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Coefficient Bounds for A Subclass of Analytic Functions Associated with Chebyshev Polynomials and Deniz-Orhan Operator

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Abstract. In this paper, we obtain initial coefficients $|a_2|$ and $|a_3|$ for a certain subclass by means of Chebyshev polynomials expansions of analytic functions defined by Deniz-Orhan operator in $\mathcal{D}$.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of the functions of the form:

$$f(z) = z + \int_2^{\infty} a_n z^n \, dn \tag{1}$$

which are analytic in the open unit disc

$$D := \{z : z \in \mathbb{C} : |z| < 1\}$$

and satisfying the conditions $f(0) = 0$ and $f'(0) = 1$.
Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the form (1) which are univalent in $\mathcal{D}$.
Let f and g be analytic functions in $\mathcal{D}$. We define that the function f is subordinate to g in $\mathcal{D}$ and denoted by

$$f(z) \prec g(z) \quad (z \in \mathcal{D}),$$

if there exists a Schwarz function ω, which is analytic in $\mathcal{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathcal{D})$ such that

$$f(z) = g(\omega(z)) \quad (z \in \mathcal{D}).$$

If g is a univalent function in $\mathcal{D}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{D}) \subset g(\mathcal{D}).$$

For a function for $f(z) \in \mathcal{A}$, the multiplier differential operator $D^n_{\alpha,\delta} f$ was extended by Deniz and Orhan [6] as follows:

$$D^0_{\alpha,\delta} f(z) = f(z),$$

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$$D^1_{\alpha,\delta}f(z)=D_{\alpha,\delta}f(z)=\alpha\delta z^2 f''(z)+(\alpha-\delta)zf'(z)+(1-\alpha+\delta)f(z),$$

$$\vdots$$

$$D^n_{\alpha,\delta}f(z)=D_{\alpha,\delta}\left(D^{n-1}_{\alpha,\delta}f(z)\right),$$

where $\alpha \geq \delta \geq 0$ and $n \in \mathbb{N}_0=\mathbb{N}\cup 0$. If f is given by (1) then from the definition of the operator $D^n_{\alpha,\delta}f(z)$ it is easy to see that

$$D^n_{\alpha,\delta}f(z)=z+\int_{k=2}^{\infty}\Phi^n_k a_k z^k, \tag{2}$$

where $\Phi_k=[1+(\alpha\delta k+\alpha-\delta)(k-1)]$, $\left(\Phi^n_k=[\Phi_k]^n\right)$; $\alpha \geq \delta \geq 0$ and $n \in \mathbb{N}_0$.

It should be remarked that the $D^n_{\alpha,\delta}f$ is a generalization of many other linear operators considered earlier. In particular, for $f\in\mathcal{A}$ we have the following:

1. $D^n_{1,0}f(z) \equiv D^n f(z)$ the operator defined by Salagean [18].
2. $D^n_{\alpha,0}f(z) \equiv D^n_\alpha f(z)$ the operator studied by Al-Oboudi [3].
3. $D^n_{\alpha,\delta}f(z)$ the same operator considered for $0 \leq \delta \leq \alpha \leq 1$, by Raducanu and Orhan [17].

Chebyshev polynomials play a considerable role in numerical analysis. There are four kinds of Chebyshev polynomials. The first and second kinds of Chebyshev polynomials are defined by:

$$T_n(t)=\cos n\varphi \quad \text{and} \quad U_n(t)=\frac{\sin(n+1)\varphi}{\sin\varphi} \quad (-1<t<1)$$

where n denotes the polynomial degree and $t=\cos\varphi$. For a brief history of Chebyshev polynomials of the first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer [1], [2], [4], [5], [7]-[16], [19].

Now, we define a subclass of analytic functions in $\mathcal{D}$ with the following subordination condition:

Definition 1.1. *A function $f \in \mathcal{A}$ given by* (1) *is said to be in the class $\mathcal{N}(\gamma,\lambda,t)$ for $0 \leq \gamma \leq 1$, $\lambda \in \mathbb{C}\setminus 0$ and $t \in \left(\frac{1}{2},1\right]$ if the following subordination hold:*

$$1+\frac{1}{\lambda}\left[\left(D^n_{\alpha,\delta}f(z)\right)'+\gamma z\left(D^n_{\alpha,\delta}f(z)\right)''-1\right]\prec H(z,t)=\frac{1}{1-2tz+z^2} \quad (z\in\mathcal{D}) \tag{3}$$

where

$$0\leq\gamma\leq1,\ \alpha\geq\delta\geq0, \lambda\in\mathbb{C}\setminus 0,\ n\in\mathbb{N}.$$

We consider that if $t=\cos\varphi\ \left(\frac{-\pi}{3}<\varphi<\frac{\pi}{3}\right)$, then $H(z,t)=\frac{1}{1-2\cos\varphi z+z^2}=1+\int_{n=1}^{\infty}\frac{\sin(n+1)\varphi}{\sin\varphi}z^n \quad (z\in\mathcal{D})$.

Thus, $H(z,t)=1+2\cos\varphi z+\left(3\cos^2\varphi-\sin^2\varphi\right)z^2+\cdots \quad (z\in\mathcal{D})$.

So, we write the Chebyshev polynomials of the second kind as following:

$$H(z,t)=1+U_1(t)z+U_2(t)z^2+\cdots \quad (z\in\mathcal{D}, -1<t<1)$$

where $U_{n-1}(\mathrm{t})=\frac{\sin(n\arccos t)}{\sqrt{1-t^2}}$, $n\in\mathbb{N}$ and we have $U_n(t)=2tU_{n-1}(t)-U_{n-2}(t)$,

$$U_1(t)=2t,\ U_2(t)=4t^2-1,\ U_3(t)=8t^3-4t,\ U_4(t)=16t^4-12t^2+1,\cdots \tag{4}$$

The Chebyshev polynomials $T_n(t)$, $t\in[-1,1]$ of the first kind have the generating function of the form

$$\int_{n=0}^{\infty}T_n(t)z^n=\frac{1-tz}{1-2tz+z^2} \quad (z\in\mathcal{D}).$$

There is the following connection by the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$:

$$\frac{dT_n(t)}{dt}=nU_{n-1}(t), \quad T_n(t)=U_n(t)-tU_{n-1}(t), \quad 2T_n(t)=U_n(t)-U_{n-2}(t).$$

In this paper, we obtain initial coefficients $|a_2|$ and $|a_3|$ for subclass $\mathcal{N}(\gamma,\lambda,t)$ by means of Chebyshev polynomials expansions of analytic functions defined by Deniz-Ozkan operator in $\mathcal{D}$.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{N}\left(\gamma,\lambda,t\right)$

We begin with the following result involving initial coefficient bounds for the function class $\mathcal{N}(\gamma,\lambda,t)$.

Theorem 2.1. *Let the function* $f(z)$ *given by* (1) *be in the class* $\mathcal{N}(\gamma,\lambda,t)$. *Then,*

$$|a_2| \leq \frac{t\,|\lambda|}{(1+\gamma)\,\Phi_2^n} \tag{5}$$

and

$$|a_3| \leq \frac{|\lambda|\left(4t^2+2t-1\right)}{3\,(1+2\gamma)\,\Phi_3^n} \tag{6}$$

where $\Phi_2^n = [1+(2\alpha\delta+\alpha-\delta)]^n$ *and* $\Phi_3^n = [1+2\,(3\alpha\delta+\alpha-\delta)]^n$.

Proof. Let $f\in\mathcal{N}(\gamma,\lambda,t)$. From (3), we have

$$1+\frac{1}{\lambda}\left[\left(D_{\alpha,\delta}^n f(z)\right)'+\gamma z\left(D_{\alpha,\delta}^n f(z)\right)''-1\right] = 1+U_1(t)\,p(z)+U_2(t)\,p^2(z)+... \tag{7}$$

for some analytic functions

$$p(z)=c_1z+c_2z^2+c_3z^3+\cdots \quad (z\in\mathcal{D})$$

such that $p(0)=0$, $\left|p(z)\right|<1$ $(z\in\mathcal{D})$. Then, for all $j\in\mathbb{N}$,

$$\left|c_j\right|\leq 1$$

and for all $\mu\in\mathbb{R}$

$$\left|c_2-\mu c_1^2\right| \leq \max\{1,|\mu|\}.$$

It follows from (7) that

$$1+\frac{1}{\lambda}\left[\left(D_{\alpha,\delta}^n f(z)\right)'+\gamma z\left(D_{\alpha,\delta}^n f(z)\right)''-1\right] = 1+U_1(t)\,c_1z+\left[U_1(t)\,c_2+U_2(t)\,c_1^2\right]z^2+\cdots \tag{8}$$

It follows from (8) that

$$\frac{2\,(1+\gamma)\,\Phi_2^n a_2}{\lambda}=U_1(t)\,c_1 \tag{9}$$

and

$$\frac{3\,(1+2\gamma)\,\Phi_3^n a_3}{\lambda}=U_1(t)\,c_2+U_2(t)\,c_1^2 \tag{10}$$

From (4), (8) and (9), we have

$$|a_2| \leq \frac{t\,|\lambda|}{(1+\gamma)\,\Phi_2^n}$$

By using (4) and (8) in (10), we obtain

$$|a_3| \leq \frac{|\lambda|\left(4t^2+2t-1\right)}{3\,(1+2\gamma)\,\Phi_3^n}$$

which completes the proof of Theorem 1. □

For λ= 1 in Theorem 1, we obtain the following corollary.

Corollary 2.2. *Let* $f \in \mathcal{N}(\gamma, 1, t)$. *Then*

$$|a_2| \leq \frac{t}{(1+\gamma)\,\Phi_2^n}$$

and

$$|a_3| \leq \frac{4t^2+2t-1}{3\,(1+2\gamma)\,\Phi_3^n}$$

where $\Phi_2^n = [1 + (2\alpha\delta + \alpha - \delta)]^n$ *and* $\Phi_3^n = [1 + 2\,(3\alpha\delta + \alpha - \delta)]^n$.

If we choose $\gamma = 0$ in Theorem 1, we get the following corollary.

Corollary 2.3. *Let* $f \in \mathcal{N}(0, \lambda, t)$. *Then*

$$|a_2| \leq \frac{t\,|\lambda|}{\Phi_2^n}$$

and

$$|a_3| \leq \frac{|\lambda|\left(4t^2+2t-1\right)}{3\Phi_3^n}$$

where $\Phi_2^n = [1 + (2\alpha\delta + \alpha - \delta)]^n$ *and* $\Phi_3^n = [1 + 2\,(3\alpha\delta + \alpha - \delta)]^n$.

Putting $\gamma = 0$ in Corollary 1, we have the following corollary.

Corollary 2.4. *Let* $f \in \mathcal{N}(0, 1, t)$. *Then*

$$|a_2| \leq \frac{t}{\Phi_2^n}$$

and

$$|a_3| \leq \frac{4t^2+2t-1}{3\Phi_3^n}$$

where $\Phi_2^n = [1 + (2\alpha\delta + \alpha - \delta)]^n$ *and* $\Phi_3^n = [1 + 2\,(3\alpha\delta + \alpha - \delta)]^n$.

Putting $n = 0$ in Corollary 3, we get the following result.

Corollary 2.5. *Let* $f \in \mathcal{N}(0, 1, t)$. *Then*

$$|a_2| \leq t$$

and

$$|a_3| \leq \frac{4t^2+2t-1}{3}.$$

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