TURKISH JOURNAL OF SCIENCE
VOLUME 10, ISSUE 1, 1–9
ISSN: 2587–0971

https://www.tjoscience.com

New Extensions of Grüss Type Inequalities via Generalized Proportional Fractional Integral Operators

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Abstract. In this paper, we have performed some new integral inequalities of Grüss type via generalized proportional fractional integral operators. We have used fairly elemantery integration methods, some classical integral inequalities and properties of generalized proportional fractional integral operators.

1. Introduction

In [20], Grüss inequality has been given as:
Suppose that the functions $U, V : [a_1, b_1] \to \mathbb{R}$ are positive with $A \leq U(t) \leq B$ and $C \leq V(s) \leq D$, for all $t, s \in [a_1, b_1]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} U(t)V(s)d\,tds - \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} U(t)dt \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} V(s)ds \right| \\ \leq \; & \frac{1}{4} (B - A)(D - C) \end{aligned} \tag{1}$$

where the constants $B, A, C, D \in \mathbb{R}$ and $\frac{1}{4}$ is the sharp value of inequality (1).
For new results regarding Grüss inequality, which is one of the most striking types of inequalities among integral inequalities and has attracted the attention of many researchers, see the articles [17]-[20].
Let us recall some well-known concepts. We note that the beta function $B(\alpha, \beta)$ is defined (see [15])

$$B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt & (\mathbb{R}(\alpha), \mathbb{R}(\beta) > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \end{cases}$$

where Γ is the familiar Gamma function. Here and in the following, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers and non-positive integers, respectively and let

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Received: 05 January 2025; Accepted:12 February 2025; Published: 30 April 2025
Keywords. Grüss type inequalities, Generalized Proportional Fractional integrals, Cauchy-Schwarz inequality.
2010 *Mathematics Subject Classification*. 26D15, 26A51
Cited this article as: Butt, S.I., Raza, M.A., Nadeem, M. & Karaduman, E. (2025). New Extensions of Grüss Type Inequalities via Generalized Proportional Fractional Integral Operators. Turkish Journal of Science, 10(1), 1-9.

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.
Fractional analysis has given a new orientation not only to mathematics, but also to physics, statistics, engineering and other applied sciences, with its birth dating back to ancient times and its rapid development in recent years. This development has gained a new momentum in recent years, especially with the definition of new fractional integral and derivative operators. Although new fractional operators lead to effective applications and generalizations in the field, they also have advantages over classical derivative and integral operators with their core structures and properties. Let's take a look at a few of these operators.

Definition 1.1. *(See [15]) Let $f \in L_1[a,b]$. The Riemannn-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}f(x)dx, \qquad t > a$$

and

$$J_{b-}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}f(x)dx, \qquad t < b$$

respectively. Here $\Gamma(t)$ is the Gamma function and its definition is $\Gamma(t) = \int_0^{\infty} e^{-t}t^{x-1}dx$. It is to be noted that $J_{a+}^{0}f(t) = J_{b-}^{0}f(t) = f(t)$ in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [16], Jarad et al. identified the proportional generalized fractional integrals that satisfy many important features as follows:

Definition 1.2. *The left and right generalized proportional fracitonal integral operators are respectively defined by*

$${}_{a+}\mathfrak{I}^{\alpha,\lambda}f(t) = \frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_a^t e^{\left[\frac{\lambda-1}{\lambda}(t-x)\right]}(t-x)^{\alpha-1}f(x)dx, \qquad t > a$$

and

$${}_{b-}\mathfrak{I}^{\alpha}f(t) = \frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_t^b e^{\left[\frac{\lambda-1}{\lambda}(x-t)\right]}(x-t)^{\alpha-1}f(x)dx, \qquad t < b$$

where $\lambda \in (0,1]$ and $\alpha \in \mathbb{C}$ and $\mathbb{R}(\alpha) > 0$.

The readers can find detailed information about fractional analysis studies, different usage areas of new operators and current trends in inequality theory in the articles [1]-[14].
The main motivation of this paper is to prove some new integral inequalities via generalized proportional fracitonal integral operators.

2. Main results

Lemma 2.1. *Let $0 < \lambda_1 \leq 1$ and let $m, M \in \Re$ and v be an integrable function on $[0,\infty)$. Then, we have*

$$\begin{aligned} &\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\,{}_a^{GPF}I^{\alpha,\lambda_1}v^2(t) + \left({}_a^{GPF}I^{\alpha,\lambda_1}v(t)\right)^2 \qquad (2)\\ = &\left(\frac{M}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} - {}_a^{GPF}I^{\alpha,\lambda_1}v(t)\right)\\ &\left({}_a^{GPF}I^{\alpha,\lambda_1}v(t) - \frac{m}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right)\\ &-\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\,{}_a^{GPF}I^{\alpha,\lambda_1}\Big((M-v(t))(v(t)-m)\Big)\end{aligned}$$

for all $t > 0$, $\alpha > 0$ where $a_1 = \frac{\lambda_1-1}{\lambda_1}$.

Proof. Let $m, M \in \mathfrak{R}$ and v be an integrable function on $[0,\infty)$, for all $\tau, \rho \in [0,\infty)$, we have

$$\begin{aligned} & (M - v(\rho))(v(\tau) - m) + (M - v(\tau))(v(\rho) - m) \\ & -(M - v(\tau))(v(\tau) - m) - (M - v(\rho))(v(\rho) - m) \\ = \; & v^2(\tau) + v^2(\rho) + 2v(\tau)v(\rho). \end{aligned} \tag{3}$$

Multiplying both sides of (3) by $\frac{1}{\lambda_1^\alpha \Gamma(\alpha)} e^{\frac{\lambda_1 - 1}{\lambda_1}(t-\tau)}(t-\tau)^{\alpha-1}$, then integrating the resulting identity with respect to τ from a to t ($\tau \in (0,t), t > 0$), we get

$$\begin{aligned} & \Big(M - v(\rho)\Big)\Big({}_a^{GPF}I^{\alpha,\lambda_1} v(t) - m \frac{1}{\lambda_1^\alpha \Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\Big) \\ & + \Big(M \frac{1}{\lambda_1^\alpha \Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} - {}_a^{GPF}I^{\alpha,\lambda_1} v(t)\Big) \\ & \Big(v(\rho) - m\Big) - {}_a^{GPF}I^{\alpha,\lambda_1}\Big((M - v(t))(v(t) - m)\Big) \\ & - \Big((M - v(\rho))(v(\rho) - m)\Big) \frac{1}{\lambda_1^\alpha \Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} \\ = \; & {}_a^{GPF}I^{\alpha,\lambda_1} v^2(t) + v^2(\rho) \frac{1}{\lambda_1^\alpha \Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} + 2v(\rho) {}_a^{GPF}I^{\alpha,\lambda_1} v(t). \end{aligned} \tag{4}$$

Now, again multiply both side of (4) by $\frac{1}{\lambda_1^\alpha \Gamma(\alpha)} e^{\frac{\lambda_1 - 1}{\lambda_1}(t-\rho)}(t-\rho)^{\alpha-1}$, then integrating the resulting identity with respect to ρ from a to t ($\rho \in (0,t), t > 0$), which gives (2), and the lemma is proved. □

Theorem 2.2. *Let f and g be two integrable function on $[0,\infty)$, satisfying the condition that*

$$m \le f(t) \le M, p \le g(t) \le P, m, M, p, P \in \mathfrak{R}, t \in [0,\infty), \tag{5}$$

we have

$$\begin{aligned} & \left| \frac{1}{\lambda_1^\alpha \Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} {}_a^{GPF}I^{\alpha,\lambda_1} fg(t) - {}_a^{GPF}I^{\alpha,\lambda_1} f(t) {}_a^{GPF}I^{\alpha,\lambda_1} g(t) \right| \\ \le \; & \left(\frac{1}{2\lambda_1^\alpha \Gamma(\alpha)} \sum_{k_1=0}^{\infty} \frac{a_1^{k_1}}{k_1!} \frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} \right)^2 (M - m)(P - p) \end{aligned} \tag{6}$$

where $a_1 = \frac{\lambda_1 - 1}{\lambda_1}$.

Proof. Let f and g be two function satisfying the condition (5). Define

$$H(\tau,\rho) = \Big(f(\tau) - f(\rho)\Big)\Big(g(\tau) - g(\rho)\Big), \tau, \rho \in (0,t), t > 0.$$

It follows that

$$H(\tau,\rho) = f(\tau)g(\tau) - f(\tau)g(\rho) - f(\rho)g(\tau) + f(\rho)g(\rho). \tag{7}$$

Then, multiplying (7) by $\frac{1}{\lambda_1^\alpha \Gamma(\alpha)} e^{\frac{\lambda_1 - 1}{\lambda_1}(t-\tau)}(t-\tau)^{\alpha-1}$, by integrating the resulting identity with respect to τ from

a to t, we get

$$\begin{aligned} & \frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\int_a^t H(\tau,\rho)e^{\frac{\lambda_1-1}{\lambda_1}(t-\tau)}(t-\tau)^{\alpha-1}d\tau \\ = & {}_a^{GPF}I^{\alpha,\lambda_1}fg(t)-g(\rho){}_a^{GPF}I^{\alpha,\lambda_1}f(t)-f(\rho){}_a^{GPF}I^{\alpha,\lambda_1}g(t) \\ & +f(\rho)g(\rho)\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}. \end{aligned} \tag{8}$$

Again multiplying (8) by $\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}e^{\frac{\lambda_1-1}{\lambda_1}(t-\rho)}(t-\rho)^{\alpha-1}$, which is positive because $\rho\in(0,t)$ and integrating the resulting identity with respect to ρ from a to t, we have

$$\begin{aligned} & \frac{1}{(\lambda_1^{\alpha})^2\Gamma(\alpha)^2}\int_a^t\int_a^t H(\tau,\rho)e^{\frac{\lambda_1-1}{\lambda_1}(t-\tau)}e^{\frac{\lambda_1-1}{\lambda_1}(t-\rho)}(t-\rho)^{\alpha-1}(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}d\tau d\rho \\ = & 2\left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\alpha,\lambda_1}fg(t)-{}_a^{GPF}I^{\alpha,\lambda_1}f(t){}_a^{GPF}I^{\alpha,\lambda_1}g(t)\right). \end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\alpha,\lambda_1}fg(t)-{}_a^{GPF}I^{\alpha,\lambda_1}f(t){}_a^{GPF}I^{\alpha,\lambda_1}g(t)\right)^2 \\ \leq & \left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\alpha,\lambda_1}f^2(t)-\left({}_a^{GPF}I^{\alpha,\lambda_1}f(t)\right)^2\right) \\ & \left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\alpha,\lambda_1}g^2(t)-\left({}_a^{GPF}I^{\alpha,\lambda_1}g(t)\right)^2\right). \end{aligned} \tag{9}$$

Since $\Big(M-f(t)\Big)\Big(f(t)-m\Big)\geq 0$ and $\Big(P-g(t)\Big)\Big(g(t)-p\Big)\geq 0$, we can write

$$\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\alpha,\lambda_1}\left(\Big(M-f(t)\Big)\Big(f(t)-m\Big)\right)\geq 0$$

and

$$\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\alpha,\lambda_1}\left(\Big(P-g(t)\Big)\Big(g(t)-p\Big)\right)\geq 0.$$

Thus

$$\begin{aligned} & \frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\alpha,\lambda_1}f^2(t)-\left({}_a^{GPF}I^{\alpha,\lambda_1}f(t)\right)^2 \\ \leq & \left(M\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_a^{GPF}I^{\alpha,\lambda_1}f(t)\right) \\ & \times\left({}_a^{GPF}I^{\alpha,\lambda_1}f(t)-\frac{m}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right) \end{aligned} \tag{10}$$

and

$$\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_{a}^{GPF}I^{\alpha,\lambda_1}g^2(t)-\left({}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)\right)^2 \tag{11}$$
$$\leq \left(\frac{P}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)\right)$$
$$\times\left({}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)-p\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right).$$

Combining (9), (10) and (11), using lemma (2.1), we conclude that

$$\left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_{a}^{GPF}I^{\alpha,\lambda_1}fg(t)-{}_{a}^{GPF}I^{\alpha,\lambda_1}f(t){}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)\right)^2 \tag{12}$$
$$\leq \left(\frac{M}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_{a}^{GPF}I^{\alpha,\lambda_1}f(t)\right)$$
$$\times\left({}_{a}^{GPF}I^{\alpha,\lambda_1}f(t)-\frac{m}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right) \tag{13}$$
$$\times\left(\frac{P}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)\right)$$
$$\times\left({}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)-\frac{p}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right).$$

Now, using the elementary inequality $4ab \leq (a+b)^2$, $a, b \in \Re$, we can show that

$$4\left(\frac{M}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_{a}^{GPF}I^{\alpha,\lambda_1}f(t)\right) \tag{14}$$
$$\times\left({}_{a}^{GPF}I^{\alpha,\lambda_1}f(t)-\frac{m}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right)$$
$$\leq \left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}(M-m)\right)^2$$

and

$$4\left(\frac{P}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)\right) \tag{15}$$
$$\times\left({}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)-\frac{p}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right)$$
$$\leq \left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}(P-p)\right)^2.$$

From (12), (14) and (15), we get the result (6). □

Lemma 2.3. *Let $0<\lambda_1\leq 1$ and $0<\lambda_2\leq 1$ and let f and g be two positive integrable function on $[0,\infty)$ then, for all $t>0$, $\alpha>0$, $\beta>0$, we have*

$$\begin{aligned}
&\Bigg[\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\beta,\lambda_2}fg(t) \qquad (16)\\
&+\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_a^{GPF}I^{\alpha,\lambda_1}fg(t)-{}_a^{GPF}I^{\alpha,\lambda_1}f(t){}_a^{GPF}I^{\beta,\lambda_2}g(t)-{}_a^{GPF}I^{\beta,\lambda_2}f(t){}_a^{GPF}I^{\alpha,\lambda_1}g(t)\Bigg]^2\\
\leq\;&\Bigg(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\beta,\lambda_2}f^2(t)+\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_a^{GPF}I^{\alpha,\lambda_1}f^2(t)-2{}_a^{GPF}I^{\alpha,\lambda_1}f(t){}_a^{GPF}I^{\beta,\lambda_2}f(t)\Bigg)\\
&\Bigg(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\beta,\lambda_2}g^2(t)+\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_a^{GPF}I^{\alpha,\lambda_1}g^2(t)-2{}_a^{GPF}I^{\alpha,\lambda_1}g(t){}_a^{GPF}I^{\beta,\lambda_2}g(t)\Bigg)
\end{aligned}$$

where $a_1=\frac{\lambda_1-1}{\lambda_1}$ and $a_2=\frac{\lambda_2-1}{\lambda_2}$.

Proof. Multiplying (8) by $\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}e^{\frac{\lambda_2-1}{\lambda_2}(t-\rho)}(t-\rho)^{\beta-1}$ which is positive for $\rho\in(0,t)$ and integrate with respect to ρ from a to t, then applying Cauchy-Schwarz inequality for double integral, we obtain (16). □

Lemma 2.4. *Let $0<\lambda_1\leq 1$ and $0<\lambda_2\leq 1$ and let v be a integrable function on $[0,\infty)$ and $m,M\in\Re$, then for all $t>0$, $\alpha>0$, $\beta>0$, we have*

$$\begin{aligned}
&\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\beta,\lambda_2}v^2(t) \qquad (17)\\
&+\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_a^{GPF}I^{\alpha,\lambda_1}v^2(t)\\
&-2{}_a^{GPF}I^{\alpha,\lambda_1}v(t){}_a^{GPF}I^{\beta,\lambda_2}v(t)\\
=\;&\left(M\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_a^{GPF}I^{\alpha,\lambda_1}v(t)\right)\\
&\times\left({}_a^{GPF}I^{\beta,\lambda_2}v(t)-m\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}\right)\\
&+\left(M\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}-{}_a^{GPF}I^{\beta,\lambda_2}v(t)\right)\\
&\times\left({}_a^{GPF}I^{\alpha,\lambda_1}v(t)-m\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right)\\
&-\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_a^{GPF}I^{\alpha,\lambda_1}\Big((M-v(t))(v(t)-m)\Big)\\
&-\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\beta,\lambda_2}\Big((M-v(t))(v(t)-m)\Big),
\end{aligned}$$

where $a_1=\frac{\lambda_1-1}{\lambda_1}$ and $a_2=\frac{\lambda_2-1}{\lambda_2}$.

Proof. Multiplying (4) by $\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}e^{\frac{\lambda_2-1}{\lambda_2}(t-\rho)}(t-\rho)^{\beta-1}$, which is positive for $\rho\in(0,t)$, $t>0$, then integrating the resulting identity with respect to ρ from a to t, we get

$$\begin{aligned}
&\left({}_{a}^{GPF}I^{\alpha,\lambda_1}v(t)-\frac{m}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\right)\\
&\times\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\lambda_2-1}{\lambda_2}(t-\rho)}(t-\rho)^{\beta-1}\Big(M-v(\rho)\Big)d\rho\\
&+\left(\frac{M}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}-{}_{a}^{GPF}I^{\alpha,\lambda_1}v(t)\right)\\
&\times\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\lambda_2-1}{\lambda_2}(t-\rho)}(t-\rho)^{\beta-1}\int_a^t\Big(v(\rho-m)\Big)d\rho\\
&-{}_{a}^{GPF}I^{\alpha,\lambda_1}\Big((M-v(t))(v(t)-m)\Big)\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\lambda_2-1}{\lambda_2}(t-\rho)}(t-\rho)^{\beta-1}d\rho\\
&-\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\\
&\times\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\lambda_2-1}{\lambda_2}(t-\rho)}(t-\rho)^{\beta-1}\Big(M-v(\rho)\Big)\Big(v(\rho)-m\Big)d\rho\\
=\;&\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_{a}^{GPF}I^{\alpha,\lambda_1}v^2(t)\\
&+\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_{a}^{GPF}I^{\beta,\lambda_2}v(t)\\
&+2{}_{a}^{GPF}I^{\alpha,\lambda_1}v(t){}_{a}^{GPF}I^{\beta,\lambda_2}v^2(t).
\end{aligned}\tag{18}$$

This gives the proof of lemma (2.4) □

Theorem 2.5. *Let* $0<\lambda_1\leq 1$ *and* $0<\lambda_2\leq 1$ *let f,g be two integrable function on* $[0,\infty)$*, satisfying the condition that*

$$m\leq f(t)\leq M, p\leq g(t)\leq P,\ m,M,p,P\in\mathfrak{R}$$

and $t\in[0,\infty)$. *Then for all* $T>0$, $\alpha>0$, $\beta>0$, *we have*

$$\begin{aligned}
&\left(\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_{a}^{GPF}I^{\beta,\lambda_2}fg(t)\right.\\
&+\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_{a}^{GPF}I^{\alpha,\lambda_1}fg(t)\\
&\left.-{}_{a}^{GPF}I^{\alpha,\lambda_1}f(t){}_{a}^{GPF}I^{\beta,\lambda_2}g(t)-{}_{a}^{GPF}I^{\beta,\lambda_2}f(t){}_{a}^{GPF}I^{\alpha,\lambda_1}g(t)\right)^2
\end{aligned}\tag{19}$$

$$\leq \Bigg[\Bigg(M\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} - {}_a^{GPF}I^{\alpha,\lambda_1}f(t)\Bigg)$$
$$\Bigg({}_a^{GPF}I^{\beta,\lambda_2}f(t) - m\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}\Bigg)$$
$$+\Bigg(M\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2} - {}_a^{GPF}I^{\beta,\lambda_2}f(t)\Bigg)$$
$$\Bigg({}_a^{GPF}I^{\alpha,\lambda_1}f(t) - m\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\Bigg)\Bigg]$$

$$\times\Bigg[\Bigg(P\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1} - {}_a^{GPF}I^{\alpha,\lambda_1}g(t)\Bigg)$$
$$\Bigg({}_a^{GPF}I^{\beta,\lambda_2}g(t) - p\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}\Bigg)$$
$$+\Bigg(P\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2} - {}_a^{GPF}I^{\beta,\lambda_2}g(t)\Bigg)$$
$$\Bigg({}_a^{GPF}I^{\alpha,\lambda_1}g(t) - p\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}\Bigg)\Bigg].$$

Proof. Since $\big(M-f(t)\big)\big(f(t)-m\big)\geq 0$ and $\big(P-g(t)\big)\big(g(t)-p\big)\geq 0$. Then we can write

$$\begin{aligned}&-\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\beta,\lambda_2}\big((M-f(t))(f(t)-m)\big)\\&-\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_a^{GPF}I^{\alpha,\lambda_1}\big((M-f(t))(f(t)-m)\big)\\ \leq\;& 0\end{aligned} \tag{20}$$

and

$$\begin{aligned}&-\frac{1}{\lambda_1^{\alpha}\Gamma(\alpha)}\sum_{k_1=0}^{\infty}\frac{a_1^{k_1}}{k_1!}\frac{(t-a)^{\alpha+k_1}}{\alpha+k_1}{}_a^{GPF}I^{\beta,\lambda_2}\big((P-g(t))(g(t)-p)\big)\\&-\frac{1}{\lambda_2^{\beta}\Gamma(\beta)}\sum_{k_2=0}^{\infty}\frac{a_2^{k_2}}{k_2!}\frac{(t-a)^{\beta+k_2}}{\beta+k_2}{}_a^{GPF}I^{\alpha,\lambda_1}\big((P-g(t))(g(t)-p)\big)\\ \leq\;& 0.\end{aligned} \tag{21}$$

Applying lemma (2.4) to f and g, then using lemma (2.3) and the equation (20) and (21), we get (19). □

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