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Caputo and Atangana-Baleanu-Caputo Fractional Derivative Applied to Garden Equation

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Abstract. In this study, the garden equation which is a nonlinear partial differential equation is discussed. First, we will expand the garden equation to the Caputo derivative and Atangana-Baleanu fractional derivative in the sense of Caputo. Then, we will then demonstrate the existence of the new equation with the help of the fixed point theorem. Finally, we will examine uniqueness solution for the two fractional operators.

1. Introduction

Many nonlinear partial differential equations are used to describe real world problems. Such problems are used in many branches, especially in engineering, earth sciences and physics [5–7]. For example, one of these examples is the garden equation. The garden equation is a nonlinear differential equation used to describe some dynamics in hydrodynamics and plasma physics. For example, plasma physics is the study of the state of a substance that contains charged particles and liquids under the influence of electric and magnetic fields. It is possible to create plasma by heating a gas until it breaks chemical bonds that connect electrons to parent atoms or molecules. The subject of plasma is up to date and has many different application areas such as beam storage, accelerator physics, space and astrophysics.

To describe complex problems, the concept of a fractional-order derivative and a partial differential equation are used. One of the difficulties encountered in solving such equations is to predict the future behavior of the physical problem. Using fractional derivative operators to cope with this situation helps researchers [4]. Many fractional derivative definitions are used in the literature. The Caputo version [2] of the captive derivative is mostly used to model real world problems because it allows the use of initial conditions. The problem encountered in this version, however, is singularity due to the function used to induce the local derivative. Atangana-Baleanu fractional derivative [3] in the sense of Caputo is also quite assertive in this regard. Because the kernel used in this definition is both non-local and non-singular. This allows us to get rid of the singularity problem in the Caputo fractional derivative. Atangana, Akgul and Owolabi [8] present a detailed analysis including, numerical solution, stability analysis and error analysis. Atangana and Akgul [9] tried to construct new transfer functions that would lead the Sumudu transformation to new Bode, Nichols and Nyquist plots.

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In this study, we devoted the first part to the general history and physical history of the garden equation. In the second part, the necessary definitions and theorems that will be used in the article are given. In the third section, we expanded the garden equation to the Caputo fractional derivative. Then we examined the existence and uniqueness solutions of the new equation. In the last chapter, we expanded the same equation to Atangana-Baleanu fractional derivative in the sense of Caputo. We have examined the existence and uniqueness solutions for this equation.

2. Preliminaries

We give in this section some fundamental definitions [1–3] on fractional derivative.

Definition 2.1. *The Caputo derivative of fractional derivative is defined as [2]:*

$$ {}_a^C D_t^{\nu} f(t)) = \frac{1}{\Gamma(n-\nu)} \int_a^t \frac{f^{(n)}(r)}{(t-r)^{\nu+1-n}} dr, \quad n-1 < \nu < n \in \mathbb{N}. \tag{1}$$

Definition 2.2. *The Riemann-Liouville fractional integral is defined as [1]:*

$$ J^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t f(r)(t-r)^{\nu-1} dr. \tag{2}$$

Definition 2.3. *The Riemann-Liouville fractional derivative is defined as [1]:*

$$ {}_a^R D_t^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{f(r)}{(t-r)^{\nu+1-n}} dr, \quad n-1 < \nu < n \in \mathbb{N}. \tag{3}$$

Definition 2.4. *The Sobolev space of order 1 in (a, b) is defined as [2]:*

$$ H^1(a,b) = \{u \in L^2(a,b) : u' \in L^2(a,b)\}.$$

Definition 2.5. *Let a function $u \in H^1(a,b)$ and $\nu \in (0,1)$. The AB fractional derivative in Caputo sense of order ν of u with a based point a is defined as [3]:*

$$ {}_a^{ABC} D_t^{\nu} u(t)) = \frac{B(\nu)}{1-\nu} \int_a^t u'(s) E_{\nu}\Big[-\frac{\nu}{1-\nu}(t-s)^{\nu}\Big] ds, \tag{4}$$

where $B(\nu)$ has the same properties as in Caputo and Fabrizio case, and is defined as

$$ B(\nu) = 1 - \nu + \frac{\nu}{\Gamma(\nu)},$$

$E_{\nu,\beta}(\lambda^{\nu})$ is the Mittag-Leffler function, defined in terms of a series as the following entire function

$$ E_{\nu,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\lambda^{\nu})^k}{\Gamma(\nu k + \beta)}, \qquad \nu > 0, \quad \lambda < \infty \quad and \quad \beta > 0, \quad \lambda = -\nu(1-\nu)^{-1}. \tag{5}$$

Definition 2.6. *Let a function $u \in H^1(a,b)$ and $\nu \in (0,1)$. The AB fractional derivative in Riemann-Liouville sense of order ν of u with a based point a is defined as [3]:*

$$ {}_a^{ABR} D_t^{\nu} u(t)) = \frac{B(\nu)}{1-\nu} \frac{d}{dt} \int_a^t u(s) E_{\nu}\Big[-\frac{\nu}{1-\nu}(t-s)^{\nu}\Big] ds, \tag{6}$$

when the function u is constant, we get zero.

Definition 2.7. *The Atangana-Baleanu fractional integral of order ν with base point a is defined as [3]:*

$$ {}^{AB} I_t^{\nu} u(t)) = \frac{1-\nu}{B(\nu)} u(t) + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_a^t u(s)(t-s)^{\nu-1} ds, \tag{7}$$

when the function u is constant, we get zero.

3. Garden equation with Caputo derivative

Garden equation is given by,

$$u_t(x,t) = 6(u + \epsilon^2 u^2)u_x + u_{xxx} \tag{8}$$

The initial condition is $u(x, 0) = f(x)$ for the equation (1). The equation (1) with Caputo derivative is given as below

$${}_0^C D_t^{\nu} u(x,t) = 6(u + \epsilon^2 u^2)u_x + u_{xxx}. \tag{9}$$

In this section we show existence and uniqueness solution of the equation (1). Let us present every continuous functions $G = C[a, b]$ in the Banach space defined in the closed set $[a, b]$ and consider $Z = \{\rho, a \in G, \rho(x,t) \geq 0 \quad and \quad a(x,t) \geq 0, a \leq t \leq b\}$

Definition 3.1. *[4] Let X be a Banach space with a cone H. H initiates a restricted order ≤ in E in the succeeding approach.*

$$y \geq x \Rightarrow y - x \in H$$

Now applying the fractional integral in equation (9), we obtain the following,

$$u(x,t) - u(x,0) = \frac{1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} \Big[6(u + \epsilon^2 u^2)u_x + u_{xxx} \Big] dr. \tag{10}$$

Now we can use system (10) to show the existence of equation (8). Necessary lemma for the existence of the solutions are given as Lemma 3.2. We now need to define an operator which $X : G \to G$.

$$Xu(x,t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} \phi(x, r, \rho(x,r)) dr \tag{11}$$

To be dealt with more easily, let us consider below

$$\phi(x, r, u) = 6(u + \epsilon^2 u^2)u_x + u_{xxx} \tag{12}$$

Lemma 3.2. *The mapping $X : G \to G$ is completely continuous.*

Proof. Let $N \subset G$ be bounded. There exists a constants $l > 0$ such that $||u|| < l$. Let,

$$T = max_{\substack{0 \leq t \leq 1 \\ 0 \leq u \leq l}} \phi(x, t, u(x,t))$$

$\forall u \in N$, we have,

$$\begin{aligned} ||Xu(x,t)|| &\leq \frac{1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} ||\phi(x, r, u(x,r))|| dr \\ &\leq \frac{T}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} dr \\ &= \frac{T}{\Gamma(\nu+1)} t^{\nu} \end{aligned} \tag{13}$$

Hence $X(N)$ is bounded.
Now in the following part, we will consider $t_1 < t_2$ and $u(x,t), \in N$ and ; then for a given $\epsilon > 0$ if $|t_2 - t_1| < \delta$. We have,

$$
\begin{aligned}
\|Tu(x,t_2)-Tu(x,t_1)\| =&\|\frac{1}{\Gamma(\nu)}\int_0^{t_2}(t_2-r)^{\nu-1}\|\phi(x,r,u(x,r))\|dr\\
&-\frac{1}{\Gamma(\nu)}\int_0^{t_1}(t_1-r)^{\nu-1}\|\phi(x,r,u(x,r))dr\|\\
=&\|\frac{1}{\Gamma(\nu)}\int_0^{t_2}(t_2-r)^{\nu-1}\|\phi(x,r,u(x,r))\|dr\\
&-\frac{1}{\Gamma(\nu)}\int_0^{t_2}(t_1-r)^{\nu-1}\|\phi(x,r,u(x,r))\|dr\\
&-\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2}(t_1-r)^{\nu-1}\|\phi(x,r,u(x,r))\|dr\\
\leq&\frac{1}{\Gamma(\nu)}\int_0^{t_2}\|(t_2-r)^{\nu-1}-(t_1-r)^{\nu-1}\|\,\|\rho(x,r,u(x,r))\|dr
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2}\|(t_1-r)^{\nu-1}\|\,\|\phi(x,r,u(x,r))\|dr\\
\leq&\frac{T}{\Gamma(\nu)}\int_0^{t_2}\Big((t_2-r)^{\nu-1}-(t_1-r)^{\nu-1}\Big)dr+\frac{T}{\Gamma(\nu)}\int_{t_1}^{t_2}(t_1-r)^{\nu-1}dr\\
=&\frac{T}{\Gamma(\nu)}\Big(\int_0^{t_2}(t_2-r)^{\nu-1}dr-\int_0^{t_2}(t_1-r)^{\nu-1}dr+\int_{t_1}^{t_2}(t_1-r)^{\nu-1}dr\Big)\\
=&\frac{T}{\Gamma(1+\nu)}\Big(t_2^{\nu}+(t_1-t_2)^{\nu}-t_1^{\nu}+(t_1-t_2)^{\nu}\Big)\\
\leq&\frac{2T}{\Gamma(1+\nu)}(t_1-t_2)^{\nu}+\frac{T}{\Gamma(1+\nu)}(t_1-t_1)^{\nu}\\
=&\frac{2T}{\Gamma(1+\nu)}(t_1-t_2)^{\nu}\\
<&\frac{2T}{\Gamma(1+\nu)}\delta^{\nu}\\
=&\epsilon
\end{aligned}
\tag{15}
$$

It is clear seen that, when the same steps are applied to the $a(x,t)$ function, we get same situation. Finally,

$$|Xu(x,t_2)-Xu(x,t_1)|\leq\epsilon$$

are satisfied. Where $\delta=\Big(\epsilon\Gamma(1+\nu/2T)\Big)^{1/\nu}$.Therefore $X(N)$ is equicontinuous. So that $\overline{X(N)}$ is compact via The Arzela-Ascoli theorem. □

Theorem 3.3. *Let $\chi:[u_1,u_2]\times[0,\infty)\to[0,\infty)$, then $\chi(x,t,.)$ is non-decreasing for each t in $[u_1,u_2]$. there exists a positive constants z_1 and z_2 such that $C(n)z_1\leq\chi(x,t,z_1)$, $C(n)z_2\geq\chi(x,t,z_2)$, $0\leq z_1(x,t)\leq z_2(x,t)$, $u_1\leq t\leq u_2$. This means that the new equation has a positive solution.*

Proof. We only need to consider the fixed point for operator of X. With framework of Lemma 3.2, the considered operator $X : K \to K$ is completely continuous. Let us take two arbitrary u_1 and u_2,

$$\begin{aligned} Xu_1(x,t) =& \frac{1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} \phi(x,r,u_1(x,r))dr \\ \leq& \frac{1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} \phi(x,r,u_2(x,r))dr \\ =& Xu_2(x,t) \end{aligned} \tag{16}$$

Hence X is a non-decreasing operator. So that the operator $X : \langle z_1, z_2 \rangle \to \langle z_1, z_2 \rangle$ is compact and continuous via Lemma 3.2. In that case, K is a normal cone of X. □

3.1. Uniqueness of Solution

The aim of this chapter is to prove the uniqueness of solutions to the equation (10). So the uniqueness of the solution is presented as below,

$$\begin{aligned} \|Xu_1(x,t) - Xu_2(x,t)\| =& \|\frac{1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} \Big(\phi(x,r,u_1(x,r)) - \phi(x,r,u_2(x,r))\Big) dr\| \\ \leq& \frac{1}{\Gamma(\nu)} B_1 \int_0^t (t-r)^{\nu-1} \|u_1(x,r) - u_2(x,r)\| dr \end{aligned} \tag{17}$$

So that,

$$\|Xu_1(x,t) - Xu_2(x,t)\| \leq \left\{ \frac{B_1 t^\nu}{\Gamma(\nu+1)} \right\} \|u_1(x,r) - u_2(x,r)\|$$

Therefore, if the following conditions hold,

$$\left\{ \frac{B_1 t^\nu}{\Gamma(\nu+1)} \right\} < 1$$

Then mapping X is a contraction, which implies fixed point, and thus the model has a unique positive solution.

4. Garden equation with AB derivative in Caputo sense

We present in this chapter the existence and uniqueness of solutions of the garden equation using the Atangana-Baleanu derivative. Let $\Omega = (a, b)$ be an open and bounded subset of R^n. For a given $\nu \in (0, 1)$ and functions $u(x,t) \in H^1(\Omega) \times [0, T]$. We apply the equation (8) to the Atangana-Baleanu fractional derivative,

$$ {}_0^{ABC}D_t^\nu u(x,t) = \xi(x,t,u) \tag{18}$$

where

$$\xi(x,t,u) = 6(u + \epsilon^2 u^2)u_x + u_{xxx} \tag{19}$$

Using the Atangana-Baleanu integral to (18) it yields

$$u(x,t) = u(x,0) + \frac{1-\nu}{B(\nu)} \xi(x,t,u(x,t)) + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_0^t \xi(x,r,u(x,r))(t-r)^{\nu-1} dr \tag{20}$$

for all $t \in [0, T]$.

Theorem 4.1. *If the inequality (21) hold, ξ satisfies Lipshitz condition and contraction.*

$$0 < 3\varphi_1 d_1 + 2\epsilon^2 \varphi_1 d_2 + \varphi_1^3 \leq 1 \tag{21}$$

Proof. We would like to start with the kernel ξ. Let η and κ are two functions, the following equation is written:

$$\begin{aligned}\xi(x,t,\eta)-\xi(x,t,\kappa)=&\Big(6(\eta+\epsilon^2\eta^2)\eta_x+\eta_{xxx}\Big)-\Big(6(\kappa+\epsilon^2\kappa^2)\kappa_x+\kappa_{xxx}\Big)\\ &6(\eta\eta_x-\kappa\kappa_x)+6\epsilon^2(\kappa^2\kappa_x-\eta^2\eta_x)+(\eta_{xxx}-\kappa_{xxx})\end{aligned} \tag{22}$$

Then, applying the norm on both sides gives

$$\begin{aligned}\|\xi(x,t,\eta)-\xi(x,t,\kappa)\|=&\|6(\eta\eta_x-\kappa\kappa_x)+6\epsilon^2(\kappa^2\kappa_x-\eta^2\eta_x)+(\eta_{xxx}-\kappa_{xxx})\|\\ \le&\, 6\|\eta\eta_x-\kappa\kappa_x\|+6\epsilon^2\|\kappa^2\kappa_x-\eta^2\eta_x\|+\|\eta_{xxx}-\kappa_{xxx}\|\\ \le&\, 3\|\partial_x(\eta^2-\kappa^2)\|+2\epsilon^2\|\partial_x(\eta^3-\kappa^3)\|+\|\partial_{xxx}(\eta-\kappa)\|\end{aligned} \tag{23}$$

Using the Lipschitz condition for the first order derivative function ∂_x; we can find φ_1 such that

$$\begin{aligned}\|\xi(x,t,\eta)-\xi(x,t,\kappa)\|\le&3\varphi_1\|\eta^2-\kappa^2\|+2\epsilon^2\varphi_1\|\eta^3-\kappa^3\|+\varphi_1^3\|\eta-\kappa\|\\ \le&\, 3\varphi_1\|\eta+\kappa\|\cdot\|\eta-\kappa\|+2\epsilon^2\varphi_1\Big(\|\eta\|^2+\|\eta\|\cdot\|\kappa\|+\|\kappa\|^2\Big)\|\eta-\kappa\|+\varphi_1^3\|\eta-\kappa\|\\ \le&\Big[3\varphi_1d_1+2\epsilon^2\varphi_1d_2+\varphi_1^3\Big]\|\eta-\kappa\|\end{aligned} \tag{24}$$

So the following inequality can be written.

$$\|\xi(x,t,\eta)-\xi(x,t,\kappa)\|\le K\|(\eta(x,t)-\kappa(x,t)\|. \tag{25}$$

where

$$K=\Big(3\varphi_1d_1+2\epsilon^2\varphi_1d_2+\varphi_1^3\Big)$$

Therefore ξ satisfies the Lipschitz condition. Then we can say that it is a contraction.
In the another case, the following inequality can be written because our kernel is linear,

$$\xi_2(x,t,v_1)-\xi_2(x,t,v_2)\le(c\vartheta_1^2+d)\|v_1(x,t)-v_2(x,t)\|$$

Hence, the proof is complete. We can now show that the uniqueness of the solution. □

4.1. Uniqueness of solution

The uniqueness solution for equation (20) is presented as below. Let $u_1,u_2\in H^1$ be two solutions of (20). Let $u=u_1-u_2$. the following equation can be written,

$$u=\frac{1-\nu}{B(\nu)}\Big(\xi(x,t,u_1(x,t))-\xi(x,t,u_2(x,t))\Big)+\frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t\Big(\xi(x,r,u_1(x,r))-\xi(x,r,u_2(x,r))\Big)dr,$$

If the norms of both sides are taken, by the Gronwall inequality [20],

$$\|u\|\le\frac{1-\nu}{B(\nu)}\|\xi(x,t,u_1(x,t))-\xi(x,t,u_2(x,t))\|+\frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t\|\xi(x,r,u_1(x,r))-\xi(x,r,u_2(x,r))\|dr\le K_1\int_0^t\|\xi(x,t,u_1(x,t))\|_{H^1}dr.$$

Finally, the equation (20) has a unique solution for the equation u.

References

[1] I. Podlubny, Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999.
[2] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, Geophys. J. R. Astr. Soc. 13 (1967) 529–539.
[3] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, Therm. Sci. 20 (2016) 763—769.
[4] T. Yamamoto, X. Chen, An existence and nonexistence theorem for solutions of nonlinear systems and its application to algebraic equations, Journal of computational and applied mathematics 30 (1990) 87–97.
[5] J. H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities. Computer methods in applied mechanics and engineering, 167(1-2), (1998) 69-73.
[6] S. Liao, An optimal homotopy-analysis approach for strongly nonlinear differential equations. Communications in Nonlinear Science and Numerical Simulation, 15(8), (2010) 2003-2016.
[7] N. A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos, Solitons & Fractals, 24(5), (2005) 1217-1231.
[8] A. Atangana, A. Akgül, A. and K. M. Owolabi, Analysis of fractal fractional differential equations. Alexandria Engineering Journal, (2020).
[9] A. Atangana and A. Akgül, Can transfer function and Bode diagram be obtained from Sumudu transform. Alexandria Engineering Journal, (2020).