

The Newton Inequality for Twice Differentiable and s -Convex Functions

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Abstract. This paper focuses on the development of several new Newton-type inequalities for functions whose second derivatives are s -convex in the Breckner sense. By utilizing the setting of Riemann integrals and incorporating a summation parameter $p \geq 1$, we derive refined error estimates for Newton-Cotes-like quadrature formulas. The results obtained in this study provide a broad generalization that covers various convexity classes, including P -functions and classical convex functions as special cases. Furthermore, the established inequalities offer potential applications in numerical analysis for improving the precision of quadrature rules.

1. Introduction

In the field of numerical integration, Simpson's inequalities play a pivotal role in estimating the error of quadrature formulas. Two of the most prominent rules in this domain are:

- **Simpson's 1/3 rule:**

$$\frac{1}{b-a} \int_a^b f(t) dt \approx \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].$$

- **Simpson's 3/8 rule (Second Simpson's formula):**

$$\frac{1}{b-a} \int_a^b f(t) dt \approx \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right],$$

which is also widely recognized as the Newton-Cotes quadrature formula.

In recent years, a significant amount of research has been dedicated to refining these inequalities. For instance, Gao and Shi [7] established various Newton-type inequalities specifically for functions whose second derivatives are convex. The exploration of these concepts has extended into quantum calculus; Luangboon et al. [9] utilized (p, q) -calculus to derive Simpson and Newton-type inequalities, while Ali et

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al. [1] introduced similar bounds for quantum differentiable convex functions. Furthermore, Erden et al. [6] provided error estimations for Newton-type formulas using Lipschitzian mappings and functions of bounded variation. For a broader perspective on recent generalizations of Newton's inequality, one may refer to the works of Iftikhar et al. [8], Butt et al. [5], and Ali et al. [2].

For a function that is four times continuously differentiable, the classical error bound for the Simpson 3/8 rule is given by:

$$\left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{(b-a)^4}{6480} \|f^{(4)}\|_{\infty}. \quad (1)$$

When considering lower-order differentiability, Gao [7] proved that for twice differentiable functions:

$$\text{Error} \leq \frac{(b-a)^2 [|f''(a)| + |f''(b)|]}{384}, \quad (2)$$

whereas Sitthiwirattam et al. [12] established a bound for continuously differentiable functions:

$$\text{Error} \leq \frac{25(b-a) [|f'(a)| + |f'(b)|]}{576}. \quad (3)$$

The evolution of these inequalities is closely tied to the generalization of convexity. Mitrinovic et al. [10] provided the foundation for classical convex functions, while Pearce and Rubinov [11] introduced P -functions. A significant generalization was offered by Breckner [4] through the definition of s -convexity:

Definition 1.1 ([4]). Let $s \in [0, 1]$. A non-negative function $\Phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be s -convex if

$$\Phi(\tau x_1 + (1-\tau)x_2) \leq \tau^s \Phi(x_1) + (1-\tau)^s \Phi(x_2) \quad (4)$$

holds for all $x_1, x_2 \in I$ and $\tau \in (0, 1)$.

Notably, if $s = 1$, this definition reduces to standard convexity [10], and if $s = 0$, it reduces to the class of P -functions [11]. To facilitate further analysis in this framework, Benaissa and Sarikaya [3] established the following auxiliary result:

Lemma 1.2 ([3]). For $\tau \in (0, 1)$ and $s \in [0, 1]$, the following inequality holds:

$$\tau^s + (1-\tau)^s \leq 2^{1-s}. \quad (5)$$

The proof relies on the analysis of the function $g_s(\tau) = \tau^s + (1-\tau)^s - 2^{1-s}$. By examining its first derivative $g'_s(\tau) = s\tau^{s-1} - s(1-\tau)^{s-1}$, it is observed that the function reaches its maximum at $\tau = 1/2$. Since $g_s(1/2) = 0$, the inequality $\tau^s + (1-\tau)^s \leq 2^{1-s}$ is maintained for all $\tau \in (0, 1)$.

Motivated by these developments, in this paper, we establish a new version of Newton-type inequalities for s -convex functions using a summation parameter $p \geq 1$ within the framework of Riemann integrals.

2. The basic identity

Lemma 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function such that $f'' \in L_1([a, b])$, then the following identity holds.

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \\ &= \frac{(b-a)^2}{4} \left\{ \int_0^{\frac{1}{3}} \left(t^2 - \frac{1}{4}t\right) [f''((1-t)a + tb) + f''(ta + (1-t)b)] dt \right. \\ &+ \int_{\frac{1}{3}}^{\frac{2}{3}} \left(t^2 - t + \frac{1}{4}\right) [f''((1-t)a + tb) + f''(ta + (1-t)b)] dt \\ &+ \left. \int_{\frac{2}{3}}^1 \left(t^2 - \frac{7}{4}t + \frac{3}{4}\right) [f''((1-t)a + tb) + f''(ta + (1-t)b)] dt \right\}. \end{aligned} \quad (6)$$

Proof. By using the integration by parts, we obtain

$$\begin{aligned} H_1 &= \int_0^{\frac{1}{3}} \left(t^2 - \frac{1}{4}t\right) [f''((1-t)a + tb) + f''(ta + (1-t)b)] dt \\ &= \left(\frac{1}{b-a}\right) \left(t^2 - \frac{1}{4}t\right) [f'((1-t)a + tb) - f'(ta + (1-t)b)] \Big|_0^{\frac{1}{3}} \\ &\quad - \frac{1}{b-a} \int_0^{\frac{1}{3}} \left(2t - \frac{1}{4}\right) [f'((1-t)a + tb) - f'(ta + (1-t)b)] dt \\ &= \frac{1}{b-a} \left(\frac{1}{36}\right) \left(f'\left(\frac{2a+b}{3}\right) - f'\left(\frac{a+2b}{3}\right)\right) \\ &\quad - \frac{1}{(b-a)^2} \left(2t - \frac{1}{4}\right) [f((1-t)a + tb) + f(ta + (1-t)b)] \Big|_0^{\frac{1}{3}} \\ &\quad + \frac{2}{(b-a)^2} \int_0^{\frac{1}{3}} [f((1-t)a + tb) + f(ta + (1-t)b)] dt \\ &= \frac{1}{b-a} \left(\frac{1}{36}\right) \left(f'\left(\frac{2a+b}{3}\right) - f'\left(\frac{a+2b}{3}\right)\right) \\ &\quad - \frac{1}{(b-a)^2} \left(\frac{5}{12}\right) \left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right) - \frac{1}{4(b-a)^2} (f(a) + f(b)) \\ &\quad + \frac{2}{(b-a)^2} \int_0^{\frac{1}{3}} [f((1-t)a + tb) + f(ta + (1-t)b)] dt. \end{aligned}$$

According to the preceding details , we deduce

$$\begin{aligned} H_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left(t^2 - t + \frac{1}{4} \right) [f''((1-t)a + tb) + f''(ta + (1-t)b)] dt \\ &= -\frac{1}{b-a} \left(\frac{1}{18} \right) \left(f' \left(\frac{2a+b}{3} \right) - f' \left(\frac{a+2b}{3} \right) \right) \\ &\quad - \frac{1}{(b-a)^2} \left(\frac{2}{3} \right) \left(f \left(\frac{2a+b}{3} \right) + f \left(\frac{a+2b}{3} \right) \right) \\ &\quad + \frac{2}{(b-a)^2} \int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1} [f((1-t)a + tb) + f(ta + (1-t)b)] dt, \end{aligned}$$

and

$$\begin{aligned} H_3 &= \int_{\frac{2}{3}}^1 \left(t^2 - \frac{7}{4}t + \frac{3}{4} \right) [f''((1-t)a + tb) + f''(ta + (1-t)b)] dt \\ &= +\frac{1}{b-a} \left(\frac{1}{36} \right) \left(f' \left(\frac{2a+b}{3} \right) - f' \left(\frac{a+2b}{3} \right) \right) \\ &\quad - \frac{1}{(b-a)^2} \left(\frac{5}{12} \right) \left(f \left(\frac{2a+b}{3} \right) + f \left(\frac{a+2b}{3} \right) \right) - \frac{1}{(b-a)^2} \left(\frac{1}{4} \right) (f(b) + f(a)) \\ &\quad + \frac{2}{(b-a)^2} \int_{\frac{2}{3}}^1 [f((1-t)a + tb) + f(ta + (1-t)b)] dt. \end{aligned}$$

Consequently

$$\begin{aligned} H_1 + H_2 + H_3 &= \frac{2}{(b-a)^2} \int_0^1 [f((1-t)a + tb) + f(ta + (1-t)b)] dt \\ &\quad - \frac{1}{2(b-a)^2} \left[f(a) + 3f \left(\frac{2a+b}{3} \right) + 3f \left(\frac{a+2b}{3} \right) + f(b) \right], \end{aligned}$$

since

$$\int_0^1 f((1-t)a + tb) dt = \int_0^1 f(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b f(t) dt,$$

we deduce

$$\begin{aligned} H_1 + H_2 + H_3 &= \frac{4}{(b-a)^3} \int_a^b f(t) dt \\ &\quad - \frac{1}{2(b-a)^2} \left[f(a) + 3f \left(\frac{2a+b}{3} \right) + 3f \left(\frac{a+2b}{3} \right) + f(b) \right]. \end{aligned}$$

To get the desired result, multiply the previous equality by $\frac{(b-a)^2}{4}$. $\square$

3. Newton-type inequalities via s-convex functions

We need the following inequality to prove the next results.

Let $A, B \geq 0$ and $\eta > 0$:

$$A^\eta + B^\eta \leq \max(1, 2^{1-\eta})(A + B)^\eta. \quad (7)$$

Theorem 3.1. Let $p \geq 1$ and assume that f are defined as in Lemma 2.1. If $|f''|^p$ is an s -convex mapping on $[a, b]$, the following Newton-type inequality holds.

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{192} \left(\frac{1}{2}\right)^{\frac{s}{p}} \left[|f''(a)|^p + |f''(b)|^p \right]^{\frac{1}{p}} \end{aligned} \quad (8)$$

where $s \in [0, 1]$.

Proof. Let $p \geq 1$, $0 \leq \alpha < \beta \leq 1$ and $w \in L_p(\alpha, \beta)$, using Hölder inequality gives:

$$\begin{aligned} & \int_\alpha^\beta |w(t)| \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] dt \\ & = \int_\alpha^\beta \left(|w(t)|^{1-\frac{1}{p}} \right) \left(|w(t)|^{\frac{1}{p}} \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] \right) dt \\ & \leq \left(\int_\alpha^\beta |w(t)| dt \right)^{1-\frac{1}{p}} \\ & \quad \times \left[\left(\int_\alpha^\beta |w(t)| |f''((1-t)a + tb)|^p dt \right)^{\frac{1}{p}} + \left(\int_\alpha^\beta |w(t)| |f''(ta + (1-t)b)|^p dt \right)^{\frac{1}{p}} \right]. \end{aligned}$$

The inequality (7) yields $A^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(A + B)^{\frac{1}{p}}$, thus

$$\begin{aligned} & \int_\alpha^\beta |w(t)| \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] dt \\ & \leq \left(\int_\alpha^\beta |w(t)| dt \right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \\ & \quad \times \left[\int_\alpha^\beta |w(t)| \left(|f''((1-t)a + tb)|^p + |f''(ta + (1-t)b)|^p \right) dt \right]^{\frac{1}{p}}. \end{aligned}$$

Since $|f''|^p$ is an s -convex function, applying the inequality (5), we obtain:

$$|f''((1-t)a + tb)|^p \leq (1-t)^s |f''(a)|^p + t^s |f''(b)|^p,$$

then

$$\begin{aligned} & |f''((1-t)a + tb)|^p + |f''(ta + (1-t)b)|^p \\ & \leq [(1-t)^s + t^s] (|f''(a)|^p + |f''(b)|^p) \\ & \leq 2 \left(\frac{1}{2}\right)^s (|f''(a)|^p + |f''(b)|^p), \end{aligned}$$

hence

$$\begin{aligned} & \int_{\alpha}^{\beta} |w(t)| \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] dt \\ & \leq \left(\int_{\alpha}^{\beta} |w(t)| dt \right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[\int_{\alpha}^{\beta} |w(t)| 2 \left(\frac{1}{2} \right)^s (|f''(a)|^p + |f''(b)|^p) dt \right]^{\frac{1}{p}}. \end{aligned}$$

As a result for all $0 \leq \alpha < \beta \leq 1$, we have

$$\begin{aligned} & \int_{\alpha}^{\beta} |w(t)| \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] dt \\ & \leq \left(\int_{\alpha}^{\beta} |w(t)| dt \right) 2 \left(\frac{1}{2} \right)^{\frac{s}{p}} (|f''(a)|^p + |f''(b)|^p)^{\frac{1}{p}}. \end{aligned} \quad (9)$$

Using the modulus of identity (6) and applying the inequality (9), we deduce

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{4} \left\{ \int_0^{\frac{1}{3}} \left| t^2 - \frac{1}{4}t \right| \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] dt \right. \\ & \quad + \int_{\frac{1}{3}}^{\frac{2}{3}} \left| t^2 - t + \frac{1}{4} \right| \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] dt \\ & \quad \left. + \int_{\frac{2}{3}}^1 \left| t^2 - \frac{7}{4}t + \frac{3}{4} \right| \left[|f''((1-t)a + tb)| + |f''(ta + (1-t)b)| \right] dt \right\} \\ & \leq \frac{(b-a)^2}{2} \left(\frac{1}{2} \right)^{\frac{s}{p}} (|f''(a)|^p + |f''(b)|^p)^{\frac{1}{p}} \\ & \quad \times \left\{ \int_0^{\frac{1}{3}} \left| t^2 - \frac{1}{4}t \right| dt + \int_{\frac{1}{3}}^{\frac{2}{3}} \left| t^2 - t + \frac{1}{4} \right| dt + \int_{\frac{2}{3}}^1 \left| t^2 - \frac{7}{4}t + \frac{3}{4} \right| dt \right\}. \end{aligned}$$

Since

$$\int_0^{\frac{1}{3}} \left| t^2 - \frac{1}{4}t \right| dt + \int_{\frac{1}{3}}^{\frac{2}{3}} \left| t^2 - t + \frac{1}{4} \right| dt + \int_{\frac{2}{3}}^1 \left| t^2 - \frac{7}{4}t + \frac{3}{4} \right| dt = \frac{1}{96},$$

the required inequality is realized. $\square$

Put $p = 1$ in the above Theorem 3.1, we get the following Corollary.

Corollary 3.2. Let $s \in [0, 1]$ and assume that the assumptions of Lemma 6 hold. If $|f'|$ is a s -convex mapping on $[a, b]$, then the following Newton-type inequality holds

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\ & \leq \frac{(b-a)^2}{192} \left(\frac{1}{2} \right)^s (|f''(a)| + |f''(b)|) \end{aligned} \quad (10)$$

where $s \in [0, 1]$.

4. Newton-type Inequalities for Convex Functions

If we choose $s = 1$ in the Theorem 3.1 and Corollary 3.2, we obtain the results bellow.

Corollary 4.1. *Let $p \geq 1$ and assume that f are defined as in Lemma 2.1. If $|f''|^p$ is a convex mapping on $[a, b]$, the following Newton-type inequality holds*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \leq \frac{(b-a)^2}{192} \left[\frac{|f''(a)|^p + |f''(b)|^p}{2} \right]^{\frac{1}{p}}. \quad (11)$$

Corollary 4.2. *Assume that the assumptions of Lemma 6 hold. If $|f'|$ is a convex mapping on $[a, b]$, then the following Newton-type inequality holds*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \leq \frac{(b-a)^2}{384} (|f''(a)| + |f''(b)|). \quad (12)$$

This is the same as the inequality (2).

5. Newton-type Inequalities Involving P -functions

If we choose $s = 0$ in the Theorem 3.1 and Corollary 3.2, we obtain the new results involving the class P -functions.

Corollary 5.1. *Let $p \geq 1$ and assume that f are defined as in Lemma 2.1. If $|f''|^p$ is a P -function mapping on $[a, b]$, the following Newton-type inequality holds*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \leq \frac{(b-a)^2}{192} \left[|f''(a)|^p + |f''(b)|^p \right]^{\frac{1}{p}}. \quad (13)$$

Corollary 5.2. *Assume that the assumptions of Lemma 6 hold. If $|f'|$ is P -function mapping on $[a, b]$, then the following Newton-type inequality holds*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \leq \frac{(b-a)^2}{192} (|f''(a)| + |f''(b)|). \quad (14)$$

6. Applications

Some new inequalities are obtained for the previously inequality (11).

For any positive values $\lambda_1, \lambda_2, a_1, a_2 > 0$, we consider the following means:

- The weighted arithmetic mean:

$$W(\lambda_1, \lambda_2, a, b) = \frac{\lambda_1 a + \lambda_2 b}{\lambda_1 + \lambda_2}.$$

- The arithmetic mean:

$$A(a, b) = \frac{a + b}{2}.$$

- The harmonic mean:

$$H(a, b) = \frac{2ab}{a + b}.$$

- The n -logarithmic mean:

$$L_n(a, b) = \left(\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \right)^{\frac{1}{n}}, \quad n \in \mathbf{R} - \{-1, 0\}, \quad b \neq a.$$

- The logarithmic mean:

$$L(a, b) = \left(\frac{b-a}{\ln b - \ln a} \right), \quad n \in \mathbf{R} - \{-1, 0\}, \quad b \neq a.$$

Proposition 6.1. Let $b > a > 0$, $p \geq 1$ and $n > 2$. Then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{4} A(a^n, b^n) + \frac{3}{8} [W^n(1, 2, a, b) + W^n(2, 1, a, b)] - L_n^n(a, b) \right| \\ & \leq \frac{(b-a)^2 n(n-1)}{192} A^{\frac{1}{p}}(a^{p(n-2)}, b^{p(n-2)}). \end{aligned} \quad (15)$$

Proof. Let $t > 0$ and consider the function $f(t) = t^n$, so $f''(t) = n(n-1)t^{n-2}$.

With regard to

$$(|f''(t)|^p)'' = n^p (n-1)^p p(n-2) (p(n-2)-1) t^{p(n-2)-2} \geq 0,$$

then the function $|f''(t)|^p$ is convex. $\square$

Remark 6.2. The inequality (15) generalized result in [7, Proposition 3.1], just put $p = 1$.

Proposition 6.3. Let $b > a > 0$, $p \geq 1$. Then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{4} H^{-1}(a, b) + \frac{3}{8} [W^{-1}(1, 2, a, b) + W^{-1}(2, 1, a, b)] - L^{-1}(a, b) \right| \\ & \leq \frac{(b-a)^2}{96} H^{-p}(a^{3p}, b^{3p}). \end{aligned} \quad (16)$$

Proof. Let $t > 0$, and consider the function $f(t) = \frac{1}{t}$, then $f''(t) = \frac{2}{t^3}$.

With gives us

$$(|f''(t)|^p)'' = 2^p (3p) (3p+1) \frac{1}{t^{3p+2}} \geq 0,$$

hence, the function $|f''(t)|^p$ is convex. $\square$

Remark 6.4. The inequality (16) generalized result in [7, Proposition 3.2], just put $p = 1$.

7. Conclusion

In conclusion, this study successfully establishes Newton-type inequalities for twice differentiable and s -convex functions using the Riemann integral. Moreover, it introduces novel Newton-type inequalities that incorporate a summation parameter $p \geq 1$ to address different cases of convexity. These findings contribute to the existing body of knowledge on mathematical inequalities and provide a broader framework for analyzing convex functions. The results have potential applications in various mathematical and applied fields where convexity and differential properties are of interest.

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