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Certain Subclasses of p- Valent Functions Defined by New Multiplier Transformations Associated with Borel Distribution Functions

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Abstract. In the paper, the new multiplier transformations $\mathcal{J}_p^\delta(\kappa,\lambda,\mu,l)$ $(0<\kappa\leq 1,\ \delta,l\geq 0,\ \lambda\geq\mu\geq 0;\ p\in\mathbb{N})$ of multivalent functions is defined. Making use of the operator $\mathcal{B}_{p,\kappa}^\delta(\lambda,\mu,l)$ two new subclasses $\mathcal{P}^\delta_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ and $\widetilde{\mathcal{P}}^\delta_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ of multivalent analytic functions are introduced and investigated in the open unit disk. Some interesting relations and characteristics such as neighborhoods, partial sums and quasi-convolution properties of functions belonging to each of these subclasses $\mathcal{P}^\delta_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ and $\widetilde{\mathcal{P}}^\delta_{\lambda,\mu,l}(A,B;\sigma,p)$ are investigated.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}(n,p)$ denote the class of functions normalized by

$$f(z)=z^p+\sum_{k=n+p}^{\infty}a_kz^k \quad (p,n\in\mathbb{N}:=\{1,2,3,...\}) \tag{1}$$

which are analytic and $p-valent$ in the open unit disk $\mathcal{U}=\{z:\ z\in\mathbb{C} \text{ and } |z|<1\}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then, we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0)=0$, $|w(z)|<1$ such that $f(z)=g(w(z))$ $(z\in\mathcal{U})$. We denote this subordination $f\prec g$ or $f(z)\prec g(z)$ $(z\in\mathcal{U})$. In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0)=g(0)$, $f(\mathcal{U})\subset g(\mathcal{U})$.

For $f\in\mathcal{A}(n,p)$ given by (1) and $g(z)$ given by:

$$g(z)=z^p+\sum_{k=n+p}^{\infty}b_kz^k \quad (p,n\in\mathbb{N}:=\{1,2,3,...\})$$

their convolution (or Hadamard product), denoted by $(f*g)$, is defined as

$$(f*g)(z):=z^p+\sum_{k=n+p}^{\infty}a_kb_kz^k=:(g*f)(z) \quad (z\in\mathcal{U}).$$

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Note that $f * g \in \mathcal{A}(n,p)$. In particular, we set

$$\mathcal{A}(p,1) := \mathcal{A}_p,\ \ \mathcal{A}(1,n) := \mathcal{A}(n),\ \ \mathcal{A}(1,1) := \mathcal{A}_1 = \mathcal{A}.$$

For a function f in $\mathcal{A}(n,p)$, Deniz and Orhan [1] defined the *multiplier transformations* $\mathcal{J}_p^{\delta}(\lambda,\mu,l)$ as follows:

Definition 1.1. *[1] Let $f \in \mathcal{A}(n,p)$. For the parameters $\delta,\lambda,\mu,l \in \mathbb{R}$, $\lambda \geq \mu \geq 0$ and $\delta, l \geq 0$ define the multiplier transformations $\mathcal{J}_p^{\delta}(\lambda,\mu,l)$ on $\mathcal{A}(n,p)$ by the following*

$$\mathcal{J}_p^{0}(\lambda,\mu,l)f(z) = f(z)$$

$$(p+l)\mathcal{J}_p^{1}(\lambda,\mu,l)f(z) = \lambda\mu z^2 f''(z) + (\lambda-\mu+(1-p)\lambda\mu)\, zf'(z) + (p(1-\lambda+\mu)+l)\, f(z)$$

$$\begin{aligned}(p+l)\mathcal{J}_p^{2}(\lambda,\mu,l)f(z) &= \lambda\mu z^2[\mathcal{J}_p^{1}(\lambda,\mu,l)f(z)]'' + (\lambda-\mu+(1-p)\lambda\mu)\, z[\mathcal{J}_p^{1}(\lambda,\mu,l)f(z)]' \\ &\quad + (p(1-\lambda+\mu)+l)\,\mathcal{J}_p^{1}(\lambda,\mu,l)f(z)\end{aligned}$$

$$\mathcal{J}_p^{\delta_1}(\lambda,\mu,l)(\mathcal{J}_p^{\delta_2}(\lambda,\mu,l)f(z)) = \mathcal{J}_p^{\delta_2}(\lambda,\mu,l)(\mathcal{J}_p^{\delta_1}(\lambda,\mu,l)f(z))$$

for $z \in \mathcal{U}$ and $p,n \in \mathbb{N} := \{1,2,...\}$.

If f is given by (1) then from the definition of the multiplier transformations $\mathcal{J}_p^{\delta}(\lambda,\mu,l)$, we can easily see that

$$\mathcal{J}_p^{\delta}(\lambda,\mu,l)f(z) = z^p + \sum_{k=n+p}^{\infty}\left[\frac{(k-p)(\lambda\mu k+\lambda-\mu)+p+l}{p+l}\right]^{\delta} a_k z^k.$$

Probability distributions are applied in a wide range of scientific areas, such as neural networks, economic forecasting, radiationless sources, andmeteorology, and are used to describe several real-life phenomena. Inmathematics, the concept is extensively used to study singular structures of Laplacian eigenfunctions, derivatives of distributions, orthogonal polynomials, transmission eigenfunctions, and impulse functions (see, for example, [2–6]).

The Borel distribution (BD) was introduced by Wanas et al. [7] as

$$P(X=\mu) = \frac{(\mu\kappa)^{\mu-1}\, e^{-\mu\kappa}}{\mu!},\ \ 0<\kappa\leq 1,\ \ \mu = 1,2,3,\ldots.$$

Furthermore, they introduced the series

$$M_\kappa(z) = z^p + \sum_{k=n+p}^{\infty}\frac{[\kappa(k-p)]^{k-p-1}\, e^{-\kappa(k-p)}}{(k-p)!} z^k,\ \ 0<\kappa\leq 1,\ \ p\in\mathbb{N},$$

whose coefficients are probabilities of the BD.

The research on inclusion relations of analytic functions in certain special sets is a subject that has its origin at the beginning of the study of geometric function theory. Ruscheweyh in [8] studied neighborhood and inclusion relations of univalent functions. Srivastava et al. [9] investigated the inclusion properties of multivalent functions. The authors in [10] derived inclusion symmetric relations for (q,d)-neighborhoods of analytic univalent functions. For further results, please see [11–13] and works cited therein. Recently, various subclasses of univalent functions in geometric function theory have been investigated (for details, see [10, 14–17]).

Let us consider the linear operator $B_\kappa f(z) : \mathcal{A}(n,p) \to \mathcal{A}(n,p)$ as

$$B_\kappa f(z) = M_\kappa(z) * f(z) = z^p + \sum_{k=n+p}^{\infty}\frac{[\kappa(k-p)]^{k-p-1}\, e^{-\kappa(k-p)}}{(k-p)!} a_k z^k,\ \ 0<\kappa\leq 1,\ \ p\in\mathbb{N}.$$

For $\delta \geq 0$, we define the operator $\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l) f(z) := \mathcal{J}^{\delta}_{p}(\lambda,\mu,l) B_{\kappa} f(z) : \mathcal{A}(n,p) \to \mathcal{A}(n,p)$ as

$$B_{\kappa} f(z) := \mathcal{B}^{0}_{p,\kappa}(\lambda,\mu,l) f(z) = z^{p} + \sum_{k=n+p}^{\infty} \frac{[\kappa(k-p)]^{k-p-1} e^{-\kappa(k-p)}}{(k-p)!} a_k z^k,$$

$$\begin{aligned}(p+l)\mathcal{B}^{1}_{p,\kappa}(\lambda,\mu,l) f(z) &= \lambda\mu z^2 \left(\mathcal{B}^{0}_{p,\kappa}(\lambda,\mu,l) f(z)\right)'' + (\lambda-\mu+(1-p)\lambda\mu) z \left(\mathcal{B}^{0}_{p,\kappa}(\lambda,\mu,l) f(z)\right)' \\ &\quad + (p(1-\lambda+\mu)+l)\, \mathcal{B}^{0}_{p,\kappa}(\lambda,\mu,l) f(z)\end{aligned} \tag{2}$$

$$\begin{aligned}(p+l)\mathcal{B}^{2}_{p,\kappa}(\lambda,\mu,l) f(z) &= \lambda\mu z^2 \left(\mathcal{B}^{1}_{p,\kappa}(\lambda,\mu,l) f(z)\right)'' + (\lambda-\mu+(1-p)\lambda\mu) z \left(\mathcal{B}^{1}_{p,\kappa}(\lambda,\mu,l) f(z)\right)' \\ &\quad + (p(1-\lambda+\mu)+l)\, \mathcal{B}^{1}_{p,\kappa}(\lambda,\mu,l) f(z)\end{aligned}$$

$$\mathcal{B}^{\delta_1}_{p,\kappa}(\lambda,\mu,l)\left(\mathcal{B}^{\delta_2}_{p,\kappa}(\lambda,\mu,l) f(z)\right) = \mathcal{B}^{\delta_2}_{p,\kappa}(\lambda,\mu,l)\left(\mathcal{B}^{\delta_1}_{p,\kappa}(\lambda,\mu,l) f(z)\right)$$

for $z \in \mathcal{U}$ and $p, n \in \mathbb{N} := \{1, 2, ...\}$.

If f is given by (1) then from the definition of the multiplier transformations $\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l) f(z)$, we can easily see that

$$\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l) f(z) = z^p + \sum_{k=n+p}^{\infty} \Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l) a_k z^k,$$

where

$$\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l) = \frac{[\kappa(k-p)]^{k-p-1} e^{-\kappa(k-p)}}{(k-p)!} \left[\frac{(k-p)(\lambda\mu k+\lambda-\mu)+p+l}{p+l}\right]^{\delta}. \tag{3}$$

Now, by making use of the operator $\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l) f(z)$, we define a new subclass of functions belonging to the class $\mathcal{A}(n,p)$.

Definition 1.2. $0 < \kappa \leq 1,\ \lambda \geq \mu \geq 0;\ l, \delta \geq 0;\ p \in \mathbb{N}$ *and for the parameters* σ, A *and* B *such that*

$$-1 \leq A < B \leq 1, \quad 0 < B \leq 1 \text{ and } 0 \leq \sigma < p,$$

we say that a function $f(z) \in \mathcal{A}(n,p)$ *is in the class* $\mathcal{P}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ *if it satisfies the following subordination condition:*

$$\frac{1}{p-\sigma}\left(\frac{\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l) f(z)\right]'}{z^{p-1}} - \sigma\right) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}). \tag{4}$$

If the following inequality holds true,

$$\left|\frac{\frac{\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l) f(z)\right]'}{z^{p-1}} - p}{B\frac{\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l) f(z)\right]'}{z^{p-1}} - [pB+(A-B)(p-\sigma)]}\right| < 1 \quad (z \in \mathcal{U}) \tag{5}$$

the inequality (5) is equivalent the subordination condition (4).

Furthermore, we say that a function $f(z) \in \mathcal{P}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ is in the subclass $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ if $f(z)$ is of the following form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} |a_k| z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, ...\}). \tag{6}$$

The main object of the present paper is to investigate the various important properties and characteristics of two subclasses of $\mathcal{A}(n,p)$ of normalized analytic functions in $\mathcal{U}$ with negative and positive coefficients, which are introduced here by making use of the multiplier transformations $\mathcal{J}^{\kappa,\delta}_{p}(\lambda,\mu,l)$ defined by (2). Several properties involving generalized neighborhoods and partial sums for functions belonging to the class $\mathcal{P}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ are investigated. Furthermore, we derive many results for the Quasi-convolution of functions belonging to the class $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$.

2. BASIC PROPERTIES OF THE FUNCTION CLASS $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$

We first determine a necessary and sufficient condition for a function $f(z) \in \mathcal{A}(n,p)$ of the form (6) to be in the class $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$.

Theorem 2.1. *Let the function $f(z) \in \mathcal{A}(n,p)$ be defined by (6). Then, the function $f(z)$ is in the class $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ if and only if*

$$\sum_{k=n+p}^{\infty} k(1+B)\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k| \leq (B-A)(p-\sigma), \tag{7}$$

where $\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)$ is given by (3).

Proof. If the condition (7) hold true, we find from (6) and (7) that

$$\begin{aligned}
&\left|\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l)\,f(z)\right]' - pz^{p-1}\right| - \left|B\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l)\,f(z)\right]' - z^{p-1}\left[pB+(A-B)(p-\sigma)\right]\right| \\
=\; &\left|-\sum_{k=n+p}^{\infty} k\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k|\,z^{k-1}\right| - \left|(B-A)(p-\sigma)z^{p-1} - B\sum_{k=n+p}^{\infty} k\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k|\,z^{k-1}\right| \\
\leq\; &\sum_{k=n+p}^{\infty} k(1+B)\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k| - (B-A)(p-\sigma) \leq 0 \quad \left(z \in \partial\mathcal{U} = \{z:\ z\in\mathbb{C} \text{ and } |z|=1\}\right).
\end{aligned}$$

Hence, by the *Maximum Modulus Theorem*, we have

$$f(z) \in \widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p).$$

Conversely, assume that the function $f(z)$ defined by (6) is in the class $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$. Then, we have

$$\left|\frac{\frac{\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l)f(z)\right]'}{z^{p-1}} - p}{B\frac{\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l)f(z)\right]'}{z^{p-1}} - \left[pB+(A-B)(p-\sigma)\right]}\right| = \left|\frac{\sum_{k=n+p}^{\infty} k\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k|\,z^{k-p}}{(B-A)(p-\sigma)z^{p-1} - B\sum_{k=n+p}^{\infty} k\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k|\,z^{k-p}}\right| < 1,$$

where $z \in \mathcal{U}$. Now, since $\left|\Re(z)\right| \leq |z|$ for all z, we have

$$\Re\left(\frac{\sum_{k=n+p}^{\infty} k\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k|\,z^{k-p}}{(B-A)(p-\sigma)z^{p-1} - B\sum_{k=n+p}^{\infty} k\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k|\,z^{k-p}}\right) < 1. \tag{8}$$

We choose values of z on the real axis so that the following expression:

$$\frac{\left[\mathcal{B}^{\delta}_{p,\kappa}(\lambda,\mu,l)\,f(z)\right]'}{z^{p-1}}$$

is real. Then, upon clearing the denominator in (8) and letting $z \to 1^-$ though real values, we get the following inequality

$$\sum_{k=n+p}^{\infty} k(1+B)\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)\,|a_k| \leq (B-A)(p-\sigma).$$

This completes the proof of Theorem 2.1. □

Remark 2.2. *Since $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ is contained in the function class $\mathcal{P}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$, a sufficient condition for $f(z)$ defined by (1) to be in the class $\mathcal{P}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$ is that it satisfies the condition (7) of Theorem 2.1.*

Corollary 2.3. *Let the function $f(z) \in \mathcal{A}(n,p)$ be defined by (6). If the function $f(z) \in \widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$, then*

$$|a_k| \leq \frac{(B-A)(p-\sigma)}{k(1+B)\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)} \quad (k,p \in \mathbb{N}).$$

The result is sharp for the function $f(z)$ given by:

$$f(z) = z^p - \frac{(B-A)(p-\sigma)}{k(1+B)\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)} z^k \quad (k,p \in \mathbb{N}).$$

We next prove the following growth and distortion properties for the class $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$.

Theorem 2.4. *If a function $f(z)$ be defined by (6) is in the class $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$, then*

$$\left(\frac{p!}{(p-q)!} - \frac{(B-A)(p-\sigma)(n+p-1)!}{(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)(n+p-q)!}|z|^n\right)|z|^{p-q} \tag{9}$$
$$\leq \left|f^{(q)}(z)\right| \leq \left(\frac{p!}{(p-q)!} + \frac{(B-A)(p-\sigma)(n+p-1)!}{(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)(n+p-q)!}|z|^n\right)|z|^{p-q}$$

for $q \in \mathbb{N}_0,\ p > q$ and all $z \in \mathcal{U}$. The result is sharp for the function $f(z)$ given by:

$$f(z) = z^p - \frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi^{n+p}_{p}(\delta,\lambda,\mu,l)} z^{n+p} \quad (p \in \mathbb{N}). \tag{10}$$

Proof. In view of Theorem 2.1, we have

$$\frac{(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)(n+p)!}\sum_{k=n+p}^{\infty} k!\,|a_k| \leq \sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}|a_k| \leq 1,$$

which readily yields

$$\sum_{k=n+p}^{\infty} k!\,|a_k| \leq \frac{(B-A)(p-\sigma)(n+p-1)!}{(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)} \quad (k,p \in \mathbb{N}). \tag{11}$$

Now, by differentiating both sides of (6) $q-$times with respect to z, we obtain

$$f^{(q)}(z) = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}a_k z^{k-q} \quad (q \in \mathbb{N}_0;\ p > q). \tag{12}$$

Theorem 2.4 follows readily from (11) and (12).
Finally, it is easy to see that the bounds in (9) are attained for the function $f(z)$ given by (10). □

3. INCLUSION RELATIONS INVOLVING NEIGHBORHOODS

We follow earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [18], Ruscheweyh [8] and others including Srivastava et al. [9, 19], Orhan [20, 21], Deniz and Orhan [22], and Aouf et al. [23] (see also [24]).

Firstly, we define the $(n,\eta)-$neighborhood of function $f(z) \in \mathcal{A}(n,p)$ of the form (1) by means of Definition 3.1 below.

Definition 3.1. *For $\eta > 0$ and a non-negative sequence $\mathcal{S} = \{s_k\}_{k=1}^{\infty}$, where*

$$s_k := \frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)} \quad (k \in \mathbb{N}).$$

The $(n,\eta)-$neighborhood of a function $f(z) \in \mathcal{A}(n,p)$ of the form (1) is defined as follows:

$$\mathcal{N}_{n,p}^{\eta}(f) := \left\{ g: \ g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{A}(n,p) \text{ and } \sum_{k=n+p}^{\infty} s_k \left|b_k - a_k\right| \le \eta \ (\eta > 0) \right\}. \tag{13}$$

For $s_k = k$, Definition 3.1 would correspond to the $\mathcal{N}_\eta-$neighborhood considered by Ruscheweyh [8].

Our first result based upon the familiar concept of neighborhood defined by (13).

Theorem 3.2. *Let $f(z) \in \mathcal{P}_{\kappa,\lambda,\mu,l}^{\delta}(A,B;\sigma,p)$ be given by (1). If f satisfies the inclusion condition:*

$$\left(f(z) + \varepsilon z^p\right)(1+\varepsilon)^{-1} \in \mathcal{P}_{\kappa,\lambda,\mu,l}^{\delta}(A,B;\sigma,p) \quad \left(\varepsilon \in \mathbb{C};\ |\varepsilon| < \eta;\ \eta > 0\right), \tag{14}$$

then

$$\mathcal{N}_{n,p}^{\eta}(f) \subset \mathcal{P}_{\kappa,\lambda,\mu,l}^{\delta}(A,B;\sigma,p).$$

Proof. It is not difficult to see that a function f belongs to $\mathcal{P}_{\kappa,\lambda,\mu,l}^{\delta}(A,B;\sigma,p)$ if and only if

$$\frac{\left[\mathcal{J}_p^{\delta}(\kappa,\lambda,\mu,l)f(z)\right]' - pz^{p-1}}{B\left[\mathcal{J}_p^{\delta}(\kappa,\lambda,\mu,l)f(z)\right]' - z^{p-1}\left[pB + (A-B)(p-\sigma)\right]} \ne \tau \quad \left(z \in \mathcal{U};\ \tau \in \mathbb{C},\ |\tau| = 1\right),$$

which is equivalent to

$$(f * h)(z)/z^p \ne 0 \quad (z \in \mathcal{U}), \tag{15}$$

where for convenience,

$$h(z) := z^p + \sum_{k=n+p}^{\infty} c_k z^k = z^p + \sum_{k=n+p}^{\infty} \frac{k(1+\tau B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{\tau(B-A)(p-\sigma)} z^k. \tag{16}$$

We easily find from (16) that

$$|c_k| \le \left| \frac{k(1+\tau B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{\tau(B-A)(p-\sigma)} \right| \le \frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)} \quad (k \in \mathbb{N}).$$

Furthermore, under the hypotheses of theorem, (14) and (15) yield the inequality

$$\frac{\left((f(z)+\varepsilon z^p)(1+\varepsilon)^{-1}\right) * h(z)}{z^p} \ne 0 \quad (z \in \mathcal{U})$$

or

$$\frac{f(z) * h(z)}{z^p} \ne \varepsilon \quad (z \in \mathcal{U}),$$

which is equivalent to

$$\frac{f(z) * h(z)}{z^p} \ge \eta \quad (z \in \mathcal{U};\ \eta > 0).$$

Now, if we let

$$g(z) := z^p + \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{N}_{n,p}^{\eta}(f),$$

then we have

$$\left|\frac{(f(z)-g(z))*h(z)}{z^p}\right| = \left|\sum_{k=n+p}^{\infty}(a_k-b_k)c_k z^{k-p}\right|$$

$$\leq \sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}|a_k-b_k|\,|z|^{k-p} < \eta \;\; (z\in\mathcal{U};\; \eta>0).$$

Thus, for any complex number τ such that $|\tau| = 1$, we have

$$(g*h)(z)/z^p \neq 0 \;\; (z\in\mathcal{U}),$$

which implies that $g \in \mathcal{P}_{\kappa,\lambda,\mu,l}^{\delta}(A,B;\sigma,p)$. The proof is complete. □

We now define the (n,η)−neighborhood of a function $f(z)\in\mathcal{A}(n,p)$ of the form (6) as follows.

Definition 3.3. *For $\eta > 0$, the (n,η)− neighborhood of a function $f(z)\in\mathcal{A}(n,p)$ of the form (6) is given by*

$$\widetilde{\mathcal{N}}_{n,p}^{\eta}(f) := \left\{ g:\; g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{A}(n,p) \text{ and } \sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}\big||b_k|-|a_k|\big| \leq \eta \;\; (\eta>0)\right\}. \quad (17)$$

Next, we prove Theorem 3.4.

Theorem 3.4. *If the function $f(z)$ defined by (6) is in the class $\widetilde{\mathcal{P}}_{\kappa,\lambda,\mu,l}^{\delta+1}(A,B;\sigma,p)$, then*

$$\widetilde{\mathcal{N}}_{n,p}^{\eta}(f) \subset \widetilde{\mathcal{P}}_{\kappa,\lambda,\mu,l}^{\delta}(A,B;\sigma,p)$$

where

$$\eta := \frac{n[\lambda\mu(n+p)+\lambda-\mu]}{n[\lambda\mu(n+p)+\lambda-\mu]+p+l}.$$

The result is the best possible in the sense that η cannot be increased.

Proof. For a function $f(z)\in\widetilde{\mathcal{P}}_{\kappa,\lambda,\mu,l}^{\delta+1}(A,B;\sigma,p)$ of the form (6) Theorem 2.1 immediately yields

$$\sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}|a_k| \leq \frac{p+l}{n[\lambda\mu(n+p)+\lambda-\mu]+p+l}. \quad (18)$$

Similarly, by taking

$$g(z) := z^p - \sum_{k=n+p}^{\infty}|b_k|z^k \in \widetilde{\mathcal{N}}_{n,p}^{\eta}(f) \;\; \left(\eta = \frac{n[\lambda\mu(n+p)+\lambda-\mu]}{n[\lambda\mu(n+p)+\lambda-\mu]+p+l}\right),$$

we find from the definition (17) that

$$\sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}\big||b_k|-|a_k|\big| \leq \eta \;\; (\eta>0). \quad (19)$$

With the help of (18) and (19), we have

$$\begin{aligned}\sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}|b_k| &\leq \sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}|a_k| \\ &\quad + \sum_{k=n+p}^{\infty}\frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}\big||b_k|-|a_k|\big| \\ &\leq \frac{p+l}{n[\lambda\mu(n+p)+\lambda-\mu]+p+l}+\eta = 1.\end{aligned}$$

Hence, in view of the Theorem 2.1 again, we see that $g(z) \in \widetilde{\mathcal{P}}^{\delta+1}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$.

To show the sharpness of the assertion of Theorem 3.4, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^p - \left[\frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta+1,\lambda,\mu,l)}\right] z^{n+p} \in \widetilde{\mathcal{P}}^{\delta+1}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$$

and

$$g(z) = z^p - \left[\frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta+1,\lambda,\mu,l)} + \frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)}\eta^*\right] z^{n+p}$$

where $\eta^* > \eta$. Clearly, the function $g(z)$ belong to $\widetilde{\mathcal{N}}^{\eta^*}_{n,p}(f)$. On the other hand, we find from Theorem 2.1 that $g(z) \notin \widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$. This evidently completes the proof of Theorem 3.4. □

4. PARTIAL SUMS OF THE FUNCTION CLASS $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$

Following the earlier work by Silverman [25] and recently Liu [26] and Deniz and Orhan [22], in this section we investigate the ratio of real parts of functions involving (6) and its sequence of partial sums defined by

$$\psi_m(z) = \begin{cases} z^p, & m = 1,2,\dots,n+p-1; \\ z^p - \sum_{k=n+p}^{m} |a_k| z^k, & m = n+p, n+p+1,\dots \end{cases} \quad (k \geq n+p;\ n,p \in \mathbb{N}) \tag{20}$$

and determine sharp lower bounds for $\Re\{f(z)/\psi_m(z)\}$ and $\Re\{\psi_m(z)/f(z)\}$.

Theorem 4.1. *Let $f \in \mathcal{A}(n,p)$ and $\psi_m(z)$ be given by (6) and (20), respectively. Suppose also that*

$$\sum_{k=n+p}^{\infty} \theta_k |a_k| \leq 1 \quad \left(\text{where } \theta_k = \frac{k(1+B)\Phi^k_{p,\kappa}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma)}\right). \tag{21}$$

Then for $m \geq k+p$, we have

$$\Re\left(\frac{f(z)}{\psi_m(z)}\right) > 1 - \frac{1}{\theta_{m+1}} \tag{22}$$

and

$$\Re\left(\frac{\psi_m(z)}{f(z)}\right) > \frac{\theta_{m+1}}{1+\theta_{m+1}}. \tag{23}$$

The results are sharp for every m with the extremal functions given by:

$$f(z) = z^p - \frac{1}{\theta_{m+1}} z^{m+1}. \tag{24}$$

Proof. From the hypothesis of the theorem 4.1, we see that

$$\theta_{k+1} > \theta_k > 1 \ \ (k \geq n+p).$$

Therefore, we have

$$\sum_{k=n+p}^{m} |a_k| + \theta_{m+1} \sum_{k=m+1}^{\infty} |a_k| \leq \sum_{k=n+p}^{\infty} \theta_k |a_k| \leq 1 \tag{25}$$

using hypothesis (21) again.

We set

$$\omega(z) = \theta_{m+1}\left[\frac{f(z)}{\psi_m(z)} - \left(1 - \frac{1}{\theta_{m+1}}\right)\right] = 1 - \frac{\theta_{m+1}\sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{1 - \sum_{k=n+p}^{m} |a_k| z^{k-p}}. \tag{26}$$

By applying (25) and (26), we find that

$$\left|\frac{\omega(z)-1}{\omega(z)+1}\right| = \left|\frac{-\theta_{m+1}\sum_{k=m+1}^{\infty}|a_k|\,z^{k-p}}{2-2\sum_{k=n+p}^{m}|a_k|\,z^{k-p}-\theta_{m+1}\sum_{k=m+1}^{\infty}|a_k|\,z^{k-p}}\right|$$

$$\leq \frac{\theta_{m+1}\sum_{k=m+1}^{\infty}|a_k|}{2-2\sum_{k=n+p}^{m}|a_k|\,z^{k-p}-\theta_{m+1}\sum_{k=m+1}^{\infty}|a_k|} \leq 1 \quad (z\in\mathcal{U};\ k\geq n+p),$$

which shows that $\Re\left(\omega(z)\right)>0$ $(z\in\mathcal{U})$. From (26), we immediately obtain the inequality (22).

To confirm that the function f given by (24) gives a sharp result, we observe for $z\to 1^{-}$ that

$$\frac{f(z)}{\psi_m(z)} = 1-\frac{1}{\theta_{m+1}}z^{m-p+1} \to 1-\frac{1}{\theta_{m+1}},$$

which shows that the bound in (22) is the best possible. Similarly, if we set

$$\phi(z) = (1+\theta_{m+1})\left[\frac{\psi_m(z)}{f(z)}-\frac{\theta_{m+1}}{1+\theta_{m+1}}\right]=1+\frac{(1+\theta_{m+1})\sum_{k=m+1}^{\infty}|a_k|\,z^{k-p}}{1-\sum_{k=n+p}^{m}|a_k|\,z^{k-p}},$$

and make use of (25), we can deduce that

$$\left|\frac{\phi(z)-1}{\phi(z)+1}\right| = \left|\frac{(1+\theta_{m+1})\sum_{k=m+1}^{\infty}|a_k|\,z^{k-p}}{2-2\sum_{k=n+p}^{m}|a_k|\,z^{k-p}+(\theta_{m+1}-1)\sum_{k=m+1}^{\infty}|a_k|\,z^{k-p}}\right|$$

$$\leq \frac{(1+\theta_{m+1})\sum_{k=m+1}^{\infty}|a_k|}{2-2\sum_{k=n+p}^{m}|a_k|\,z^{k-p}-(\theta_{m+1}-1)\sum_{k=m+1}^{\infty}|a_k|} \leq 1 \quad (z\in\mathcal{U};\ k\geq n+p),$$

which leads us immediately to assertion (23) of the theorem.

The bound in (23) is sharp with the extremal function given by (24). The proof of theorem is thus complete. □

5. PROPERTIES ASSOCIATED WITH QUASI-CONVOLUTION

In this part, we present results concerning the Quasi-convolution of a function that is in the class $\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma,p)$.

For the functions $f_j(z)\in\mathcal{A}(n,p)$ given by:

$$f_j(z) = z^p-\sum_{k=n+p}^{\infty}\left|a_{k,j}\right|z^k \quad \left(j=\overline{1,m},\ p\in\mathbb{N}\right),$$

we denote by $(f_1\bullet f_2)(z)$ the Quasi-convolution of functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1\bullet f_2)(z) = z^p-\sum_{k=n+p}^{\infty}\left|a_{k,1}\right|\left|a_{k,2}\right|z^k.$$

Theorem 5.1. *If $f_j(z)\in\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\sigma_j,p)$ $(j=\overline{1,m})$, then*

$$(f_1\bullet f_2\bullet\ldots\bullet f_m)(z)\in\widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A,B;\Upsilon,p),$$

where

$$\Upsilon := p-\frac{\prod_{j=1}^{m}(B-A)(p-\sigma_j)}{(B-A)[(n+p)(1+B)\Phi_{p,\kappa}^{n+p}(\delta,\lambda,\mu,l)]^{m-1}}. \tag{27}$$

The result is sharp for the functions $f_j(z)$ given by:

$$f_j(z) = z^p - \frac{(B-A)(p-\sigma_j)}{(n+p)(1+B)\Phi_{p,\kappa}^{n+p}(\delta,\lambda,\mu,l)} z^{p+n} \quad (j = \overline{1,m}). \tag{28}$$

Proof. For $m = 1$, we see that $\Upsilon = \sigma_1$. For $m = 2$, Theorem 2.1 gives

$$\sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\sigma_j)} \left|a_{k,j}\right| \leq 1 \quad (j = 1,2).$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}} \sqrt{\left|a_{k,1}\right|\left|a_{k,2}\right|} \leq 1. \tag{29}$$

To prove the case when $m = 2$, we have to find the largest Υ such that

$$\sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{(B-A)(p-\Upsilon)} \left|a_{k,1}\right|\left|a_{k,2}\right| \leq 1,$$

or such that

$$\frac{\left|a_{k,1}\right|\left|a_{k,2}\right|}{(B-A)(p-\Upsilon)} \leq \frac{\sqrt{\left|a_{k,1}\right|\left|a_{k,2}\right|}}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}. \tag{30}$$

This is equivalent to

$$\sqrt{\left|a_{k,1}\right|\left|a_{k,2}\right|} \leq \frac{(B-A)(p-\Upsilon)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}.$$

Further, by using (29), we need to find the largest Υ such that

$$\frac{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)} \leq \frac{(B-A)(p-\Upsilon)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}$$

or, equivalently, that

$$\frac{1}{(B-A)(p-\Upsilon)} \leq \frac{k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}.$$

It follows from (30) that

$$\Upsilon \leq p - \frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)}.$$

Now, defining the function $\chi(k)$ by:

$$\chi(k) = p - \frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)k(1+B)\Phi_{p,\kappa}^{k}(\delta,\lambda,\mu,l)},$$

we see that $\chi'(k) \geq 0$ for $k \geq p+n$. This implies that

$$\Upsilon \leq \chi(n+p) = p - \frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)(n+p)(1+B)\Phi_{p,\kappa}^{n+p}(\delta,\lambda,\mu,l)}.$$

Therefore, the result is true for $m = 2$.

Suppose that the result is true for any positive integer m. Then, we have $(f_1 \bullet f_2 \bullet ... \bullet f_m \bullet f_{m+1})(z) \in \widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A, B; \gamma, p)$, when

$$\gamma = p - \frac{(B-A)(p-\Upsilon)(B-A)(p-\sigma_{m+1})}{(B-A)(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)}$$

where Υ is given by (27). After a simple calculation, we have

$$\gamma \leq p - \frac{\prod_{j=1}^{m+1}(B-A)(p-\sigma_j)}{(B-A)[(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)]^m}.$$

Thus, the result is true for $m+1$. Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer m.

Finally, taking the functions $f_j(z)$ defined by (28), we have

$$\begin{aligned}(f_1 \bullet f_2 \bullet ... \bullet f_m)(z) &= z^p - \left\{\prod_{j=1}^{m} \frac{(B-A)(p-\sigma_j)}{(p+n)(1+B)\Phi^{p+n}_{p,\kappa}(\delta,\lambda,\mu,l)}\right\} z^{p+n} \\ &= z^p - \mathrm{A}_{p+n} z^{p+n},\end{aligned}$$

which shows that

$$\begin{aligned}\sum_{k=p+n}^{\infty} \frac{k(1+B)\Phi^{k}_{p,\kappa}(\delta,\lambda,\mu,l)}{(B-A)(p-\Upsilon)} \mathrm{A}_k &= \frac{(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)}{(B-A)(p-\Upsilon)} \mathrm{A}_{p+n} \\ &= \frac{(n+p)(1+B)\Phi^{n+p}_{p}(\delta,\lambda,\mu,l)}{(B-A)(p-\Upsilon)} \\ &\quad \times \left\{\prod_{j=1}^{2} \frac{(B-A)(p-\sigma_j)}{(p+n)(1+B)\Phi^{p+n}_{p,\kappa}(\delta,\lambda,\mu,l)}\right\}.\end{aligned}$$

Consequently, the result is sharp. □

Putting $\sigma_j = \sigma$ $(j = \overline{1,m})$ in Theorem 5.1, we have Corollary 5.2

Corollary 5.2. *If* $f_j(z) \in \widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A, B; \sigma, p)$ $(j = \overline{1,m})$, *then*

$$(f_1 \bullet f_2 \bullet ... \bullet f_m)(z) \in \widetilde{\mathcal{P}}^{\delta}_{\kappa,\lambda,\mu,l}(A, B; \Upsilon, p),$$

where

$$\Upsilon := p - \frac{[(B-A)(p-\sigma)]^m}{(B-A)[(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)]^{m-1}}.$$

The result is sharp for the functions $f_j(z)$ *given by:*

$$f_j(z) = \frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi^{n+p}_{p,\kappa}(\delta,\lambda,\mu,l)} z^{p+n} \quad (j = \overline{1,m}).$$

Conclusion 5.3. *In this study, new subclasses of p-valent analytic functions associated with Borel distribution functions were introduced by means of a generalized multiplier transformation. Several fundamental properties of these subclasses were investigated, including coefficient estimates, growth and distortion bounds, neighborhood inclusions, partial sum results, and quasi-convolution properties. It is expected that the techniques and results presented here will stimulate further research on multivalent function classes associated with probability distributions and related operator theory.*

References

[1] Deniz, E., Orhan, H. (2011). Certain Subclasses of Multivalent Functions Defined by New Multiplier Transformations. Arab. J. Sci. Eng., 36, 1091–1112.
[2] Bain, L., Engelhardt, M. (1992). Introduction to Probability and Mathematical Statistics; Duxburry Press: Belmont, CA, USA, Volume 1.
[3] Blasten, E., Liu, H. (2021). Scattering by curvatures, radiationless sources, transmission eigenfunctions, and inverse scattering problems. SIAM J. Math. Anal., 53, 3801–3837.
[4] Sakar, F.M., Hussain, S., Ahmed, I. (2022). Application of Gegenbauer polynomials for bi-univalent functions defined by subordination. Turk. J. Math., 46, 1089–1098.
[5] Yin, W., Yang, W., Liu, H. (2020). A neural network scheme for recovering scattering obstacles with limited phaseless far-field data. Comput. Phys., 417, 109594.
[6] Cao, X., Diao, H., Liu, H., Zou, J. (2020). On nodal and generalized singular structures of Laplacian eigen functions and applications to inverse scattering problems. J. Math. Pures Appl., 143, 116-–161.
[7] Wanas, A.K., Khuttar, J.A. (2020). Applications of Borel distribution series on analytic functions. Earthline J. Math. Sci., 4, 71–82.
[8] Ruscheweyh, S. (1981). Neighorhoods of univalent functions. Proc. Am. Math. Soc., 81, 521–527.
[9] Srivastava, H. M., Orhan, H. (2007). Coefficient inequalities and inclusion relations fore some families of analytic and multivalent functions. Appl. Math. Lett., 20, 686–691.
[10] Amini, E., Fardi, M., Al-Omari, S., Nonlaopon, K. (2022). Results on univalent functions defined by q-analogues of Salagean and Ruscheweh operators. Symmetry, 14, 1725.
[11] Mahmood, S., Khan, I., Srivastava, H.M., Malik, S.N. (2019). Inclusion relations for certain families of integral operators associated with conic regions. J. Inequal. Appl., 2019, 59.
[12] Lashin, A.M.Y., Aouf, M.K., Badghaish, A.O., Bajamal, A.Z. (2022). Some Inclusion Relations of Certain Subclasses of Strongly Starlike, Convex and Close-to-Convex Functions Associated with a Pascal Operator. Symmetry, 14, 1079.
[13] Lashin, A.Y., Badghaish, A.O., Algethami, B.M. (2022). Inclusion relations for some classes of analytic functions involving Pascal distribution series. J. Inequal. Appl., 2022, 161.
[14] Amini, E., Al-Omari, S., Nonlaopon, K., Baleanu, D. (2022). Estimates for coefficients of bi-univalent functions associated with a Fractional q-Difference Operator. Symmetry, 14, 879.
[15] Srivastava, H.M., Wanas, A.K., Güney, H.Ö. (2021). New Families of Bi-univalent Functions Associated with the Bazilevič Functions and the λ-Pseudo-Starlike Functions. Iran. J. Sci. Technol. Trans. Sci., 45, 1799—1804.
[16] Jahangiri, J.M., Hamidi, S.G. (2016). Advances on the coefficients of bi-prestarlike functions. Comptes Rendus Math., 354, 980–985.
[17] Amini, E., Fardi, M., Al-Omari, S.;, Saadeh, R. (2023). Certain differential subordination results for univalent functions associated with q-Salagean operators. AIMS Mathematics, 8, 15892-–15906.
[18] Goodman, A. W. (1957). Univalent functions and nonanalytic curves. Proceedings of the American Mathematical Society, 8(3), 598–601.
[19] Srivastava, H. M., Patel, J. (2005). Some subclasses of multivalent functions involving a certain linear operator. Journal of mathematical analysis and applications, 310(1), 209–228.
[20] Orhan, H. (2007). On neighborhoods of analytic functions defined by using Hadamard product. Novi Sad Journal of Mathematics, 37(1), 17–25.
[21] Orhan, H. (2009). Neighborhoods of a certain class of p-valent functions with negative coefficients defined by using a differential operator. Math. Ineq. Appl, 12(2), 335–349.
[22] Deniz, E., Orhan, H. (2010). Some properties of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator. Czechoslovak mathematical journal, 60(3), 699–713.
[23] Aouf, M. K., Silverman, H., Srivastava, H. M. (2008). Some families of linear operators associated with certain subclasses of multivalent functions. Computers and Mathematics with Applications, 55(3), 535–549.
[24] Catas, A. (2009). Neighborhoods of a certain class of analytic functions with negative coefficients. Banach Journal of Mathematical Analysis, 3(1), 111–121.
[25] Silverman, H. (1997). Partial sums of starlike and convex functions. Journal of Mathematical Analysis and Applications, 209(1), 221-227.
[26] Liu, J. L. (2007). Further properties of a certain subclass of analytic and multivalent functions. Applied mathematics and computation, 187(1), 290–294.