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Steffensen-Type Integral Inequalities for Functions with s-Convex Absolute n-th Derivatives

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Abstract. In this paper, we establish new Steffensen-type integral inequalities for functions whose absolute nth derivatives are s-convex in the second sense. By deriving an appropriate integral representation and employing inequalities such as Hölder, power-mean, and Young, we obtain several explicit upper bounds expressed in terms of endpoint values of the absolute n-th derivative of f. The obtained results generalize Steffensen-type inequalities to the s-convex framework and provide a unified treatment of different analytical techniques within this setting.

1. INTRODUCTION

Let $I \subseteq \mathbb{R}$ be a nonempty interval. For a fixed parameter $s \in (0, 1]$, a function $f : I \to \mathbb{R}$ is said to be *s-convex in the second sense* if for all $x_1, x_2 \in I$ and for all $t \in [0, 1]$, the inequality

$$f(tx_1 + (1-t)x_2) \leq t^s f(x_1) + (1-t)^s f(x_2) \tag{1}$$

holds. If the inequality in (1) is reversed, then f is said to be s-concave.

The classical Hermite–Hadamard inequality admits the following generalization for s-convex functions. If $f : I \to \mathbb{R}$ is s-convex in the second sense and f is integrable on $[a, b] \subset I$, then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{2}$$

For further insights into s-convex functions and related variants of the Hermite–Hadamard inequality, the reader is referred to the following references: [2–6, 15].

In 1918, J. F. Steffensen established the celebrated inequality given in [1]

Theorem 1.1. *Suppose that f and g are integrable functions defined on (a, b), f is decreasing and for each $t \in (a, b)$, $0 \leq g(t) \leq 1$. Then, the following inequality*

$$\int_{b-\lambda}^{b} f(t)\, dt \leq \int_a^b f(t)\, g(t)\, dt \leq \int_a^{a+\lambda} f(t)\, dt \tag{3}$$

holds, where $\lambda = \int_a^b g(t)\, dt$.

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The Steffensen inequality has since become a central topic of investigation, and numerous generalizations and related developments can be found in [8–14].

In [7] , Alomari derived the following Steffensen-type identities by employing an integral representation introduced in [12].

Theorem 1.2. *[7]Let $f, g : [a,b] \subset \mathbb{R}^{+} \to \mathbb{R}$ be integrable such that $0 \leq g(t) \leq 1$, for all $t \in [a,b]$ such that $\int_a^b g(t) f'(t)\,dt$ exists. If f is absolutely continuous on $[a,b]$ such that $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1)$ then we have*

$$\begin{aligned} & \left| \int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)\,g(t)\,dt \right| \\ \leq\ & \frac{1}{(s+1)(s+2)} \left[\lambda^2 \left| f'(a) \right| + (b-a-\lambda)^2 \left| f'(b) \right| \right] \\ +\ & \frac{1}{s+2} \left[\lambda^2 + (b-a-\lambda)^2 \right] f'(a+\lambda) \end{aligned} \tag{4}$$

and

$$\begin{aligned} & \left| \int_a^{b} f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt \right| \\ \leq\ & \frac{1}{(s+1)(s+2)} \left[\lambda^2 \left| f'(b) \right| + (b-a-\lambda)^2 \left| f'(a) \right| \right] \\ & + \frac{1}{s+2} \left[\lambda^2 + (b-a-\lambda)^2 \right] f'(b-\lambda). \end{aligned} \tag{5}$$

where $\lambda := \int_a^b g(t)\,dt$.

In [13], Ekinci et al. established the following four results for functions whose absolute value of the nth derivative is convex where $\lambda := \int_a^b g(t)\,dt$ and

$$\begin{aligned} T_n^1 &= \int_a^b g(x) \frac{(a+\lambda-x)^{n-1}}{(n-1)!} f^{(n-1)}(x)\,dx, \\ T_n^2 &= \int_a^b g(x) \frac{(b-\lambda-x)^{n-1}}{(n-1)!} f^{(n-1)}(x)\,dx, \\ I_n^1 &= \int_a^{a+\lambda} f(x)\,dx - \sum_{k=1}^{n-2} \frac{\lambda^k}{k!} f^{(k-1)}(a), \\ I_n^2 &= \int_{b-\lambda}^b f(x)\,dx + \sum_{k=1}^{n-1} \frac{(-\lambda)^k}{k!} f^{(k)}(b). \end{aligned}$$

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be differentiable n times and assume that $|f^{(n)}|$ is convex on $[a,b]$. Let $g : [a,b] \to [0,1]$ be integrable and set $\lambda = \int_a^b g(t)\,dt$. Then the following inequalities hold:*

$$\begin{aligned} & \left| \int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t) g(t)\,dt + \sum_{k=1}^{n-1} \left(I_k^1 - T_k^1 \right) \right| \\ \leq & \frac{n\lambda^{n+1}}{(n+2)!} |f^{(n)}(a)| + \frac{n(b-a-\lambda)^{n+1}}{(n+2)!} |f^{(n)}(b)| \\ & + \frac{2\left(\lambda^{n+1} + (b-a-\lambda)^{n+1}\right)}{(n+2)!} |f^{(n)}(a+\lambda)|. \end{aligned} \tag{6}$$

and

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\left(T_k^2 - I_k^2\right)\right| \tag{7}$$

$$\leq \frac{n\lambda^{n+1}}{(n+2)!}\,|f^{(n)}(b)|\,|f^{(n)}(a+\lambda)| + \frac{n(b-a-\lambda)^{n+1}}{(n+2)!}\,|f^{(n)}(a)| + \frac{2\left(\lambda^{n+1} + (b-a-\lambda)^{n+1}\right)}{(n+2)!}. \tag{8}$$

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable and assume that $|f^{(n)}|^q$ is convex on $[a,b]$. Let $g : [a,b] \to [0,1]$ be integrable and define $\lambda = \int_a^b g(t)\,dt$. Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. Then the following two inequalities hold:*

$$\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right)\right|$$
$$\leq \frac{\lambda^{n+1}}{(n-1)!\,2^{1/q}}\left(\frac{p!\,(p(n-1))!}{(pn+1)!}\right)^{1/p}\left(|f^{(n)}(a)|^q + |f^{(n)}(a+\lambda)|^q\right)^{1/q} \tag{9}$$
$$+ \frac{(b-a-\lambda)^{n+1}}{(n-1)!\,2^{1/q}}\left(\frac{p!\,(p(n-1))!}{(pn+1)!}\right)^{1/p}\left(|f^{(n)}(a+\lambda)|^q + |f^{(n)}(b)|^q\right)^{1/q}$$

and

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\left(T_k^2 - I_k^2\right)\right|$$
$$\leq \frac{(b-a-\lambda)^{n+1}}{(n-1)!\,2^{1/q}}\left(\frac{p!\,(p(n-1))!}{(pn+1)!}\right)^{1/p}\left(|f^{(n)}(a)|^q + |f^{(n)}(b-\lambda)|^q\right)^{1/q} \tag{10}$$
$$+ \frac{\lambda^{n+1}}{(n-1)!\,2^{1/q}}\left(\frac{p!\,(p(n-1))!}{(pn+1)!}\right)^{1/p}\left(|f^{(n)}(b-\lambda)|^q + |f^{(n)}(b)|^q\right)^{1/q}.$$

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable and let $g : [a,b] \to [0,1]$ be integrable. Define $\lambda := \int_a^b g(t)\,dt$,and assume that $|f^{(n)}|^q$ is convex on $[a,b]$ for some $q \geq 1$. Then the following inequalities hold:*

$$\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right)\right|$$
$$\leq \frac{\lambda^{n+1}}{(n+1)!}\left[\frac{n}{n+2}\left|f^{(n)}(a)\right|^q + \frac{2}{n+2}\left|f^{(n)}(a+\lambda)\right|^q\right]^{1/q}$$
$$+\frac{(b-a-\lambda)^{n+1}}{(n+1)!}\left[\frac{n}{n+2}\left|f^{(n)}(a+\lambda)\right|^q + \frac{2}{n+2}\left|f^{(n)}(b)\right|^q\right]^{1/q} \tag{11}$$

and

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\left(T_k^2 - I_k^2\right)\right|$$
$$\leq \frac{(b-a-\lambda)^{n+1}}{(n+1)!}\left[\frac{n}{n+2}\left|f^{(n)}(a)\right|^q + \frac{2}{n+2}\left|f^{(n)}(b-\lambda)\right|^q\right]^{1/q}$$
$$+\frac{\lambda^{n+1}}{(n+1)!}\left[\frac{n}{n+2}\left|f^{(n)}(b-\lambda)\right|^q + \frac{2}{n+2}\left|f^{(n)}(b)\right|^q\right]^{1/q}. \tag{12}$$

Theorem 1.6. *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable and let $g : [a,b] \to [0,1]$ be integrable. Define $\lambda := \int_a^b g(t)\,dt,\qquad \mu := b-a-\lambda$,and let $q > 1$ and $p = \frac{q}{q-1}$. Assume that $|f^{(n)}|^q$ is convex on $[a,b]$. Then the following inequalities hold:*

$$\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right)\right|$$
$$\leq \frac{1}{(n-1)!}\left\{\frac{\Gamma(p+1)\,\Gamma\big(p(n-1)+1\big)}{p\,\Gamma(pn+2)}\left(\lambda^{pn+1}+\mu^{pn+1}\right)\right.$$
$$\left.+\frac{1}{2q}\left[\lambda\left(\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(a+\lambda)\right|^q\right) + \mu\left(\left|f^{(n)}(a+\lambda)\right|^q + \left|f^{(n)}(b)\right|^q\right)\right]\right\} \tag{13}$$

and

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\left(T_k^2 - I_k^2\right)\right|$$
$$\leq \frac{1}{(n-1)!}\left\{\frac{\Gamma(p+1)\,\Gamma\big(p(n-1)+1\big)}{p\,\Gamma(pn+2)}\left(\lambda^{pn+1}+\mu^{pn+1}\right)\right.$$
$$\left.+\frac{1}{2q}\left[\mu\left(\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b-\lambda)\right|^q\right) + \lambda\left(\left|f^{(n)}(b-\lambda)\right|^q + \left|f^{(n)}(b)\right|^q\right)\right]\right\}. \tag{14}$$

The Gamma and Beta functions will play a fundamental role in the analysis carried out in this work. For $x > 0$, the Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,$$

and serves as a continuous extension of the factorial, since $\Gamma(n) = (n-1)!$ for every positive integer n. The Beta function is defined through

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt, \qquad \alpha,\beta > 0,$$

and is linked to the Gamma function via the classical relation

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

These special functions arise naturally in the evaluation of several kernel integrals, and they will be used throughout the sequel to present our results in a clear and compact manner.

2. MAIN RESULTS

In order to obtain the results of this section, we require the following lemma introduced by Ekinci et al. [13].

Lemma 2.1. *Let $f, g : [a, b] \subset \mathbb{R}^{+} \to \mathbb{R}$ be integrable such that $0 \le g(t) \le 1$ for all $t \in [a, b]$ and $\int_a^b g(t) f^{(k)}(t)\, dt$ exists for $k = 1, 2, \ldots, n$. Then, we have the following identities:*

$$\begin{aligned} &\int_a^{a+\lambda} f(t)\, dt - \int_a^b f(t) g(t)\, dt + \sum_{k=1}^{n-1} \left(I_k^1 - T_k^1 \right) \\ &= - \int_a^{a+\lambda} \left(\int_a^x (1 - g(t))\, dt \right) \frac{(a + \lambda - x)^{n-1}}{(n-1)!} f^{(n)}(x)\, dx \\ &\quad - \int_{a+\lambda}^b \left(\int_x^b g(t)\, dt \right) \frac{(a + \lambda - x)^{n-1}}{(n-1)!} f^{(n)}(x)\, dx, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\int_a^b f(t) g(t)\, dt - \int_{b-\lambda}^b f(t)\, dt + \sum_{k=1}^{n-1} \left(T_k^2 - I_k^2 \right) \\ &= - \int_a^{b-\lambda} \left(\int_a^x g(t)\, dt \right) \frac{(b - \lambda - x)^{n-1}}{(n-1)!} f^{(n)}(x)\, dx \\ &\quad - \int_{b-\lambda}^b \left(\int_x^b (1 - g(t))\, dt \right) \frac{(b - \lambda - x)^{n-1}}{(n-1)!} f^{(n)}(x)\, dx, \end{aligned} \tag{16}$$

where

$$\lambda := \int_a^b g(t)\, dt$$

and

$$\begin{aligned} T_n^1 &= \int_a^b g(x) \frac{(a + \lambda - x)^{n-1}}{(n-1)!} f^{(n-1)}(x)\, dx, \\ T_n^2 &= \int_a^b g(x) \frac{(b - \lambda - x)^{n-1}}{(n-1)!} f^{(n-1)}(x)\, dx, \\ I_n^1 &= \int_a^{a+\lambda} f(x)\, dx - \sum_{k=1}^{n-2} \frac{\lambda^k}{k!} f^{(k-1)}(a), \\ I_n^2 &= \int_{b-\lambda}^b f(x)\, dx + \sum_{k=1}^{n-1} \frac{(-\lambda)^k}{k!} f^{(k)}(b). \end{aligned}$$

Throughout the paper, the notations T_n^1, T_n^2, I_n^1 and I_n^2 will be used as defined above.

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be n-times differentiable on $[a, b]$. Assume that $|f^{(n)}|$ is s–convex in the second sense on $[a, b]$ for some $s \in (0, 1]$. Let $g : [a, b] \to [0, 1]$ be integrable and set $\lambda = \int_a^b g(t)\, dt$, $\mu = b - a - \lambda$. Then, the* *following inequalities hold:*

$$\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right)\right|$$
$$\leq \frac{\lambda^{n+1}}{(n-1)!(n+s)(n+s+1)}\,|f^{(n)}(a)| + \frac{\Gamma(s+2)}{\Gamma(n+s+2)}\left(\lambda^{n+1} + \mu^{n+1}\right)|f^{(n)}(a+\lambda)| \tag{17}$$
$$+ \frac{\mu^{n+1}}{(n-1)!(n+s)(n+s+1)}\,|f^{(n)}(b)|,$$

and

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\left(T_k^2 - I_k^2\right)\right|$$
$$\leq \frac{\lambda^{n+1}}{(n-1)!(n+s)(n+s+1)}\,|f^{(n)}(b)| + \frac{\Gamma(s+2)}{\Gamma(n+s+2)}\left(\lambda^{n+1} + \mu^{n+1}\right)|f^{(n)}(a+\lambda)| \tag{18}$$
$$+ \frac{\mu^{n+1}}{(n-1)!(n+s)(n+s+1)}\,|f^{(n)}(a)|.$$

Proof. From Lemma 2.1, we have

$$L := \int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right) = K_n^1,$$

where

$$K_n^1 = -\int_a^{a+\lambda}\left(\int_a^x (1-g(t))\,dt\right)\frac{(x-a-\lambda)^{n-1}}{(n-1)!}\,f^{(n)}(x)\,dx$$
$$-\int_{a+\lambda}^b\left(\int_x^b g(t)\,dt\right)\frac{(x-a-\lambda)^{n-1}}{(n-1)!}\,f^{(n)}(x)\,dx.$$

Since $0 \leq g(t) \leq 1$, the inner integrals satisfy $0 \leq \int_a^x (1-g(t))\,dt \leq x-a$ and $0 \leq \int_x^b g(t)\,dt \leq b-x$. Hence taking absolute values and using the triangle inequality gives

$$|L| \leq I_1 + I_2,$$

where

$$I_1 = \frac{1}{(n-1)!}\int_a^{a+\lambda}(x-a)(a+\lambda-x)^{n-1}|f^{(n)}(x)|\,dx,$$

$$I_2 = \frac{1}{(n-1)!}\int_{a+\lambda}^b (b-x)(x-a-\lambda)^{n-1}|f^{(n)}(x)|\,dx.$$

Applying the substitutions

$$x = t(a+\lambda) + (1-t)a, \qquad x = kb + (1-k)(a+\lambda),$$

and using the s-convexity of $|f^{(n)}|$, we obtain

$$\begin{aligned}
|I_1| &\leq \frac{\lambda^{n+1}}{(n-1)!}\int_0^1 t(1-t)^{n-1}\Big((1-t)^s|f^{(n)}(a)| + t^s|f^{(n)}(a+\lambda)|\Big)\,dt \\
&= \frac{\lambda^{n+1}}{(n-1)!}\left[|f^{(n)}(a)|\int_0^1 t(1-t)^{n-1+s}\,dt + |f^{(n)}(a+\lambda)|\int_0^1 t^{s+1}(1-t)^{n-1}\,dt\right], \\
|I_2| &\leq \frac{\mu^{n+1}}{(n-1)!}\int_0^1 k^{n-1}(1-k)\Big((1-k)^s|f^{(n)}(a+\lambda)| + k^s|f^{(n)}(b)|\Big)\,dk \\
&= \frac{\mu^{n+1}}{(n-1)!}\left[|f^{(n)}(a+\lambda)|\int_0^1 k^{n-1}(1-k)^{1+s}\,dk + |f^{(n)}(b)|\int_0^1 k^{n-1+s}(1-k)\,dk\right].
\end{aligned}$$

Evaluating these integrals using the properties of the Beta and Gamma functions yields

$$\begin{aligned}
\int_0^1 t(1-t)^{n-1+s}\,dt &= B(2,n+s) = \frac{\Gamma(n+s)\Gamma(2)}{\Gamma(n+s+2)} = \frac{1}{(n+s)(n+s+1)}, \\
\int_0^1 t^{s+1}(1-t)^{n-1}\,dt &= B(s+2,n) = \frac{\Gamma(s+2)\Gamma(n)}{\Gamma(n+s+2)} = \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}, \\
\int_0^1 k^{n-1}(1-k)^{1+s}\,dk &= B(n,s+2) = \frac{\Gamma(n)\Gamma(s+2)}{\Gamma(n+s+2)} = \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}, \\
\int_0^1 k^{n-1+s}(1-k)\,dk &= B(n+s,2) = \frac{\Gamma(n+s)\Gamma(2)}{\Gamma(n+s+2)} = \frac{1}{(n+s)(n+s+1)}.
\end{aligned}$$

Inserting these values into the estimates of I_1 and I_2 yields inequality (17). The proof is completed. Inequality (18) follows by a similar argument. □

Remark 2.3. *Theorem 1.2 follows directly from Theorem 2.2 by taking $n = 1$.*

Remark 2.4. *Theorem 2.2 reduces to Theorem 1.3 for $s = 1$.*

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable on $[a,b]$. Assume that $|f^{(n)}|^q$ is s–convex in the second sense on $[a,b]$ for some $s \in (0,1]$, where $p > 1$ and $q = \frac{p}{p-1}$. Let $g : [a,b] \to [0,1]$ be integrable and set $\lambda = \int_a^b g(t)\,dt$, $\mu = b - a - \lambda$. Then, the following inequalities hold:*

$$\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right)\right| \\
\leq&\frac{B(p+1,p(n-1)+1)^{1/p}}{(n-1)!(s+1)^{1/q}} \\
\times&\left[\lambda^{n+1}\Big(|f^{(n)}(a)|^q + |f^{(n)}(a+\lambda)|^q\Big)^{1/q} + \mu^{n+1}\Big(|f^{(n)}(a+\lambda)|^q + |f^{(n)}(b)|^q\Big)^{1/q}\right],
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\left|\int_a^{b} f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\left(T_k^2 - I_k^2\right)\right| \\
\leq&\frac{B(p+1,p(n-1)+1)^{1/p}}{(n-1)!(s+1)^{1/q}} \\
\times&\left[\lambda^{n+1}\Big(|f^{(n)}(b)|^q + |f^{(n)}(b-\lambda)|^q\Big)^{1/q} + \mu^{n+1}\Big(|f^{(n)}(b-\lambda)|^q + |f^{(n)}(a)|^q\Big)^{1/q}\right],
\end{aligned} \tag{20}$$

where

$$B(p+1, p(n-1)+1) = \frac{p!\,(p(n-1))!}{(pn+1)!}.$$

Proof. Invoking the first identity in Lemma 2.1, we may express

$$|L| \leq I_1 + I_2,$$

where

$$I_1 = \frac{1}{(n-1)!}\int_a^{a+\lambda} (x-a)(a+\lambda-x)^{n-1}|f^{(n)}(x)|\,dx,$$

$$I_2 = \frac{1}{(n-1)!}\int_{a+\lambda}^{b} (b-x)(x-a-\lambda)^{n-1}|f^{(n)}(x)|\,dx.$$

Employing Hölder's inequality with conjugate exponents p and $q = \frac{p}{p-1}$, we obtain

$$I_1 \leq \frac{1}{(n-1)!}\left(\int_a^{a+\lambda} \left[(x-a)(a+\lambda-x)^{n-1}\right]^p dx\right)^{1/p} \left(\int_a^{a+\lambda} |f^{(n)}(x)|^q\,dx\right)^{1/q},$$

$$I_2 \leq \frac{1}{(n-1)!}\left(\int_{a+\lambda}^{b} \left[(b-x)(x-a-\lambda)^{n-1}\right]^p dx\right)^{1/p} \left(\int_{a+\lambda}^{b} |f^{(n)}(x)|^q\,dx\right)^{1/q}.$$

Now, applying the changes of variables

$$x = a+\lambda t, \qquad x = a+\lambda+\mu t,$$

we arrive at

$$\int_a^{a+\lambda} \left[(x-a)(a+\lambda-x)^{n-1}\right]^p dx = \lambda^{pn+1}B(p+1, p(n-1)+1),$$

$$\int_{a+\lambda}^{b} \left[(b-x)(x-a-\lambda)^{n-1}\right]^p dx = \mu^{pn+1}B(p+1, p(n-1)+1),$$

where

$$B(p+1, p(n-1)+1) = \frac{p!\,(p(n-1))!}{(pn+1)!}.$$

Next, we use the s–convexity of $|f^{(n)}|^q$. For $x = a+\lambda t$ with $t \in [0,1]$, we have

$$|f^{(n)}(a+\lambda t)|^q = |f^{(n)}\Big((1-t)a + t(a+\lambda)\Big)|^q \leq (1-t)^s|f^{(n)}(a)|^q + t^s|f^{(n)}(a+\lambda)|^q.$$

Integrating over $t \in [0,1]$ and using

$$\int_0^1 t^s\,dt = \int_0^1 (1-t)^s\,dt = \frac{1}{s+1},$$

we obtain

$$\int_a^{a+\lambda} |f^{(n)}(x)|^q\,dx = \lambda\int_0^1 |f^{(n)}(a+\lambda t)|^q\,dt \leq \frac{\lambda}{s+1}\Big(|f^{(n)}(a)|^q + |f^{(n)}(a+\lambda)|^q\Big).$$

Similarly, for $x = a+\lambda+\mu t$ with $t \in [0,1]$,

$$|f^{(n)}(a+\lambda+\mu t)|^q = |f^{(n)}\Big((1-t)(a+\lambda) + tb\Big)|^q \leq (1-t)^s|f^{(n)}(a+\lambda)|^q + t^s|f^{(n)}(b)|^q,$$

and hence

$$\int_{a+\lambda}^{b} |f^{(n)}(x)|^q\,dx = \mu\int_0^1 |f^{(n)}(a+\lambda+\mu t)|^q\,dt \le \frac{\mu}{s+1}\Big(|f^{(n)}(a+\lambda)|^q + |f^{(n)}(b)|^q\Big).$$

Therefore,

$$\begin{aligned}I_1 &\le \frac{1}{(n-1)!}\,\lambda^{n+\frac{1}{p}}\,B(p+1,p(n-1)+1)^{1/p}\left(\frac{\lambda}{s+1}\Big(|f^{(n)}(a)|^q + |f^{(n)}(a+\lambda)|^q\Big)\right)^{1/q}\\ &= \frac{\lambda^{n+1}}{(n-1)!}\,\frac{B(p+1,p(n-1)+1)^{1/p}}{(s+1)^{1/q}}\Big(|f^{(n)}(a)|^q + |f^{(n)}(a+\lambda)|^q\Big)^{1/q},\end{aligned}$$

and similarly,

$$\begin{aligned}I_2 &\le \frac{1}{(n-1)!}\,\mu^{n+\frac{1}{p}}\,B(p+1,p(n-1)+1)^{1/p}\left(\frac{\mu}{s+1}\Big(|f^{(n)}(a+\lambda)|^q + |f^{(n)}(b)|^q\Big)\right)^{1/q}\\ &= \frac{\mu^{n+1}}{(n-1)!}\,\frac{B(p+1,p(n-1)+1)^{1/p}}{(s+1)^{1/q}}\Big(|f^{(n)}(a+\lambda)|^q + |f^{(n)}(b)|^q\Big)^{1/q}.\end{aligned}$$

Combining these estimates with $|L| \le I_1 + I_2$ yields the desired inequality.
Inequality (20) can be proved by following an analogous argument. □

Remark 2.6. *Theorem 2.5 reduces to Theorem 1.4 for $s = 1$.*

Theorem 2.7. *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable on $[a,b]$. Assume that $|f^{(n)}|^q$ is s–convex in the second sense on $[a,b]$ for some $s \in (0,1]$, where $q \ge 1$. Let $g : [a,b] \to [0,1]$ be integrable and set $\lambda = \int_a^b g(t)\,dt$, $\quad \mu = b-a-\lambda$. Then, the following inequalities hold:*

$$\begin{aligned}&\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\Big(I_k^1 - T_k^1\Big)\right|\\ &\le \frac{1}{(n-1)!}\left(\frac{1}{n(n+1)}\right)^{1-\frac{1}{q}}\left\{\lambda^{n+1}\left[\frac{|f^{(n)}(a)|^q}{(n+s)(n+s+1)} + \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}|f^{(n)}(a+\lambda)|^q\right]^{1/q}\right.\\ &\left.+\,\mu^{n+1}\left[\frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}|f^{(n)}(a+\lambda)|^q + \frac{|f^{(n)}(b)|^q}{(n+s)(n+s+1)}\right]^{1/q}\right\},\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\left|\int_a^{b} f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\Big(T_k^2 - I_k^2\Big)\right|\\ &\le \frac{1}{(n-1)!}\left(\frac{1}{n(n+1)}\right)^{1-\frac{1}{q}}\left\{\lambda^{n+1}\left[\frac{|f^{(n)}(b)|^q}{(n+s)(n+s+1)} + \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}|f^{(n)}(b-\lambda)|^q\right]^{1/q}\right.\\ &\left.+\,\mu^{n+1}\left[\frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}|f^{(n)}(b-\lambda)|^q + \frac{|f^{(n)}(a)|^q}{(n+s)(n+s+1)}\right]^{1/q}\right\}.\end{aligned} \tag{22}$$

Proof. From the first identity in Lemma 2.1 we may write

$$L := \int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\Big(I_k^1 - T_k^1\Big) = K_n^1.$$

Taking absolute values and using the representation of K_n^1 together with the triangle inequality, we obtain

$$|L| \le I_1 + I_2,$$

where

$$I_1 := \frac{1}{(n-1)!}\int_a^{a+\lambda}(x-a)(a+\lambda-x)^{n-1}\left|f^{(n)}(x)\right|dx,$$

$$I_2 := \frac{1}{(n-1)!}\int_{a+\lambda}^{b}(b-x)(x-a-\lambda)^{n-1}\left|f^{(n)}(x)\right|dx.$$

We begin with I_1. Define

$$g_1(x) = (x-a)(a+\lambda-x)^{n-1}, \qquad x \in [a, a+\lambda],$$

so that

$$I_1 = \frac{1}{(n-1)!}\int_a^{a+\lambda} g_1(x)\left|f^{(n)}(x)\right|dx.$$

By the power–mean inequality, for $q \geq 1$,

$$\int_a^{a+\lambda} g_1(x)\left|f^{(n)}(x)\right|dx \leq \left(\int_a^{a+\lambda} g_1(x)\,dx\right)^{1-\frac{1}{q}}\left(\int_a^{a+\lambda} g_1(x)\left|f^{(n)}(x)\right|^q dx\right)^{1/q},$$

and hence

$$I_1 \leq \frac{1}{(n-1)!}\left(\int_a^{a+\lambda} g_1(x)\,dx\right)^{1-\frac{1}{q}}\left(\int_a^{a+\lambda} g_1(x)\left|f^{(n)}(x)\right|^q dx\right)^{1/q}. \tag{23}$$

A direct calculation with $y = x - a$ gives

$$\int_a^{a+\lambda} g_1(x)\,dx = \int_0^{\lambda} y(\lambda - y)^{n-1}dy = \frac{\lambda^{n+1}}{n(n+1)}.$$

Moreover, with the change of variable $x = a + \lambda t$, we have

$$\int_a^{a+\lambda} g_1(x)\left|f^{(n)}(x)\right|^q dx = \lambda^{n+1}\int_0^1 t(1-t)^{n-1}\left|f^{(n)}(a+\lambda t)\right|^q dt.$$

Now assume that $\left|f^{(n)}\right|^q$ is s–convex in the second sense on $[a, b]$. Then for $t \in [0, 1]$,

$$\left|f^{(n)}(a+\lambda t)\right|^q = \left|f^{(n)}\big((1-t)a + t(a+\lambda)\big)\right|^q \leq (1-t)^s\left|f^{(n)}(a)\right|^q + t^s\left|f^{(n)}(a+\lambda)\right|^q.$$

Substituting this estimate into the above integral yields

$$\begin{aligned}\int_a^{a+\lambda} g_1(x)\left|f^{(n)}(x)\right|^q dx \leq \lambda^{n+1}\Bigg[&\left|f^{(n)}(a)\right|^q\int_0^1 t(1-t)^{n-1+s}\,dt\\ &+\left|f^{(n)}(a+\lambda)\right|^q\int_0^1 t^{s+1}(1-t)^{n-1}\,dt\Bigg].\end{aligned}$$

Evaluating these integrals via standard Beta–Gamma identities gives

$$\int_0^1 t(1-t)^{n-1+s}\,dt = \frac{1}{(n+s)(n+s+1)}, \int_0^1 t^{s+1}(1-t)^{n-1}\,dt = \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}.$$

Therefore,

$$\int_a^{a+\lambda} g_1(x)\left|f^{(n)}(x)\right|^q dx \leq \lambda^{n+1}\left[\frac{\left|f^{(n)}(a)\right|^q}{(n+s)(n+s+1)} + \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}\left|f^{(n)}(a+\lambda)\right|^q\right].$$

Combining this with (23) and $\int_a^{a+\lambda} g_1(x)\,dx = \frac{\lambda^{n+1}}{n(n+1)}$, we obtain

$$\begin{aligned}
I_1 &\le \frac{1}{(n-1)!}\left(\frac{\lambda^{n+1}}{n(n+1)}\right)^{1-\frac{1}{q}}\left(\lambda^{n+1}\left[\frac{\left|f^{(n)}(a)\right|^q}{(n+s)(n+s+1)} + \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}\left|f^{(n)}(a+\lambda)\right|^q\right]\right)^{1/q} \\
&= \frac{\lambda^{n+1}}{(n-1)!}\left(\frac{1}{n(n+1)}\right)^{1-\frac{1}{q}}\left[\frac{\left|f^{(n)}(a)\right|^q}{(n+s)(n+s+1)} + \frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}\left|f^{(n)}(a+\lambda)\right|^q\right]^{1/q}.
\end{aligned} \tag{24}$$

We next estimate I_2. Let

$$g_2(x) = (b-x)(x-a-\lambda)^{n-1}, \qquad x \in [a+\lambda, b],$$

so that

$$I_2 = \frac{1}{(n-1)!}\int_{a+\lambda}^b g_2(x)\left|f^{(n)}(x)\right| dx.$$

In an analogous manner, by applying the power–mean inequality to I_2 and invoking the s-convexity of $|f^{(n)}|^q$, we obtain

$$\begin{aligned}
I_2 &\le \frac{1}{(n-1)!}\left(\frac{\mu^{n+1}}{n(n+1)}\right)^{1-\frac{1}{q}}\left(\mu^{n+1}\left[\frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}\left|f^{(n)}(a+\lambda)\right|^q + \frac{\left|f^{(n)}(b)\right|^q}{(n+s)(n+s+1)}\right]\right)^{1/q} \\
&= \frac{\mu^{n+1}}{(n-1)!}\left(\frac{1}{n(n+1)}\right)^{1-\frac{1}{q}}\left[\frac{\Gamma(s+2)(n-1)!}{\Gamma(n+s+2)}\left|f^{(n)}(a+\lambda)\right|^q + \frac{\left|f^{(n)}(b)\right|^q}{(n+s)(n+s+1)}\right]^{1/q}.
\end{aligned} \tag{25}$$

Finally, combining $|L| \le I_1 + I_2$ with (24) and (25) yields the asserted estimate, and the proof is completed. Inequality (22) can be proved by following an analogous argument. □

Remark 2.8. *Theorem 2.7 reduces to Theorem 1.5 for $s = 1$.*

Theorem 2.9. *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable on $[a,b]$. Assume that $|f^{(n)}|^q$ is s–convex in the second sense on $[a,b]$ for some $s \in (0,1]$, where $q > 1$ and $p = \frac{q}{q-1}$. Let $g : [a,b] \to [0,1]$ be integrable and set $\lambda = \int_a^b g(t)\,dt$, $\mu = b-a-\lambda$. Then, the following inequalities hold:*

$$\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right)\right| \\
&\le \frac{1}{(n-1)!}\left[\frac{\lambda^{pn+1}}{p}B\Big(p+1, p(n-1)+1\Big) + \frac{\lambda}{q(s+1)}\Big(|f^{(n)}(a)|^q + |f^{(n)}(a+\lambda)|^q\Big)\right] \\
&+ \frac{1}{(n-1)!}\left[\frac{\mu^{pn+1}}{p}B\Big(p+1, p(n-1)+1\Big) + \frac{\mu}{q(s+1)}\Big(|f^{(n)}(a+\lambda)|^q + |f^{(n)}(b)|^q\Big)\right],
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{k=1}^{n-1}\left(T_k^2 - I_k^2\right)\right| \\
&\le \frac{1}{(n-1)!}\left[\frac{\lambda^{pn+1}}{p}B\Big(p+1, p(n-1)+1\Big) + \frac{\lambda}{q(s+1)}\Big(|f^{(n)}(b)|^q + |f^{(n)}(b-\lambda)|^q\Big)\right] \\
&+ \frac{1}{(n-1)!}\left[\frac{\mu^{pn+1}}{p}B\Big(p+1, p(n-1)+1\Big) + \frac{\mu}{q(s+1)}\Big(|f^{(n)}(b-\lambda)|^q + |f^{(n)}(a)|^q\Big)\right],
\end{aligned} \tag{27}$$

where

$$B\Big(p+1, p(n-1)+1\Big) = \frac{\Gamma(p+1)\Gamma(p(n-1)+1)}{\Gamma(pn+2)}.$$

Proof. Invoking the first equality in Lemma (2.1), we can write

$$L := \int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{k=1}^{n-1}\left(I_k^1 - T_k^1\right) = K_n^1.$$

Combining the integral representation of K_n^1 with the triangle inequality yields

$$|L| \le I_1 + I_2,$$

where

$$I_1 := \frac{1}{(n-1)!}\int_a^{a+\lambda} (x-a)(a+\lambda-x)^{n-1}\left|f^{(n)}(x)\right|dx,$$
$$I_2 := \frac{1}{(n-1)!}\int_{a+\lambda}^{b} (b-x)(x-a-\lambda)^{n-1}\left|f^{(n)}(x)\right|dx.$$

Define

$$A_1(x) = (x-a)(a+\lambda-x)^{n-1}, \qquad x \in [a, a+\lambda],$$
$$A_2(x) = (b-x)(x-a-\lambda)^{n-1}, \qquad x \in [a+\lambda, b].$$

Let $q > 1$ and $p = \frac{q}{q-1}$. By Young's inequality,

$$uv \le \frac{u^p}{p} + \frac{v^q}{q} \qquad (u, v \ge 0),$$

we have, for x in the corresponding intervals,

$$A_1(x)\left|f^{(n)}(x)\right| \le \frac{A_1(x)^p}{p} + \frac{\left|f^{(n)}(x)\right|^q}{q}, \qquad A_2(x)\left|f^{(n)}(x)\right| \le \frac{A_2(x)^p}{p} + \frac{\left|f^{(n)}(x)\right|^q}{q}.$$

Integrating these inequalities over $[a, a+\lambda]$ and $[a+\lambda, b]$, respectively, and multiplying by $\frac{1}{(n-1)!}$, we obtain

$$\begin{aligned} I_1 &\le \frac{1}{(n-1)!}\left[\frac{1}{p}\int_a^{a+\lambda} A_1(x)^p dx + \frac{1}{q}\int_a^{a+\lambda}\left|f^{(n)}(x)\right|^q dx\right], \\ I_2 &\le \frac{1}{(n-1)!}\left[\frac{1}{p}\int_{a+\lambda}^{b} A_2(x)^p dx + \frac{1}{q}\int_{a+\lambda}^{b}\left|f^{(n)}(x)\right|^q dx\right]. \end{aligned} \tag{28}$$

We next evaluate the p-power integrals of A_1 and A_2. Using the change of variable $x = a + \lambda t$ on $[a, a+\lambda]$, we find

$$\int_a^{a+\lambda} A_1(x)^p dx = \lambda^{pn+1}\int_0^1 t^p(1-t)^{p(n-1)}dt = \lambda^{pn+1}B(p+1, p(n-1)+1),$$

and with $x = a + \lambda + \mu t$ on $[a+\lambda, b]$, we similarly obtain

$$\int_{a+\lambda}^{b} A_2(x)^p dx = \mu^{pn+1}\int_0^1 t^{p(n-1)}(1-t)^p dt = \mu^{pn+1}B(p+1, p(n-1)+1),$$

where

$$B(p+1, p(n-1)+1) = \frac{\Gamma(p+1)\,\Gamma\big(p(n-1)+1\big)}{\Gamma(pn+2)}.$$

Finally, we estimate the remaining integrals by using the s–convexity of $\left|f^{(n)}\right|^q$. For $x = a + \lambda t$ with $t \in [0, 1]$,

$$\left|f^{(n)}(a+\lambda t)\right|^q = \left|f^{(n)}\Big((1-t)a + t(a+\lambda)\Big)\right|^q \le (1-t)^s\left|f^{(n)}(a)\right|^q + t^s\left|f^{(n)}(a+\lambda)\right|^q.$$

Integrating over $t \in [0, 1]$ and using $\int_0^1 t^s\,dt = \int_0^1 (1-t)^s\,dt = \frac{1}{s+1}$, we get

$$\int_a^{a+\lambda} \left|f^{(n)}(x)\right|^q dx = \lambda \int_0^1 \left|f^{(n)}(a+\lambda t)\right|^q dt \le \frac{\lambda}{s+1}\Big(\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(a+\lambda)\right|^q\Big).$$

Likewise, for $x = a + \lambda + \mu t$ with $t \in [0, 1]$,

$$\left|f^{(n)}(a+\lambda+\mu t)\right|^q = \left|f^{(n)}\Big((1-t)(a+\lambda) + tb\Big)\right|^q \le (1-t)^s\left|f^{(n)}(a+\lambda)\right|^q + t^s\left|f^{(n)}(b)\right|^q,$$

and hence

$$\int_{a+\lambda}^{b} \left|f^{(n)}(x)\right|^q dx = \mu \int_0^1 \left|f^{(n)}(a+\lambda+\mu t)\right|^q dt \le \frac{\mu}{s+1}\Big(\left|f^{(n)}(a+\lambda)\right|^q + \left|f^{(n)}(b)\right|^q\Big).$$

Substituting these estimates and the computed Beta integrals into (28), and then combining the resulting bounds with $|L| \le I_1 + I_2$, we obtain the desired inequality. This completes the proof.

The proof of inequality (27) is similar and therefore omitted. □

Remark 2.10. *Theorem 2.9 reduces to Theorem 1.6 for $s = 1$.*

Conclusion

In this study, a systematic framework is developed to derive Steffensen-type integral inequalities under the assumption that the absolute n-th derivative of the underlying function satisfies s-convexity in the second sense. By employing a suitable integral representation together with classical analytical tools such as Hölder, power–mean, and Young inequalities, we derived explicit upper bounds involving only endpoint values of the derivatives. The obtained results extend a variety of known Steffensen-type inequalities and recover several earlier results as special cases when $s = 1$ or $n = 1$. Moreover, the use of Beta and Gamma functions enabled us to express the bounds in a compact and unified form. The presented inequalities contribute to the growing literature on generalized convexity and provide a flexible framework for further investigations. Possible future work may include analogous results for other generalized convexity classes or fractional-type integral operators.

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