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# Generalized Bivariate Fibonacci and Lucas Polynomials By Tridiagonal Matrices

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**Abstract.** In this paper, we present certain tridiagonal matrices associated with generalized bivariate Fibonacci and Lucas polynomials and investigate the relationships between these matrices and polynomials. We show that determinants and permanents of these tridiagonal matrices are generalized bivariate Fibonacci and Lucas polynomials, which generalize known results for Fibonacci, Lucas, Pell, Jacobsthal, Fermat, Morgan-Voyce, and Vieta polynomials in both univariate and bivariate forms.

## 1. Introduction

Polynomials defined by recurrence relations as generalizations of numbers have become an important topic in modern algebra due to their wide applications in number theory, combinatorics, and matrix theory. Fibonacci and Lucas polynomials, defined by the recurrence relations $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ with initial terms $F_0(x) = 0$, $F_1(x) = 1$, and $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$ with initial terms $L_0(x) = 2$, $L_1(x) = x$ for $n \geq 2$, are the most significant generalizations of the classical Fibonacci and Lucas numbers, respectively. There are many studies on the Fibonacci and Lucas numbers, polynomials, and their generalizations in the literature [1–3].

Bivariate Fibonacci and Lucas polynomials, which are generalizations of both Fibonacci and Lucas numbers and polynomials, are defined by the recurrence relations $F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y)$ with initial terms $F_0(x,y) = 0, F_1(x,y) = 1$, and $L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y)$ with initial terms $L_0(x,y) = 2, L_1(x,y) = x$ where $x, y \neq 0, x^2 + 4y \neq 0$ for $n \geq 2$, respectively [4, 5]. Further generalizations of these polynomials are presented using polynomials with real coefficients in their recurrence relations. For the polynomials $p(\chi,\gamma)$ and $q(\chi,\gamma)$ with real coefficients, the generalized bivariate Fibonacci polynomials are defined by the recurrence relation

$$H_n(\chi,\gamma) = p(\chi,\gamma)H_{n-1}(\chi,\gamma) + q(\chi,\gamma)H_{n-2}(\chi,\gamma), \quad n \geq 2 \tag{1}$$

with initial terms $H_0(\chi,\gamma) = 0$ and $H_1(\chi,\gamma) = 1$. Similarly, the generalized bivariate Lucas polynomials are defined by the recurrence relation

$$K_n(\chi,\gamma) = p(\chi,\gamma)K_{n-1}(\chi,\gamma) + q(\chi,\gamma)K_{n-2}(\chi,\gamma), \quad n \geq 2 \tag{2}$$

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with initial terms $K_0(\chi,\gamma) = 2$ and $K_1(\chi,\gamma) = p(\chi,\gamma)$. Sequences of the generalized bivariate Fibonacci and Lucas polynomials are denoted by $\{H_n(\chi,\gamma)\}_{n\in\mathbb{N}}$ and $\{K_n(\chi,\gamma)\}_{n\in\mathbb{N}}$, respectively. A fundamental relation between these two sequences is expressed as

$$K_n(\chi,\gamma) = H_{n+1}(\chi,\gamma) + q(\chi,\gamma)H_{n-1}(\chi,\gamma). \tag{3}$$

Additionally, the main properties of the generalized bivariate Fibonacci and Lucas polynomial sequences are examined through the use of general formulas [6]. In [7], generalized identities and related sums for bivariate Fibonacci and Lucas polynomials, including even and odd terms, are presented using Binet's formula. Then, several new identities are derived using the generalized bivariate Fibonacci and Lucas polynomials, including binomial summations, closed-form expressions for power sums, general summation formulas, generating functions, and various related relations [8].

From equations (1) and (2), the special cases of the polynomials $H_n(\chi,\gamma)$ and $K_n(\chi,\gamma)$, defined using the polynomials $p(\chi,\gamma)$ and $q(\chi,\gamma)$, are listed in Table 1.

Table 1: Special cases of the polynomials $H_n(\chi,\gamma)$ and $K_n(\chi,\gamma)$

| Polynomial Type | Symbol | $H_0/K_0$ | $H_1/K_1$ | $p(x,y)$ | $q(x,y)$ |
|---|---|---|---|---|---|
| Fibonacci polynomials | $H_n(\chi,\gamma) = F_n(x)$ | 0 | 1 | $x$ | 1 |
| Lucas polynomials | $K_n(\chi,\gamma) = L_n(x)$ | 2 | $x$ | $x$ | 1 |
| $h(x)$-Fibonacci polynomials | $H_n(\chi,\gamma) = F_{h,n}(x)$ | 0 | 1 | $h(x)$ | 1 |
| $h(x)$-Lucas polynomials | $K_n(\chi,\gamma) = L_{h,n}(x)$ | 2 | $h(x)$ | $h(x)$ | 1 |
| Fibonacci polynomials with two variables | $H_n(\chi,\gamma) = F_n(x,y)$ | 0 | 1 | $x$ | $y$ |
| Lucas polynomials with two variables | $K_n(\chi,\gamma) = L_n(x,y)$ | 2 | $x$ | $x$ | $y$ |
| Pell polynomials | $H_n(\chi,\gamma) = P_n(x)$ | 0 | 1 | $2x$ | 1 |
| Pell-Lucas polynomials | $K_n(\chi,\gamma) = Q_n(x)$ | 2 | $2x$ | $2x$ | 1 |
| Jacobsthal polynomials | $H_n(\chi,\gamma) = J_n(x)$ | 0 | 1 | 1 | $2x$ |
| Jacobsthal-Lucas polynomials | $K_n(\chi,\gamma) = j_n(x)$ | 2 | 1 | 1 | $2x$ |
| Fermat polynomials | $H_n(\chi,\gamma) = \phi_n(x)$ | 0 | 1 | $3x$ | $-2$ |
| Fermat-Lucas polynomials | $K_n(\chi,\gamma) = \vartheta_n(x)$ | 2 | $3x$ | $3x$ | $-2$ |
| Morgan-Voyce first kind polynomials | $H_n(\chi,\gamma) = B_n(x)$ | 0 | 1 | $x+2$ | $-1$ |
| Morgan-Voyce second kind polynomials | $K_n(\chi,\gamma) = C_n(x)$ | 2 | $x+2$ | $x+2$ | $-1$ |
| Vieta polynomials | $H_n(\chi,\gamma) = V_n(x)$ | 0 | 1 | $x$ | $-1$ |
| Vieta-Lucas polynomials | $K_n(\chi,\gamma) = v_n(x)$ | 2 | $x$ | $x$ | $-1$ |

Since all the results obtained in this study are presented for the entire family of the generalized bivariate Fibonacci and Lucas polynomials, the values provided in Table 1 can be directly substituted into corresponding theorems for any specific polynomial in both univariate and bivariate forms.

In the literature, matrix theory, via structures such as determinantal and permanental representations of specific matrices, has played a significant role in studies on the generalized Fibonacci and Lucas number and polynomial sequences, allowing the derivation of various properties of these sequences [9–12].

Let $A = [a_{i,j}]$ be an $n \times n$ matrix and $S_n$ be the symmetric group of permutations over the set $\{1, 2, \ldots, n\}$. The determinant of $A$, denoted by $\det(A)$, is defined as

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where $\operatorname{sgn}(\sigma)$ denotes the signature of the permutation $\sigma$, which equals $+1$ if $\sigma$ is an even permutation and $-1$ if it is an odd permutation. Similarly, the permanent of $A$, denoted by $\operatorname{per}(A)$, is defined as

$$\operatorname{per}(A) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where the summation extends over all permutations $\sigma \in S_n$. Let $A = [a_{i,j}]$ be an $m \times n$ real matrix with row vectors $r_1, r_2, \ldots, r_m$. The matrix $A$ is called contractible on column k if it contains exactly two nonzero entries in column k. Suppose that $A$ is contractible on column $k$ such that $a_{i,k} \neq 0$ and $a_{j,k} \neq 0$ for $i \neq j$. Then, the $(m-1) \times (n-1)$ matrix $A_{i,j:k}$, obtained from $A$ by replacing row $i$ with $a_{j,k}r_i + a_{i,k}r_j$ and subsequently deleting row $j$ and column $k$, is called the contraction of $A$ on column $k$ relative to rows $i$ and $j$. Similarly, if $A$ is contractible on row $k$ with $a_{k,i} \neq 0$ and $a_{k,j} \neq 0$ for $i \neq j$, the matrix $A_{k:i,j} = [A^T_{i,j:k}]^T$ is called the contraction of $A$ on row $k$ relative to columns $i$ and $j$. Then, it is established that

$$\operatorname{per}(A) = \operatorname{per}(B) \tag{4}$$

where $A$ is a nonnegative integral matrix of order $n > 1$ and $B$ is a contraction of $A$ [13].

An $n \times n$ matrix $A_n = [a_{i,j}]$ is called a tridiagonal matrix if $a_{i,j} = 0$ whenever $|i - j| > 1$, defined as follows

$$A_n = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \ddots & \vdots \\ 0 & a_{3,2} & a_{3,3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{bmatrix}.$$

Then, for $A_n$, the $n \times n$ tridiagonal matrix, the determinant is given by

$$\det(A_n) = a_{n,n}\det(A_{n-1}) - a_{n,n-1}a_{n-1,n}\det(A_{n-2}), \quad n \geq 2 \tag{5}$$

where $\det(A_1) = a_{1,1}$ and $\det(A_2) = a_{2,2}a_{1,1} - a_{2,1}a_{1,2}$ [14]. Specifically, tridiagonal matrices have been extensively used to establish a direct correspondence with generalized Fibonacci and Lucas numbers and polynomials. Researchers have investigated the connections between such number and polynomial sequences and various matrices through determinantal and permanental representations. In [14], Fibonacci and Lucas numbers were represented as determinants of tridiagonal matrices, highlighting the connections between these classical sequences and matrix structures. In [15], the permanents of certain tridiagonal matrices were investigated with applications to Fibonacci and Lucas numbers. In [16], permanental representations of Fibonacci and Lucas $p$-numbers were established, along with their connections to certain combinatorial structures. In [17], two $n \times n$ tridiagonal matrix families were studied, and the relationships between the permanents of these matrices and the Pell and Jacobsthal sequences were presented. In [18], the period of the generalized Fibonacci sequence over a finite ring was studied, and connections with tridiagonal matrices were investigated. In [19], determinantal and permanental representations of $q$-Fibonacci polynomials were presented, providing a matrix-based approach to their properties. In [20], Pell polynomials $P_n(x, s, q)$ and their connections with tridiagonal matrices were studied, highlighting structural properties and matrix representations.

The aim of this study is to investigate the relationship between the generalized bivariate Fibonacci and Lucas polynomials and various tridiagonal matrices. For this purpose, we present several determinantal and permanental representations of the generalized bivariate Fibonacci and Lucas polynomials using different tridiagonal matrices.

## 2. The Determinantal Representations

In this section, we define certain tridiagonal matrices and show that the determinants of these matrices give terms of the generalized bivariate Fibonacci and Lucas polynomial sequences, $\{H_n(\chi, \gamma)\}_{n \in \mathbb{N}}$ and $\{K_n(\chi, \gamma)\}_{n \in \mathbb{N}}$, respectively.

The following theorems present the determinantal representations of the generalized bivariate Fibonacci polynomials.

**Theorem 2.1.** *Let $H_n(\chi,\gamma)$ be the nth generalized bivariate Fibonacci polynomial, and let $E_n = (e_{rs})$ be an $n \times n$ tridiagonal matrix whose entries are defined as*

$$e_{rs} = \begin{cases} 1, & \text{if } r = s-1, \\ p(\chi,\gamma), & \text{if } r = s, \\ -q(\chi,\gamma), & \text{if } r = s+1, \\ 0, & \text{otherwise,} \end{cases}$$

*where $p(\chi,\gamma)$ and $q(\chi,\gamma)$ are polynomials with real coefficients. Explicitly,*

$$E_n = \begin{bmatrix} p(\chi,\gamma) & 1 & 0 & 0 & \cdots & 0 & 0 \\ -q(\chi,\gamma) & p(\chi,\gamma) & 1 & 0 & \cdots & 0 & 0 \\ 0 & -q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & 0 & -q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & 0 & \cdots & -q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

*Let $F_n = (E_n)^T$. Then, for $n \geq 1$, it holds that*

$$\det(E_n) = \det(F_n) = H_{n+1}(\chi,\gamma).$$

*Proof.* We prove the result by mathematical induction on $n$. We first show that $\det(E_n) = H_{n+1}(\chi,\gamma)$ for $n \geq 1$. The result is true for $n = 1, 2$. Now, assume that it holds for $n \in \mathbb{Z}^+$, namely, $\det(E_n) = H_{n+1}(\chi,\gamma)$. We show it is true for $n+1$. Using the induction hypothesis and equations (1) and (5), we obtain

$$\begin{aligned} \det(E_{n+1}) &= e_{n+1,n+1}\det(E_n) - e_{n+1,n}e_{n,n+1}\det(E_{n-1}) \\ &= p(\chi,\gamma)\det(E_n) - (-q(\chi,\gamma)(1))\det(E_{n-1}) \\ &= p(\chi,\gamma)\det(E_n) + q(\chi,\gamma)\det(E_{n-1}) \\ &= p(\chi,\gamma)H_{n+1}(\chi,\gamma) + q(\chi,\gamma)H_n(\chi,\gamma) \\ &= H_{n+2}(\chi,\gamma). \end{aligned}$$

Hence, by induction, $\det(E_n) = H_{n+1}(\chi,\gamma)$ for all $n \geq 1$. Next, consider $F_n = (E_n)^T$. Since the determinant is invariant under transposition, $\det(E_n) = \det(F_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\det(E_n) = \det(F_n) = H_{n+1}(\chi,\gamma)$$

which completes the proof. □

The following theorem generalizes the previous result to the case where the matrix entries are complex.

**Theorem 2.2.** *Let $H_n(\chi,\gamma)$ be the nth generalized bivariate Fibonacci polynomial, and let $G_n = (g_{rs})$ be an $n \times n$ tridiagonal matrix whose entries are defined by*

$$g_{rs} = \begin{cases} i, & \text{if } r = s-1, \\ p(\chi,\gamma), & \text{if } r = s, \\ i\,q(\chi,\gamma), & \text{if } r = s+1, \\ 0, & \text{otherwise,} \end{cases}$$

*where* $i = \sqrt{-1}$ *and* $p(\chi,\gamma), q(\chi,\gamma)$ *are polynomials with real coefficients. Explicitly,*

$$G_n = \begin{bmatrix} p(\chi,\gamma) & i & 0 & 0 & \cdots & 0 & 0 \\ i\,q(\chi,\gamma) & p(\chi,\gamma) & i & 0 & \cdots & 0 & 0 \\ 0 & i\,q(\chi,\gamma) & p(\chi,\gamma) & i & \cdots & 0 & 0 \\ 0 & 0 & i\,q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & i \\ 0 & 0 & 0 & 0 & \cdots & i\,q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

*Let* $T_n = (G_n)^T$*. Then, for* $n \geq 1$*, it holds that*

$$\det(G_n) = \det(T_n) = H_{n+1}(\chi,\gamma).$$

*Proof.* We prove the result by mathematical induction on $n$. We first show that $\det(G_n) = H_{n+1}(\chi,\gamma)$ for $n \geq 1$. The result is true for $n = 1, 2$. Now, assume that it holds for $n \in \mathbb{Z}^+$, namely, $\det(G_n) = H_{n+1}(\chi,\gamma)$. We show it is true for $n+1$. Using the induction hypothesis and equations (1) and (5), we obtain

$$\begin{aligned} \det(G_{n+1}) &= g_{n+1,n+1}\det(G_n) - g_{n+1,n}g_{n,n+1}\det(G_{n-1}) \\ &= p(\chi,\gamma)\det(G_n) - (i\,q(\chi,\gamma)(i))\det(G_{n-1}) \\ &= p(\chi,\gamma)\det(G_n) - (i^2 q(\chi,\gamma))\det(G_{n-1}) \\ &= p(\chi,\gamma)\det(G_n) + q(\chi,\gamma)\det(G_{n-1}) \\ &= p(\chi,\gamma)H_{n+1}(\chi,\gamma) + q(\chi,\gamma)H_n(\chi,\gamma) \\ &= H_{n+2}(\chi,\gamma). \end{aligned}$$

Hence, by induction, $\det(G_n) = H_{n+1}(\chi,\gamma)$ for all $n \geq 1$. Next, consider $T_n = (G_n)^T$. Since the determinant is invariant under transposition, $\det(G_n) = \det(T_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\det(G_n) = \det(T_n) = H_{n+1}(\chi,\gamma)$$

which completes the proof. □

Next, we present the theorems that provide determinantal representations of the generalized bivariate Lucas polynomials.

**Theorem 2.3.** *Let* $K_n(\chi,\gamma)$ *be the nth generalized bivariate Lucas polynomial, and let* $U_n = (u_{rs})$ *be an* $n \times n$ *tridiagonal matrix whose entries are defined by*

$$u_{rs} = \begin{cases} 2, & \text{if } r = 1, s = 2, \\ 1, & \text{if } r = s-1 \text{ and } r \neq 1, \\ p(\chi,\gamma), & \text{if } r = s, \\ -q(\chi,\gamma), & \text{if } r = s+1, \\ 0, & \text{otherwise,} \end{cases}$$

*where* $p(\chi,\gamma)$ *and* $q(\chi,\gamma)$ *are polynomials with real coefficients. Explicitly,*

$$U_n = \begin{bmatrix} p(\chi,\gamma) & 2 & 0 & 0 & \cdots & 0 & 0 \\ -q(\chi,\gamma) & p(\chi,\gamma) & 1 & 0 & \cdots & 0 & 0 \\ 0 & -q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & 0 & -q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & 0 & \cdots & -q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

*Let* $V_n = (U_n)^T$*. Then, for* $n \geq 1$*, it holds that*

$$\det(U_n) = \det(V_n) = K_n(\chi, \gamma).$$

*Proof.* We prove the result by mathematical induction on $n$. We first show that $\det(U_n) = K_n(\chi, \gamma)$ for $n \geq 1$. The result is true for $n = 1, 2$. Now, assume that it holds for $n \in \mathbb{Z}^+$, namely, $\det(U_n) = K_n(\chi, \gamma)$. We show it is true for $n + 1$. Using the induction hypothesis and equations (2) and (5), we obtain

$$\begin{aligned}
\det(U_{n+1}) &= u_{n+1,n+1}\det(U_n) - u_{n+1,n}u_{n,n+1}\det(U_{n-1}) \\
&= p(\chi,\gamma)\det(U_n) - (-q(\chi,\gamma)(1))\det(U_{n-1}) \\
&= p(\chi,\gamma)\det(U_n) + q(\chi,\gamma)\det(U_{n-1}) \\
&= p(\chi,\gamma)K_n(\chi,\gamma) + q(\chi,\gamma)K_{n-1}(\chi,\gamma) \\
&= K_{n+1}(\chi,\gamma).
\end{aligned}$$

Hence, by induction, $\det(U_n) = K_n(\chi, \gamma)$ for all $n \geq 1$. Next, consider $V_n = (U_n)^T$. Since the determinant is invariant under transposition, $\det(U_n) = \det(V_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\det(U_n) = \det(V_n) = K_n(\chi, \gamma)$$

which completes the proof. □

The following theorem generalizes the previous result to the case where the matrix entries are complex.

**Theorem 2.4.** *Let* $K_n(\chi, \gamma)$ *be the nth generalized bivariate Lucas polynomial, and let* $W_n = (w_{rs})$ *be an* $n \times n$ *tridiagonal matrix whose entries are defined by*

$$w_{rs} = \begin{cases} 2i, & \text{if } r = 1, s = 2, \\ i, & \text{if } r = s - 1 \text{ and } r \neq 1, \\ p(\chi,\gamma), & \text{if } r = s, \\ i\,q(\chi,\gamma), & \text{if } r = s + 1, \\ 0, & \text{otherwise,} \end{cases}$$

*where* $i = \sqrt{-1}$ *and* $p(\chi, \gamma), q(\chi, \gamma)$ *are polynomials with real coefficients. Explicitly,*

$$W_n = \begin{bmatrix} p(\chi,\gamma) & 2i & 0 & 0 & \cdots & 0 & 0 \\ i\,q(\chi,\gamma) & p(\chi,\gamma) & i & 0 & \cdots & 0 & 0 \\ 0 & i\,q(\chi,\gamma) & p(\chi,\gamma) & i & \cdots & 0 & 0 \\ 0 & 0 & i\,q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & i \\ 0 & 0 & 0 & 0 & \cdots & i\,q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

*Let* $Z_n = (W_n)^T$*. Then, for* $n \geq 1$*, it holds that*

$$\det(W_n) = \det(Z_n) = K_n(\chi, \gamma).$$

*Proof.* We prove the result by mathematical induction on $n$. We first show that $\det(W_n) = K_n(\chi, \gamma)$ for $n \geq 1$. The result is true for $n = 1, 2$. Now, assume that it holds for $n \in \mathbb{Z}^+$, namely, $\det(W_n) = K_n(\chi, \gamma)$. We show it is true for $n+1$. Using the induction hypothesis and equations (2) and (5), we obtain

$$\begin{aligned}\det(W_{n+1}) &= w_{n+1,n+1}\det(W_n) - w_{n+1,n}w_{n,n+1}\det(W_{n-1})\\ &= p(\chi,\gamma)\det(W_n) - (i\,q(\chi,\gamma)(i))\det(W_{n-1})\\ &= p(\chi,\gamma)\det(W_n) - (i^2 q(\chi,\gamma))\det(W_{n-1})\\ &= p(\chi,\gamma)\det(W_n) + q(\chi,\gamma)\det(W_{n-1})\\ &= p(\chi,\gamma)K_n(\chi,\gamma) + q(\chi,\gamma)K_{n-1}(\chi,\gamma)\\ &= K_{n+1}(\chi,\gamma).\end{aligned}$$

Hence, by induction, $\det(W_n) = K_n(\chi,\gamma)$ for all $n \geq 1$. Next, consider $Z_n = (W_n)^T$. Since the determinant is invariant under transposition, $\det(W_n) = \det(Z_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\det(W_n) = \det(Z_n) = K_n(\chi,\gamma)$$

which completes the proof. □

## 3. The Permanental Representations

In this section, we define certain tridiagonal matrices and show that the permanents of these matrices give terms of the generalized bivariate Fibonacci and Lucas polynomial sequences, $\{H_n(\chi,\gamma)\}_{n\in\mathbb{N}}$ and $\{K_n(\chi,\gamma)\}_{n\in\mathbb{N}}$, respectively.

The following theorems present the permanental representations of the generalized bivariate Fibonacci polynomials.

**Theorem 3.1.** *Let $H_n(\chi,\gamma)$ be the nth generalized bivariate Fibonacci polynomial, and let $\mathcal{E}_n = (e_{rs})$ be an $n \times n$ tridiagonal matrix whose entries are defined by*

$$e_{rs} = \begin{cases} 1, & \text{if } r = s-1,\\ p(\chi,\gamma), & \text{if } r = s,\\ q(\chi,\gamma), & \text{if } r = s+1,\\ 0, & \text{otherwise,}\end{cases}$$

*where $p(\chi,\gamma)$ and $q(\chi,\gamma)$ are polynomials with real coefficients. Explicitly,*

$$\mathcal{E}_n = \begin{bmatrix} p(\chi,\gamma) & 1 & 0 & 0 & \cdots & 0 & 0\\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & 0 & \cdots & 0 & 0\\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0\\ 0 & 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1\\ 0 & 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma)\end{bmatrix}.$$

*Let $\mathcal{F}_n = (\mathcal{E}_n)^T$. Then, for $n \geq 1$, it holds that*

$$\operatorname{per}(\mathcal{E}_n) = \operatorname{per}(\mathcal{F}_n) = H_{n+1}(\chi,\gamma).$$

*Proof.* We first show that $\operatorname{per}(\mathcal{E}_n) = H_{n+1}(\chi,\gamma)$ for $n \geq 1$ by using the matrix contraction method. Let $\mathcal{E}_n^{(r)}$ denote the $r$th contraction of $\mathcal{E}_n$ for $1 \leq r \leq n-2$. By contracting $\mathcal{E}_n$ on its first column, we get

$$\mathcal{E}_n^{(1)} = \begin{bmatrix} p^2(\chi,\gamma)+q(\chi,\gamma) & p(\chi,\gamma) & 0 & \cdots & 0 & 0\\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0\\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1\\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma)\end{bmatrix}$$

and

$$= \begin{bmatrix} H_3(\chi,\gamma) & H_2(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

where $H_3(\chi,\gamma) = p^2(\chi,\gamma) + q(\chi,\gamma)$ and $H_2(\chi,\gamma) = p(\chi,\gamma)$. Similarly, by contracting $\mathcal{E}_n^{(1)}$ on the first column, we obtain

$$\mathcal{E}_n^{(2)} = \begin{bmatrix} p^3(\chi,\gamma) + 2p(\chi,\gamma)q(\chi,\gamma) & p^2(\chi,\gamma) + q(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

and

$$= \begin{bmatrix} H_4(\chi,\gamma) & H_3(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

where $H_4(\chi,\gamma) = p^3(\chi,\gamma) + 2p(\chi,\gamma)q(\chi,\gamma)$. Continuing this process iteratively, the $r$th contraction of $\mathcal{E}_n$ is obtained by

$$\mathcal{E}_n^{(r)} = \begin{bmatrix} H_{r+2}(\chi,\gamma) & H_{r+1}(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

for $3 \leq r \leq n-4$. Specifically, for $r = n-3$, the contraction gives

$$\mathcal{E}_n^{(n-3)} = \begin{bmatrix} H_{n-1}(\chi,\gamma) & H_{n-2}(\chi,\gamma) & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

By further contracting $\mathcal{E}_n^{(n-3)}$ on its first column and using equation (1), we obtain

$$\mathcal{E}_n^{(n-2)} = \begin{bmatrix} p(\chi,\gamma)H_{n-1}(\chi,\gamma) + q(\chi,\gamma)H_{n-2}(\chi,\gamma) & H_{n-1}(\chi,\gamma) \\ q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix} = \begin{bmatrix} H_n(\chi,\gamma) & H_{n-1}(\chi,\gamma) \\ q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

Finally, from equations (1) and (4), we have

$$\operatorname{per}(\mathcal{E}_n) = \operatorname{per}(\mathcal{E}_n^{(n-2)}) = p(\chi,\gamma)H_n(\chi,\gamma) + q(\chi,\gamma)H_{n-1}(\chi,\gamma) = H_{n+1}(\chi,\gamma).$$

Next, consider $\mathcal{F}_n = (\mathcal{E}_n)^T$. Since the permanent is invariant under transposition, $\operatorname{per}(\mathcal{E}_n) = \operatorname{per}(\mathcal{F}_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\operatorname{per}(\mathcal{E}_n) = \operatorname{per}(\mathcal{F}_n) = H_{n+1}(\chi,\gamma),$$

which completes the proof. □

The following theorem generalizes the previous result to the case where the matrix entries are complex.

**Theorem 3.2.** *Let $H_n(\chi,\gamma)$ be the nth generalized bivariate Fibonacci polynomial, and let $\mathcal{G}_n = (g_{rs})$ be an $n \times n$ tridiagonal matrix whose entries are defined by*

$$g_{rs} = \begin{cases} -i, & \text{if } r = s-1, \\ p(\chi,\gamma), & \text{if } r = s, \\ iq(\chi,\gamma), & \text{if } r = s+1, \\ 0, & \text{otherwise,} \end{cases}$$

*where $i = \sqrt{-1}$ and $p(\chi,\gamma)$, $q(\chi,\gamma)$ are polynomials with real coefficients. Explicitly,*

$$\mathcal{G}_n = \begin{bmatrix} p(\chi,\gamma) & -i & 0 & 0 & \cdots & 0 & 0 \\ i\,q(\chi,\gamma) & p(\chi,\gamma) & -i & 0 & \cdots & 0 & 0 \\ 0 & i\,q(\chi,\gamma) & p(\chi,\gamma) & -i & \cdots & 0 & 0 \\ 0 & 0 & i\,q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & -i \\ 0 & 0 & 0 & 0 & \cdots & i\,q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

*Let $\mathcal{T}_n = (\mathcal{G}_n)^T$. Then, for $n \geq 1$, it holds that*

$$\text{per}(\mathcal{G}_n) = \text{per}(\mathcal{T}_n) = H_{n+1}(\chi,\gamma).$$

*Proof.* We prove the result by mathematical induction on $n$. We first show that $\text{per}(\mathcal{G}_n) = H_{n+1}(\chi,\gamma)$ by computing all permanents using the Laplace expansion with respect to the first row for $n \geq 1$. The result is true for $n = 1, 2$. Now, assume that it is true for $n \in \mathbb{Z}^+$, namely, $\text{per}(\mathcal{G}_n) = H_{n+1}(\chi,\gamma)$. We show that it is true for $n+1$. By applying the Laplace expansion for permanents with the first row of $\mathcal{G}_{n+1}$, we obtain

$$\text{per}(\mathcal{G}_{n+1}) = p(\chi,\gamma)\,\text{per}\begin{bmatrix} p(\chi,\gamma) & -i & 0 & \cdots & 0 \\ iq(\chi,\gamma) & p(\chi,\gamma) & -i & \cdots & 0 \\ 0 & iq(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) \end{bmatrix}_{n\times n} + (-i)\,\text{per}\begin{bmatrix} iq(\chi,\gamma) & -i & 0 & \cdots & 0 \\ 0 & p(\chi,\gamma) & -i & \cdots & 0 \\ 0 & iq(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) \end{bmatrix}_{n\times n}$$

By expanding the second permanent with its first column, we obtain

$$\begin{aligned} \text{per}(\mathcal{G}_{n+1}) &= p(\chi,\gamma)\,\text{per}(\mathcal{G}_n) + (-i)(iq(\chi,\gamma))\,\text{per}\begin{bmatrix} p(\chi,\gamma) & -i & \cdots & 0 \\ iq(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(\chi,\gamma) \end{bmatrix}_{(n-1)\times(n-1)} \\ &= p(\chi,\gamma)\,\text{per}(\mathcal{G}_n) + q(\chi,\gamma)\,\text{per}(\mathcal{G}_{n-1}). \end{aligned}$$

By the inductive hypothesis and using equation (1), it follows that

$$\text{per}(\mathcal{G}_{n+1}) = p(\chi,\gamma)H_{n+1}(\chi,\gamma) + q(\chi,\gamma)H_n(\chi,\gamma) = H_{n+2}(\chi,\gamma).$$

Next, consider $\mathcal{T}_n = (\mathcal{G}_n)^T$. Since the permanent is invariant under transposition, $\text{per}(\mathcal{G}_n) = \text{per}(\mathcal{T}_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\text{per}(\mathcal{G}_n) = \text{per}(\mathcal{T}_n) = H_{n+1}(\chi,\gamma)$$

which completes the proof □

Next, we present the theorems that provide permanental representations of the generalized bivariate Lucas polynomials.

**Theorem 3.3.** *Let $K_n(\chi,\gamma)$ be the nth generalized bivariate Lucas polynomial, and let $\mathcal{U}_n = (u_{rs})$ be an $n \times n$ tridiagonal matrix whose entries are defined by*

$$u_{rs} = \begin{cases} 2, & \text{if } r = 1, s = 2, \\ 1, & \text{if } r = s-1 \text{ and } r \neq 1, \\ p(\chi,\gamma), & \text{if } r = s, \\ q(\chi,\gamma), & \text{if } r = s+1, \\ 0, & \text{otherwise,} \end{cases}$$

*where $p(\chi,\gamma)$ and $q(\chi,\gamma)$ are polynomials with real coefficients. Explicitly,*

$$\mathcal{U}_n = \begin{bmatrix} p(\chi,\gamma) & 2 & 0 & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & 0 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

*Let $\mathcal{V}_n = (\mathcal{U}_n)^T$. Then, for $n \geq 1$, it holds that*

$$\text{per}(\mathcal{U}_n) = \text{per}(\mathcal{V}_n) = K_n(\chi,\gamma).$$

*Proof.* We first demonstrate that $\text{per}(\mathcal{U}_n) = K_n(\chi,\gamma)$ for $n \geq 1$ by employing the matrix contraction method. Let $\mathcal{U}_n^{(r)}$ denote the $r$th contraction of $\mathcal{U}_n$ for $1 \leq r \leq n-2$. By contracting $\mathcal{U}_n$ on its first column, we obtain

$$\mathcal{U}_n^{(1)} = \begin{bmatrix} p^2(\chi,\gamma) + 2q(\chi,\gamma) & p(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

and

$$= \begin{bmatrix} K_2(\chi,\gamma) & K_1(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

where $K_2(\chi,\gamma) = p^2(\chi,\gamma) + 2q(\chi,\gamma)$ and $K_1(\chi,\gamma) = p(\chi,\gamma)$. Similarly, by contracting $\mathcal{U}_n^{(1)}$ on its first column, we get

$$\mathcal{U}_n^{(2)} = \begin{bmatrix} p^3(\chi,\gamma) + 3p(\chi,\gamma)q(\chi,\gamma) & p^2(\chi,\gamma) + 2q(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

and

$$= \begin{bmatrix} K_3(\chi,\gamma) & K_2(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

where $K_3(\chi,\gamma) = p^3(\chi,\gamma) + 3p(\chi,\gamma)q(\chi,\gamma)$. Continuing this process iteratively, the $r$th contraction of $\mathcal{U}_n$ is obtained by

$$\mathcal{U}_n^{(r)} = \begin{bmatrix} K_{r+1}(\chi,\gamma) & K_r(\chi,\gamma) & 0 & \cdots & 0 & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 & \cdots & 0 & 0 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) & 1 \\ 0 & 0 & 0 & \cdots & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}$$

for $3 \le r \le n-4$. Specifically, for $r = n-3$, the contraction gives

$$\mathcal{U}_n^{(n-3)} = \begin{bmatrix} K_{n-2}(\chi,\gamma) & K_{n-3}(\chi,\gamma) & 0 \\ q(\chi,\gamma) & p(\chi,\gamma) & 1 \\ 0 & q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

By further contracting $\mathcal{U}_n^{(n-3)}$ on its first column and using equation (2), we obtain

$$\mathcal{U}_n^{(n-2)} = \begin{bmatrix} p(\chi,\gamma)K_{n-2}(\chi,\gamma) + q(\chi,\gamma)K_{n-3}(\chi,\gamma) & K_{n-2}(\chi,\gamma) \\ q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix} = \begin{bmatrix} K_{n-1}(\chi,\gamma) & K_{n-2}(\chi,\gamma) \\ q(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

Finally, from equations (2) and (4), we have

$$\mathrm{per}(\mathcal{U}_n) = \mathrm{per}(\mathcal{U}_n^{(n-2)}) = p(\chi,\gamma)K_{n-1}(\chi,\gamma) + q(\chi,\gamma)K_{n-2}(\chi,\gamma) = K_n(\chi,\gamma).$$

Next, consider $\mathcal{V}_n = (\mathcal{U}_n)^T$. Since the permanent is invariant under transposition, $\mathrm{per}(\mathcal{U}_n) = \mathrm{per}(\mathcal{V}_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\mathrm{per}(\mathcal{U}_n) = \mathrm{per}(\mathcal{V}_n) = K_n(\chi,\gamma),$$

which completes the proof. □

The following theorem generalizes the previous result to the case where the matrix entries are complex.

**Theorem 3.4.** *Let $K_n(\chi,\gamma)$ be the nth generalized bivariate Lucas polynomial, and let $\mathcal{W}_n = (w_{rs})$ be an $n \times n$ tridiagonal matrix whose entries are defined by*

$$w_{rs} = \begin{cases} -2i, & \text{if } r = 1, s = 2, \\ -i, & \text{if } r = s-1 \text{ and } r \neq 1, \\ p(\chi,\gamma), & \text{if } r = s, \\ iq(\chi,\gamma), & \text{if } r = s+1, \\ 0, & \text{otherwise}, \end{cases}$$

*where* $i = \sqrt{-1}$ *and* $p(\chi,\gamma)$ *and* $q(\chi,\gamma)$ *are polynomials with real coefficients. Explicitly,*

$$\mathcal{W}_n = \begin{bmatrix} p(\chi,\gamma) & -2i & 0 & 0 & \cdots & 0 & 0 \\ iq(\chi,\gamma) & p(\chi,\gamma) & -i & 0 & \cdots & 0 & 0 \\ 0 & iq(\chi,\gamma) & p(\chi,\gamma) & -i & \cdots & 0 & 0 \\ 0 & 0 & iq(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p(\chi,\gamma) & -i \\ 0 & 0 & 0 & 0 & \cdots & iq(\chi,\gamma) & p(\chi,\gamma) \end{bmatrix}.$$

*Let* $\mathcal{Z}_n = (\mathcal{W}_n)^T$*. Then, for* $n \geq 1$*, it holds that*

$$\text{per}(\mathcal{W}_n) = \text{per}(\mathcal{Z}_n) = K_n(\chi,\gamma).$$

*Proof.* We prove the result by mathematical induction on $n$. We first show that $\text{per}(\mathcal{W}_n) = K_n(\chi,\gamma)$ by computing all permanents using the Laplace expansion with respect to the first row for $n \geq 1$. The result is true for $n = 1, 2$. Now, assume that it is true for $n \in \mathbb{Z}^+$, namely, $\text{per}(\mathcal{W}_n) = K_n(\chi,\gamma)$. We show that it holds for $n+1$ by utilizing the permanental representation of the generalized bivariate Fibonacci polynomials established in Theorem 3.2. By applying the Laplace expansion for permanents with the first row of $\mathcal{W}_{n+1}$, we obtain

$$\text{per}(\mathcal{W}_{n+1}) = p(\chi,\gamma)\,\text{per}\begin{bmatrix} p(\chi,\gamma) & -i & 0 & \cdots & 0 \\ iq(\chi,\gamma) & p(\chi,\gamma) & -i & \cdots & 0 \\ 0 & iq(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) \end{bmatrix}_{n\times n} + (-2i)\,\text{per}\begin{bmatrix} iq(\chi,\gamma) & -i & 0 & \cdots & 0 \\ 0 & p(\chi,\gamma) & -i & \cdots & 0 \\ 0 & iq(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p(\chi,\gamma) \end{bmatrix}_{n\times n}$$

By expanding the second permanent with its first column and using Theorem 3.2, we obtain

$$\begin{aligned} \text{per}(\mathcal{W}_{n+1}) &= p(\chi,\gamma)\,\text{per}(\mathcal{G}_n) + (-2i)(iq(\chi,\gamma))\,\text{per}\begin{bmatrix} p(\chi,\gamma) & -i & \cdots & 0 \\ iq(\chi,\gamma) & p(\chi,\gamma) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(\chi,\gamma) \end{bmatrix}_{(n-1)\times(n-1)} \\ &= p(\chi,\gamma)\,\text{per}(\mathcal{G}_n) + 2q(\chi,\gamma)\,\text{per}(\mathcal{G}_{n-1}). \end{aligned}$$

By using Theorem 3.2 and the relationship in equation (3), it follows that

$$\begin{aligned} \text{per}(\mathcal{W}_{n+1}) &= p(\chi,\gamma)H_{n+1}(\chi,\gamma) + 2q(\chi,\gamma)H_n(\chi,\gamma) \\ &= H_{n+2}(\chi,\gamma) + q(\chi,\gamma)H_n(\chi,\gamma) \\ &= K_{n+1}(\chi,\gamma). \end{aligned}$$

Next, consider $\mathcal{Z}_n = (\mathcal{W}_n)^T$. Since the permanent is invariant under transposition, $\text{per}(\mathcal{W}_n) = \text{per}(\mathcal{Z}_n)$ is directly obtained. By combining this with the previously established result, we obtain

$$\text{per}(\mathcal{W}_n) = \text{per}(\mathcal{Z}_n) = K_n(\chi,\gamma)$$

which completes the proof. □

## 4. Conclusion and Suggestion

In this study, generalized bivariate Fibonacci and Lucas polynomial sequences are investigated via defined tridiagonal matrices. Using matrix-based methods, determinantal and permanental representations for these sequences are established. It is shown that the determinants and permanents of these matrices give the generalized bivariate Fibonacci and Lucas polynomials, thereby extending the Fibonacci and Lucas polynomials in both univariate and bivariate forms and generalizing several well-known results in the literature.

It would be interesting to explore multivariate generalizations, their applications in combinatorics, graph theory, and coding theory, and further study of determinantal and permanental properties of related polynomial sequences.

## References

[1] Horadam, A. F. (1961). A generalized Fibonacci sequence. *The American Mathematical Monthly*, 68(5), 455–459.
[2] Çağman, N. (2021). Repdigits as product of Fibonacci and Pell numbers. *Turkish Journal of Science*, 6(1), 31–35.
[3] Taşyurdu, Y. (2022). Generalized Fibonacci numbers with five parameters. *Thermal Science*, 26(2), 495–505.
[4] Hoggatt Jr., V. E., Bicknell, M. (1973). Generalized Fibonacci Polynomials and Zeckendorf's Theorem. *The Fibonacci Quarterly*, 11(4), 399–419.
[5] Hoggatt Jr., V. E., Long, C. T. (1974). Divisibility properties of generalized Fibonacci polynomials. *The Fibonacci Quarterly*, 12, 113–120.
[6] Koçer, E. G., Tunçez, S. (2016). Bivariate Fibonacci and Lucas like polynomials. *Gazi University Journal of Science*, 29(1), 109–113.
[7] Panwar, Y. K., Gupta, V. K., Bhandari, J. (2020). Generalized identities of bivariate Fibonacci and bivariate Lucas polynomials. *Journal of Amasya University, Institute of Science and Technology*, 1(2), 146–154.
[8] Yılmaz, N. (2024). Some new results for the generalized bivariate Fibonacci and Lucas polynomials. *Gazi University Journal of Science*, 28(1), 97–108.
[9] Cahill, N. D., D'Errico, J. R., Narayan, D. A., Narayan, J. Y. (2002). Fibonacci determinants. *The College Mathematics Journal*, 33(3), 221–225.
[10] Minc, H. (1978). *Permanents*, Vol. 6, Encyclopedia of Mathematics and its Applications. London: Addison-Wesley Publishing Company.
[11] Serre, D. (2002). *Matrices: Theory and Applications*. New York: Springer-Verlag.
[12] Ocal, A. A., Tuglu, N., Altinisik, E. (2005). On the representation of $k$-generalized Fibonacci and Lucas numbers. *Applied Mathematics and Computation*, 170(1), 584–596.
[13] Brualdi, R. A., Gibson, P. M. (1977). Convex polyhedra of doubly stochastic matrices. I. Applications of the permanent function. *Journal of Combinatorial Theory, Series A*, 22(2), 194–230.
[14] Cahill, N. D., Narayan, D. A. (2004). Fibonacci and Lucas numbers as Tridiagonal matrix determinants. *The Fibonacci Quarterly*, 42(3), 216–221.
[15] Kılıç, E., Taşçı, D. (2007). On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers. *Rocky Mountain Journal of Mathematics*, 37(6), 1953–1969.
[16] Kılıç, E., Stakhov, A. P. (2009). On the Fibonacci and Lucas $p$-numbers, their sums, families of bipartite graphs and permanents of certain matrices. *Chaos, Solitons & Fractals*, 40(5), 2210–2221.
[17] Öteleş, A. (2016). On the connections between the permanents of some Tridiagonal matrices and the Pell and Jacobsthal numbers. *International Journal of Pure and Applied Mathematics*, 111(3), 361–372.
[18] Taşyurdu, Y., Dilmen, Z. (2017). On period of generalized Fibonacci sequence over finite ring and tridiagonal matrix. *Celal Bayar University Journal of Science*, 13(1), 165–169.
[19] Gültekin, I., Taşyurdu, Y. (2014). Determinantal and permanental representation of $q$-Fibonacci polynomials. *QScience Connect*, 2014(1), 25.
[20] Gültekin, I., Taşyurdu, Y. (2014). On the Pell polynomials $P_n(x,s,q)$ and Tridiogonal matrix. *Research and Reviews: Discrete Mathematical Structures*, 1(1), 24–27.