we obtained new inequalities using the Atangana-Baleanu fractional integral operators for n-polynomial convex functions by employing an identity that had been previously proven. Some of the inequalities proved are reduced to existing inequalities in the literature for some special values of the parameters. And also, the inequalities obtained produce new results for some special values of the parameters. Researchers can obtain new inequalities by similar methods for different convexity classes, using different equation.

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