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Some Estimates for the Spin-Submanifold Twisted Dirac Operators

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Abstract. In this paper, we generalize lower bound estimates for the eigenvalue estimates of the submanifold twisted Dirac operator on a compact Riemannian Spin−submanifold proved by N. Ginoux and B. Morel in 2002.

1. Introduction

Defining some structure on the compact Riemannian manifolds as Spin and Spin^c−structure to obtain information about the topology and geometry of the manifold is the main way for mathematicians. Due to this feature, many authors have been systematically worked on these structures [2, 3, 7, 16]. One of the way to obtain these subtle information is the investigation the spectrum of the Dirac operator [4, 5, 7, 10, 13, 14]. The study of Dirac operators on the submanifolds was firstly started by E. Witten using the hypersurface Dirac operator to prove the positive energy-theorem [20]. Later on, this operator is investigated by the mathematicians and physicists to obtain subtle information about the topology and geometry of the manifolds. One of the ways to obtain this subtle information is done by investigating the spectrum of the Dirac operator [4–7, 9, 10, 14].

Obtaining lower bounds to the eigenvalues of the submanifold Dirac operator firstly was given by X. Zhang and O. Hijazi in [13] by generalized the results obtained on the hypersurfaces [12, 19]. The fundemantal tools used to estimate the lower bound are appropriately modified spinorial Levi−Civita connection and Schrödinger−Lichnerowicz formula.

In this paper, we will consider the generalization of the results for the the submanifolds obtained by N. Ginoux and B. Morel in [8] coming from the result for hypersurfaces given in [17]. In doing so, as in the papers of O. Hijazi and X. Zhang [12, 19], they started by restricted the spinor bundle of the Riemannian Spin−manifold to a submanifold equipped with an induced Riemannian metric. Then, they lifted the Levi−Civita connections defined on both the Riemannian Spin manifold and its submanifold onto the spinor bundle built on these manifolds, respectively. Finally, they defined the submanifold Dirac operator with the help of the spinorial Gauss formula. Some authors called this operator as a twisted Dirac operator [18]. In this paper we use this naming.

Later on, N. Ginoux and B. Morel in [8] obtain an estimates for the eigenvalues of the twisted Dirac operator on a compact Riemannian submanifold in terms of the scalar curvature, mean curvature and Energy−Momentum tensor.

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In this paper, by defining modified spinorial Levi−Civita connections we give estimates containing all inequalities obtained in [8] as special cases.

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2. Twisted Dirac Operator on The Submanifolds

Let $\widetilde{M}$ be an $(m+n)$−dimensional compact Riemannian Spin−manifold and M be an m−dimensional immersed oriented Riemannian Spin−submanifold in $\widetilde{M}$ equipped with the induced Riemannian metric. Let NM be the normal bundle of M. As we know, the manifolds M and $\widetilde{M}$ defines an unique Spin−structure on the normal bundle NM [8]. Through out the whole paper $\mathbb{S}_M$, $\mathbb{S}_{MN}$ and $\mathbb{S}_{\widetilde{M}}$ denotes the spinor bundles over the manifolds M, NM and $\widetilde{M}$, respectively. The restricted spinor bundle $\mathbb{S} := \mathbb{S}_{\widetilde{M}}\big|_M$ is identified as follows:

$$\mathbb{S} := \begin{cases} \mathbb{S}_M \otimes \mathbb{S}_{MN}, & \text{if } n \text{ or } m \text{ is even,} \\ \mathbb{S}_M \otimes \mathbb{S}_{MN} \oplus \mathbb{S}_M \otimes \mathbb{S}_{MN}, & \text{otherwise.} \end{cases} \tag{1}$$

On this restricted spinor bundle exist a Hermitian inner product, denoted by $\left(\, , \, \right)$, such that Clifford multiplication by a vector of $T\widetilde{M}\big|_M$ is skew−symmetric [15, 16].

As in [8] we denote the induced spinorial Levi−Civita connection on $\Gamma(\mathbb{S})$ by $\widetilde{\nabla}$ and ∇

$$\widetilde{\nabla} = \begin{cases} \left(\nabla^{\mathbb{S}_M} \otimes Id + Id \otimes \nabla^{\mathbb{S}_{MN}}\right) \oplus \left(\nabla^{\mathbb{S}_M} \otimes Id + Id \otimes \nabla^{\mathbb{S}_{MN}}\right), & \text{if } n \text{ and } m \text{ are odd,} \\ \left(\nabla^{\mathbb{S}_M} \otimes Id + Id \otimes \nabla^{\mathbb{S}_{MN}}\right), & \text{otherwise.} \end{cases}$$

For a fixed point $x \in M$, let $(e_1, ..., e_m, v_1, ..., v_n)$ be a positively oriented local orthonormal basis of $T\widetilde{M}\big|_M$ such that $(e_1, ..., e_m)$ $(resp(v_1, ..., v_n))$ is a positively oriented local orthonormal basis of TM $(resp\ NM)$. Also we have the following identification between the Clifford multiplication on $\Gamma(\mathbb{S}_M)$ and $\Gamma(\mathbb{S}_{\widetilde{M}})\big|_M$

$$X \cdot_M \Phi \quad = \quad \left(X \cdot \omega_\perp \cdot \Psi\right)_M, \tag{2}$$

where $\Phi = \Psi\big|_M$, $\Psi \in \Gamma(\mathbb{S}_{\widetilde{M}})$, and

$$\omega_\perp = \begin{cases} \omega_n, & \text{for } n \text{ even,} \\ -i\omega_n, & \text{for } n \text{ odd,} \end{cases} \tag{3}$$

here ω_n denoting the complex volume form:

$$\omega_n = i^{[\frac{n+1}{2}]} v_1 \cdot ... \cdot v_n. \tag{4}$$

The spinorial Gauss formula [1]:

$$\widetilde{\nabla}_i \Psi = \nabla_i \Psi + \frac{1}{2} \sum_{j=1}^{m} e_j \cdot h_{ij} \cdot \Psi, \tag{5}$$

where $\Psi \in \Gamma(\mathbb{S})$ and h_{ij} is the component of the second fundemantal form at x. Accordingly, Dirac operators are defined as follows:

$$\widetilde{D} = \sum_{i=1}^{m} e_i \cdot \widetilde{\nabla}_i, \; D = \sum_{i=1}^{m} e_i \cdot \nabla_i, \tag{6}$$

and the twisted Dirac operator which also called submanifold Dirac operator and denoted by D_H is defined as

$$D_H := (-1)^n \omega_\perp \cdot \widetilde{D} = (-1)^n \omega_\perp \cdot D + \frac{1}{2} H \cdot \omega_\perp \cdot \Psi, \tag{7}$$

where $H=\sum_{i=1}^{m} h(e_i,e_i)$ denotes the mean curvature vector field. Moreover, by using the fact that $H\cdot\omega_\perp\cdot=(-1)^{n-1}\omega_\perp\cdot H$ one gets $\widetilde{D}=D-\frac{1}{2}H$ and

$$\begin{aligned} D &= \omega_\perp\cdot D_H+\frac{1}{2}H\cdot\omega_\perp\cdot \\ &= \lambda_H\cdot\omega_\perp\cdot+\frac{1}{2}H\cdot, \end{aligned} \tag{8}$$

where λ_H denotes the eigenvalue of the twisted Dirac operator D_H.

Finally, for any $\Psi\in\Gamma(\mathbb{S})$, we give the well−known formula called twisted Lichnerowicz formula as follows:

$$\Big(D^2\Psi,\Psi\Big) = \Big(\nabla^*\nabla\Psi,\Psi\Big)+\frac{1}{4}\Big(R+R^N_\Psi\Big)|\Psi|^2, \tag{9}$$

where R is the scalar curvature of M and $R^N_\Psi := 2\sum_{i,j=1}^{m}\Big(e_i\cdot e_j\cdot I_d\otimes R^N_{e_i,e_j}\Psi,\frac{\Psi}{|\Psi|^2}\Big)$ on $M_\Psi:=\{x\in M:\Psi(x)\neq 0\}$, and $R^N_{e_i,e_j}$ stands for spinorial normal curvature tensor. Combining (8) and (9), we obtain

$$\int_M|\nabla\Psi|^2 = \int_M\Big(\lambda_H^2|\Psi|^2+\frac{1}{4}\|H\|^2|\Psi|^2+\lambda_H Re\Big(\omega_\perp\cdot\Psi,H\cdot\Psi\Big)-\frac{1}{4}(R+R^N_\Psi)|\Psi|^2\Big). \tag{10}$$

3. Lower bounds of Eigenvalues

In this section, two estimates are given. One of them is obtained in terms of the mean curvature and modified scalar curvature defined in [7, 12, 13] as follows:

$$R_{p,u,\Psi}=R+R^N_\Psi-4p\nabla u+4\nabla p\nabla u-4\Big(1-\frac{1}{m}\Big)p^2|du|^2, \tag{11}$$

where p and u are real valued functions defined on $\widetilde{M}$. If $p,u=0$, then $R_{p,u,\Psi}=R+R^N_\Psi$. In this case all estimates coincides with the result obtained in [8]. The other is obtained in terms of above modified scalar curvature and Energy−Momentum tensor.

In the following theorem, we give an optimal lower bound to the eigenvalues of D_H by using an appropriate modified spinorial Levi−Civita connection.

Theorem 3.1. *Let $M\subset\widetilde{M}$ be a compact Riemannian Spin−submanifold of a Riemannian Spin−manifold $(\widetilde{M},g)$. Consider a non−trivial eigenspinor field $\Psi\in\Gamma(\mathbb{S})$ such that $D_H\Psi=\lambda_H\Psi$. Assume that $m\geq 2$ and*

$$\Omega_{p,u,\Psi}=\{(p,u,\Psi)|mR_{p,u,\Psi}>(m-1)\|H\|^2>0\}, \tag{12}$$

on M_Ψ where p,u are real−valued functions. Then the following inequality is satisfied

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\Omega_{p,u,\Psi}}\ \inf_{M_\Psi}\Big(\sqrt{\frac{m}{m-1}R_{p,u,\Psi}}-\|H\|\Big)^2. \tag{13}$$

Proof. Define a modified spinorial Levi−Civita connection on $\Gamma(\mathbb{S})$ by

$$\nabla^u_i\Psi=\nabla_i\Psi+\alpha e^i\cdot H\cdot\Psi+\beta\lambda_H e_i\cdot\omega_\perp\cdot\Psi+p\nabla_i u\Psi+q\nabla_j u e_i\cdot e_j\cdot\Psi \tag{14}$$

for any real−valued functions α,β,p and q. Then, for any $\Psi\in\Gamma(\mathbb{S})$ and for any $i,\ 1\leq i\leq n$, we have

$$\begin{aligned} |\nabla^u_i\Psi|^2 &= |\nabla_i\Psi|^2+2\alpha Re\Big(\nabla_i\Psi,e_i\cdot H\cdot\Psi\Big)+2\lambda_H\beta Re\Big(\nabla_i\Psi,e_i\cdot\omega_\perp\cdot\Psi\Big) \\ &\quad+2pRe\Big(\nabla_i\Psi,\nabla_i u\Psi\Big)+2qRe\Big(\nabla_i\Psi,\nabla_j u e_i\cdot e_j\cdot\Psi\Big)+\alpha^2\|H\|^2|\Psi|^2 \\ &\quad+2\alpha\beta\lambda_H Re\Big(e_i\cdot H\cdot\Psi,e_i\cdot\omega_\perp\cdot\Psi\Big)+2\alpha pRe\Big(e_i\cdot H\cdot\Psi,\nabla_i u\Psi\Big) \\ &\quad+2\alpha qRe\Big(e_i\cdot H\cdot\Psi,\nabla_j u e_i\cdot e_j\cdot\Psi\Big)+\beta^2\lambda_H^2|\Psi|^2 \\ &\quad+2\beta p\lambda_H Re\Big(e_i\cdot\omega_\perp\cdot\Psi,\nabla_i u\Psi\Big)+2\beta q\lambda_H Re\Big(e_i\cdot\omega_\perp\cdot\Psi,\nabla_j u e_i\cdot e_j\cdot\Psi\Big) \\ &\quad+p^2|\nabla_i u|^2|\Psi|^2+2pqRe\Big(\nabla_i u\Psi,\nabla_j u e_i\cdot e_j\cdot\Psi\Big) \\ &\quad+q^2|du|^2|\Psi|^2. \end{aligned} \tag{15}$$

Summing over i and using the fact that $\left(\omega_\perp \cdot \Psi, \Psi\right) = (-1)^n\left(\Psi, \omega_\perp \cdot \Psi\right)$, we have

$$\begin{aligned} |\nabla^u\Psi|^2 \;=\; & |\nabla\Psi|^2 - 2\alpha Re\left(D\Psi, H\cdot\Psi\right) - 2\lambda_H\beta Re\left(D\Psi, \omega_\perp\cdot\Psi\right) \\ & +2p\sum_{i=1}^{m}\nabla_i u Re\left(\nabla_i\Psi, \Psi\right) + m\alpha^2\|H\|^2|\Psi|^2 + 2m\alpha\beta\lambda_H Re\left(H\cdot\Psi, \omega_\perp\cdot\Psi\right) \\ & -2\alpha p Re\left(H\cdot\Psi, du\cdot\Psi\right) + 2m\alpha q Re\left(H\cdot\Psi, du\cdot\Psi\right) + m\beta^2\lambda_H^2|\Psi|^2 \\ & -2\beta p\lambda_H Re\left(\omega_\perp\cdot\Psi, du\cdot\Psi\right) + 2m\beta q\lambda_H Re\left(\omega_\perp\cdot\Psi, du\cdot\Psi\right) + p^2|du|^2|\Psi|^2 \\ & -2pq|du|^2|\Psi|^2 + mq^2|du|^2|\Psi|^2. \end{aligned} \tag{16}$$

Taking $q = \frac{p}{m}$, $\alpha = \frac{\beta-1}{2(m\beta-1)}$, for β nowhere equal to $\frac{1}{m}$ and using the equality obtained in (10), we get

$$\begin{aligned} \int_M \left(1 + m\beta^2 - 2\beta\right)\lambda_H^2|\Psi|^2 \;\geq\; & \int_M \Big(\frac{(R + R^N_\Psi)}{4} + \left(1 - \frac{1}{m}\right)p^2|du|^2 - p\Delta u + \nabla p\nabla u\Big) \\ & -\Big(\frac{m^2\beta^2 - 2m\beta - m\beta^2 + 2\beta - 1)}{4(m\beta-1)^2}\Big)\|H\|^2\Big)|\Psi|^2. \end{aligned} \tag{17}$$

Using modified scalar curvature given in (11), we have

$$\lambda_H^2 \geq \frac{1}{4}\sup_{p,u,\beta}\inf_{M}\Big(\frac{R_{p,u,\Psi}}{1 + m\beta^2 - 2\beta} - \frac{(m-1)}{(m\beta-1)^2}\|H\|^2\Big). \tag{18}$$

Then, assuming $mR_{p,u,\Psi} > (m-1)\|H\|^2 > 0$ on M_Ψ, we can choose β so that

$$(1 - m\beta)^2 = \frac{(m-1)\|H\|}{\sqrt{\frac{m}{m-1}R_{p,u,\Psi}} - \|H\|}, \text{ on } M_\Psi. \tag{19}$$

Inserting (19) in (18) we get (13). □

As in [8], κ_1 be the lowest eigenvalue of the self−adjoint operator $\mathcal{R}^N$ defined by

$$\begin{aligned} \mathcal{R}^N : \Gamma(\mathbb{S}) \;&\longrightarrow\; \Gamma(\mathbb{S}) \\ \Psi \;&\longmapsto\; 2\sum_{i,j=1}^{m} e_i \cdot e_j \cdot Id \otimes R^N_{e_i,e_j}\Psi. \end{aligned} \tag{20}$$

Considering (20), tranforms the $R_{p,u,\Psi}$ as follows

$$R_{p,\kappa_1,u,\Psi} = R + \kappa_1 - 4p\nabla u + 4\nabla p\nabla u - 4\Big(1 - \frac{1}{m}\Big)p^2|du|^2, \tag{21}$$

By using (21), Theorem (3.1) can be stengthened as follows:

Corollary 3.2. *Under the same conditions as in Theorem* (3.1), *if* $m \geq 2$ *and*

$$\widetilde{\Omega}_{p,\kappa_1,u,\Psi} = \{(p, \kappa_1, u) | mR_{p,\kappa_1,u,\Psi} > (m-1)\|H\|^2 > 0\}$$

on M, *then*

$$\lambda_H^2 \;\geq\; \frac{1}{4}\sup_{\widetilde{\Omega}_{p,\kappa_1,u,\Psi}} \operatorname{Inf}_{M_\Psi}\Big(\sqrt{\frac{m}{m-1}R_{p,\kappa_1,u,\Psi}} - \|H\|\Big)^2. \tag{22}$$

In this part of the paper, concerning conformal change of the Riemannian metric and using the classic arguments given in [11–13], the optimal lower bounds is given for the square of the eigenvalue λ_H of the twisted submanifold Dirac operator D_H.

Consider the conformal change of the metric $\overline{g} = e^{2u}g$ given with any real−valued function u on $\widetilde{M}$. Let

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \overline{\mathbb{S}} \\ \Psi & \longmapsto & \overline{\Psi} \end{array} \tag{23}$$

be the induced isometry between the two corresponding spinor bunbles. The Hermitian metrics defined on the corresponding two spinor bundles $\mathbb{S}$ and $\overline{\mathbb{S}}$, respectively satisfies:

$$\left(\Psi, \Phi\right) = \left(\overline{\Psi}, \overline{\Phi}\right)_{\overline{g}}. \tag{24}$$

Also, the Clifford multiplication on $\overline{\mathbb{S}}$ is defined as

$$\overline{e_i}\,\overline{\cdot}\,\overline{\Psi} = \overline{e_i \cdot \Psi}, \tag{25}$$

where $\overline{e_i} = e^{-u}e_i$. Note that, $\overline{g} = e^{2u}g\big|_M$ is denoted the restriction of $\overline{g}$ to M. Under this restriction, the following identity are satisfied:

$$\overline{D}\left(e^{-\frac{(m-1)}{2}u}\overline{\Psi}\right) = e^{-\frac{(m+1)}{2}u}\overline{D\Psi}, \tag{26}$$

where $\Psi \in \Gamma(\mathbb{S})$ and $\overline{D}$ is the Dirac operator with respect to $\overline{g}$. Also, the corresponding mean curvature vector field is given by

$$\widetilde{H} = e^{-2u}\left(H - m\,grad^N u\right). \tag{27}$$

Let $grad^N u = 0$, then with respect to $\overline{g}$, the corresponding twisted Dirac operator $\overline{D}_H$ satisfies:

$$\overline{D}_H\left(e^{-\frac{(m-1)}{2}u}\overline{\Psi}\right) = e^{-\frac{(m+1)}{2}u}\overline{D_H\Psi}. \tag{28}$$

Finally, under the conformal change of the metric $\overline{g} = e^{2u}g$, $\overline{R}_{p,\kappa_1,u,\Psi}$ is written as

$$\begin{aligned} \overline{R}_{p,\kappa_1,u,\Psi} &= R + \kappa_1 + 4\Big(\frac{m-1}{2} - p\Big)\Delta u + 4\nabla p \nabla u - \Big(\big(m-1\big)(m-2) + 4(2-m)p \\ &\quad + 4\Big(1 - \frac{1}{m}\Big)p^2\Big)|du|^2. \end{aligned} \tag{29}$$

In the next theorem, we will consider the regular conformal change of the metric $\overline{g}$ with $grad^N u = 0$, on M.

Theorem 3.3. *Let $M \subset \widetilde{M}$ be a compact Riemannian Spin−submanifold of a Riemannian Spin−manifold $(\widetilde{M}, g)$. Consider a non−trivial eigenspinor field $\Psi \in \Gamma(\mathbb{S})$ such that $D_H\Psi = \lambda_H\Psi$. For any regular conformal change of the metric $\overline{g} = e^{2u}g$ on $\widetilde{M}$, assume that*

$$\overline{\Omega}_{p,\kappa_1,u,\Psi} = \{(p, \kappa_1, u) | m\overline{R}_{p,\kappa_1,u,\Psi} > (m-1)\|H\|^2 > 0\}$$

on M_Ψ. Then the following inequality is satisfied

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\overline{\Omega}_{p,\kappa_1,u,\Psi}} \operatorname{Inf}_{M_\Psi}\Big(\sqrt{\frac{m}{m-1}\overline{R}_{p,\kappa_1,u,\Psi}} - \|H\|\Big)^2. \tag{30}$$

Proof. Let $\Psi \in \Gamma(\mathbb{S})$ be an eigenspinor of D_H with eigenvalue λ_H and let $\overline{\phi} := e^{-\frac{n-1}{2}u}\overline{\Psi}$. Then, by condidering $\overline{D}_H\overline{\phi} = \lambda_H e^{-u}\overline{\Phi}$, $\widetilde{H} = e^{-u}\overline{H}$, $\overline{R}^N_{\overline{\Phi}} = e^{-2u}R^N_\Psi$ and applying $\overline{\Phi}$ to (17), we get

$$\begin{aligned} \int_M \Big(1 + m\beta^2 - 2\beta\Big)e^{-2u}\lambda_H^2|\overline{\Phi}|^2 \geq \int_M \frac{1}{4}\Big(&\overline{R}_{p,\kappa_1,u,\Psi} \\ &-\Big(\frac{m^2\beta^2 - 2m\beta - m\beta^2 + 2\beta - 1)}{(mb-1)^2}\Big)\|H\|^2\Big)e^{-2u}|\overline{\Phi}|^2. \end{aligned} \tag{31}$$

As in the proof of Theorem 3.1, by considering

$$(1 - m\beta)^2 = \frac{(m-1)\|H\|}{\sqrt{\frac{m}{m-1}\overline{R}_{p,\kappa_1,u,\Psi}} - \|H\|}, \text{ on } M_\Psi, \tag{32}$$

we get the desired estimates given in (30). □

In the following theorem we improve our estimation in terms of the Energy−Momentum tensor Q^{Ψ} defined on M_{Ψ} as follows:

$$Q^{\Psi}_{ij} = \frac{1}{2}\Big(e_i \cdot \omega_{\perp} \cdot \nabla_j \Psi + e_j \cdot \omega_{\perp} \cdot \nabla_i \Psi, \frac{\Psi}{|\Psi|^2}\Big). \tag{33}$$

Theorem 3.4. *Let $M \subset \widetilde{M}$ be a compact Riemannian Spin−submanifold of a Riemannian Spin−manifold $(\widetilde{M}, g)$. Consider a non−trivial eigenspinor field $\Psi \in \Gamma(\mathbb{S})$ such that $D_H \Psi = \lambda_H \Psi$. Assume that $m \geq 2$ and*

$$\Omega^{Q^{\Psi}}_{p,\kappa_1,u,\Psi} = \{(p,\kappa_1,u) | R_{p,\kappa_1,u,\Psi} + 4|Q^{\Psi}|^2 > \|H\|^2 > 0\}, \tag{34}$$

on M_{Ψ}. Then the following inequality is satisfied

$$\lambda_H^2 \geq \frac{1}{4} \sup_{\Omega^{Q^{\Psi}}_{p,\kappa_1,u,\Psi}} \ \mathop{Inf}_{M_{\Psi}} \Big(\sqrt{R_{p,\kappa_1,u,\Psi} + 4|Q^{\Psi}|^2} - \|H\|\Big)^2. \tag{35}$$

Proof. Define a modified spinorial Levi−Civita connection on $\Gamma(\mathbb{S})$ by

$$\begin{aligned}\nabla^{Q^{\Psi}}_i \Psi &= \nabla_i \Psi + \alpha e^i \cdot H \cdot \Psi + \beta \lambda_H e_i \cdot \omega_{\perp} \cdot \Psi + p \nabla_i u \Psi + q \nabla_j u e_i \cdot e_j \cdot \Psi \\ &\quad + Q^{\Psi}_{ij} e_j \cdot \omega_{\perp} \cdot \Psi,\end{aligned} \tag{36}$$

where α, β, p and q are real−valued functions. Then, for any $\Psi \in \Gamma(\mathbb{S})$ and for any i, $1 \leq i \leq n$, we have

$$\begin{aligned}|\nabla^{Q^{\Psi}}_i \Psi|^2 &= |\nabla_i \Psi|^2 + 2\alpha Re\Big(\nabla_i \Psi, e_i \cdot H \cdot \Psi\Big) + 2\lambda_H \beta Re\Big(\nabla_i \Psi, e_i \cdot \omega_{\perp} \cdot \Psi\Big) \\ &\quad + 2pRe\Big(\nabla_i \Psi, \nabla_i u \Psi\Big) + 2qRe\Big(\nabla_i \Psi, \nabla_j u e_i \cdot e_j \cdot \Psi\Big) \\ &\quad + 2Re\Big(\nabla_i \Psi, Q^{\Psi}_{ij} e_j \cdot \omega_{\perp} \cdot \Psi\Big) + \alpha^2 \|H\|^2 |\Psi|^2 \\ &\quad + 2\alpha\beta\lambda_H Re\Big(e_i \cdot H \cdot \Psi, e_i \cdot \omega_{\perp} \cdot \Psi\Big) + 2\alpha p Re\Big(e_i \cdot H \cdot \Psi, \nabla_i u \Psi\Big) \\ &\quad + 2\alpha q Re\Big(e_i \cdot H \cdot \Psi, \nabla_j u e_i \cdot e_j \cdot \Psi\Big) + 2\alpha Re\Big(e_i \cdot H \cdot \Psi, Q^{\Psi}_{ij} e_j \cdot \omega_{\perp} \cdot \Psi\Big) \\ &\quad + \beta^2 \lambda_H^2 |\Psi|^2 + 2\beta p \lambda_H Re\Big(e_i \cdot \omega_{\perp} \cdot \Psi, \nabla_i u \Psi\Big) \\ &\quad + 2\beta q \lambda_H Re\Big(e_i \cdot \omega_{\perp} \cdot \Psi, \nabla_j u e_i \cdot e_j \cdot \Psi\Big) \\ &\quad + 2\beta \lambda_H Re\Big(e_i \cdot \omega_{\perp} \cdot \Psi, Q^{\Psi}_{ij} e_j \cdot \omega_{\perp} \cdot \Psi\Big) + p^2 |\nabla_i u|^2 |\Psi|^2 \\ &\quad + 2pqRe\Big(\nabla_i u \Psi, \nabla_j u e_i \cdot e_j \cdot \Psi\Big) + 2pRe\Big(\nabla_i u \Psi, Q^{\Psi}_{ij} e_j \cdot \omega_{\perp} \cdot \Psi\Big) \\ &\quad + q^2 |du|^2 |\Psi|^2 + 2qRe\Big(\nabla_u e_i \cdot e_j \cdot \Psi, Q^{\Psi}_{ij} e_j \cdot \omega_{\perp} \cdot \Psi\Big) + |Q^{\Psi}_{ij}|^2 |\Psi|^2.\end{aligned} \tag{37}$$

Summing over i and using the fact that $trQ^{\Psi} = \lambda_H + \frac{1}{2} Re\Big(H \cdot \Psi, \frac{\omega_{\perp} \cdot \Psi}{|\Psi|^2}\Big)$, we have

$$\begin{aligned}|\nabla^{Q^{\Psi}} \Psi|^2 &= |\nabla \Psi|^2 - 2\lambda_H \alpha Re\Big(\omega_{\perp} \cdot \Psi, H \cdot \Psi\Big) - \alpha \|H\|^2 |\Psi|^2 - 2\lambda_H^2 \beta |\Psi|^2 \\ &\quad - \lambda_H \beta Re\Big(H \cdot \Psi, \omega_{\perp} \cdot \Psi\Big) + 2p \sum_{i=1}^{m} Re\Big(\nabla_i \Psi, \nabla_i u \Psi\Big) - 2|Q^{\Psi}|^2 |\Psi|^2 \\ &\quad + m\alpha^2 \|H\|^2 |\Psi|^2 + 2m\alpha\beta\lambda_H Re\Big(H \cdot \Psi, \omega_{\perp} \cdot \Psi\Big) - 2\alpha p Re\Big(H \cdot \Psi, du \cdot \Psi\Big) \\ &\quad + 2m\alpha q Re\Big(H \cdot \Psi, du \cdot \Psi\Big) + 2\alpha \lambda_H Re\Big(H \cdot \Psi, \omega_{\perp} \cdot \Psi\Big) \\ &\quad + \frac{\alpha Re\Big(H \cdot \Psi, \omega_{\perp} \cdot \Psi\Big)^2}{|\Psi|^4} |\Psi|^2 + m\lambda_H^2 \beta^2 |\Psi|^2 - 2\lambda_H \beta p Re\Big(\omega_{\perp} \cdot \Psi, du \cdot \Psi\Big) \\ &\quad + 2\lambda_H \beta q m Re\Big(\omega_{\perp} \cdot \Psi, du \cdot \Psi\Big) + 2\lambda_H^2 \beta |\Psi|^2 + \lambda_H \beta Re\Big(H \cdot \Psi, \omega_{\perp} \cdot \Psi\Big) \\ &\quad + p^2 |du|^2 |\Psi|^2 - 2pq |du|^2 |\Psi|^2 + mq^2 |du|^2 |\Psi|^2 + |Q^{\Psi}|^2 |\Psi|^2.\end{aligned} \tag{38}$$

Taking $q = \frac{p}{m}$, and using the equality obtained in (10), we get

$$\int_M |\nabla^{Q^\Psi}\Psi|^2 = \int_M \Big(\big(1+m\beta^2\big)\lambda_H^2|\Psi|^2 - \frac{1}{4}\big(R+R^N\big)|\Psi|^2 + \big(1-\frac{1}{m}\big)p^2|du|^2|\Psi|^2 + \big(p\Delta_u - \nabla p\nabla u\big)|\Psi|^2 + \big(1+2m\alpha\beta\big)Re\big(H\cdot\Psi,\omega_\perp\cdot\Psi\big) - |Q^\Psi|^2|\Psi|^2 + \big(\frac{1}{4}+m\alpha^2-\alpha\big)||H||^2|\Psi|^2 + \frac{\alpha Re\big(H\cdot\Psi,\omega_\perp\cdot\Psi\big)^2}{|\Psi|^4}|\Psi|^2\Big). \quad (39)$$

Using the definition given in (21) and taking $\alpha = -\frac{1}{2m\beta}$ we get

$$\int_M \Big[\big(1+m\beta^2\big)\lambda_H^2|\Psi|^2 \geq \int_M \Big(\frac{R_{p,\kappa_1,u,\Psi}}{4}|\Psi|^2 - \big(\frac{1+m\beta^2}{4m\beta^2}\big)||H||^2|\Psi|^2 - \frac{1}{2m\beta}\Big(||H||^2 - \frac{Re\big(H\cdot\Psi,\omega_\perp\cdot\Psi\big)^2}{|\Psi|^4}|\Psi|^2\Big) + |Q^\Psi|^2|\Psi|^2\Big) \quad (40)$$

Since $||H||^2 - \frac{Re\big(H\cdot\Psi,\omega_\perp\cdot\Psi\big)^2}{|\Psi|^4}|\Psi|^2 \geq 0$, we have
we have

$$\lambda_H^2 \geq \frac{1}{4}\inf_M\Big(\frac{R_{p,\kappa_1,u,\Psi}+4|Q|^2}{1+m\beta^2} - \frac{||H||^2}{m\beta^2}\Big). \quad (41)$$

If $R_{p,\kappa_1,u,\Psi} + 4|Q|^2 > ||H||^2 > 0$ on M_Ψ, we can choose β as

$$\beta = \sqrt{\frac{||H||}{\big(m\sqrt{R_{p,\kappa_1,u,\Psi}+4|Q|^2} - ||H||\big)}} \text{ on } M_\Psi. \quad (42)$$

□

In the next theorem, we will consider regular conformal change of the metric $\overline{g}$ with $grad^N u = 0$, on M as in Theorem (3.3).

Theorem 3.5. *Let $M \subset \widetilde{M}$ be a compact Riemannian Spin−submanifold of a Riemannian Spin−manifold $(\widetilde{M}, g)$. Consider a non−trivial eigenspinor field $\Psi \in \Gamma(\mathbb{S})$ such that $D_H\Psi = \lambda_H\Psi$. For any regular conformal change of the metric $\overline{g} = e^{2u}g$ on M, assume that*

$$\overline{\Omega}^{Q^\Psi}_{p,\kappa_1,u,\Psi} = \{(p,\kappa_1,u)|\overline{R}_{p,\kappa_1,u,\Psi} + 4|Q^\Psi|^2 > ||H||^2 > 0\}$$

on M_Ψ. Then the following inequality is satisfied

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\overline{\Omega}^{Q^\Psi}_{p,\kappa_1,u,\Psi}} \inf_{M_\Psi}\Big(\sqrt{\overline{R}_{p,\kappa_1,u,\Psi}+4|Q|^2} - ||H||\Big)^2. \quad (43)$$

where p, u are real−valued functions.

Proof. As in Theorem (3.3, applying $\overline{\Phi}$ to (40), we get

$$\int_M \big(1+m\beta^2\big)e^{-2u}\lambda_H^2|\overline{\Phi}|^2 \geq \int_M \frac{1}{4}\Big(\overline{R}_{p,\kappa_1,u,\Psi} + 4|Q^\Psi|^2 - \big(\frac{1+m\beta^2)}{(m\beta^2}\big)||H||^2\Big)e^{-2u}|\overline{\Phi}|^2. \quad (44)$$

As in the proof of Theorem 3.3, we finally by taking

$$\beta = \sqrt{\frac{||H||}{m\big(\sqrt{\overline{R}_{p,\kappa_1,u,\Psi}+4|Q^\Psi|^2} - ||H||\big)}} \quad (45)$$

we obtained the desired result given in (43). □

References

[1] Bär, C. (1998). Extrinsic Bounds for Eigenvalues of the Dirac Operator. *Ann. Glob. Anal. Geom.*, 16(6), 573–596.

[2] Eker, S., Değirmenci, N. (2018). Seiberg−Witten−like equations without Self−Duality on odd dimensional manifolds. *Journal of Partial Differential Equations*, 31(4), 291–303.

[3] Eker, S. (2019). The Bochner vanishing theorems on the conformal killing vector fields, *TWMS Journal of Applied and Engineering Mathematics*, 9(1) Special Issue, 114–120.

[4] Eker, S . (2020). Lower Bound Eigenvalue Problems of the Compact Riemannian Spin-Submanifold Dirac Operator. *Erzincan Üniversitesi Fen Bilimleri Enstitüsü Dergisi* , 13 (ÖZEL SAYI I) , 56–62 .

[5] Eker, S. (2020). Lower Bounds for the Eigenvalues of the Dirac Operator on Spin^c Manifolds. *Iranian Journal of Science and Technology, Transactions A: Science*, 44(1), 251—257.

[6] Friedrich, T. (1980). Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung. *Mathematische Nachrichten*, 97(1), 117–146.

[7] Friedrich, T. (2001). Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors. *Journal of Geometry and Physics*, 37(1-2), 1–14.

[8] Ginoux, N., Bertrand, M. (2002). On eigenvalue estimates for the submanifold Dirac operator. *International Journal of Mathematics*, 13(05), 533-548.

[9] Hijazi, O. (1986). A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors. *Communications in Mathematical Physics*, 104(1), 151–162.

[10] Hijazi, O. (1991). Première valeur propre de l'opérateur de Dirac et nombre de Yamabe. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 313(12), 865–868.

[11] Hijazi, O. (1995). Lower bounds for the eigenvalues of the Dirac operator. *Journal of Geometry and Physics*, 16(1), 27–38.

[12] Hijazi, O., Xiao, Z. (2001). Lower Bounds for the Eigenvalues of the Dirac Operator: Part II. The Submanifold Dirac Operator. *Annals of Global Analysis and Geometry*, 20(2), 163–181.

[13] Hijazi, O., Xiao, Z. (2001). Lower Bounds for the Eigenvalues of the Dirac Operator: Part II. The Submanifold Dirac Operator. *Annals of Global Analysis and Geometry*, 20(2), 163–181.

[14] Hijazi, O., Montiel, S., Xiao, Z. (2001). Eigenvalues of the Dirac Operator on Manifolds with Boundary. *Communications in Mathematical Physics*, 221(2), 255–265.

[15] Lawson, H.B., Michelsohn, M.L. (2016). Spin geometry. *Princeton university press*, 38.

[16] Lichnerowicz, O. (1963). Spineurs harmoniques. *C.R. Acad. Sci. Paris Ser. A–B*, 257.

[17] Morel, B. (2001). Eigenvalue estimates for the Dirac–Schrödinger operators. *Journal of Geometry and Physics*, 38(1), 1–18.

[18] Adriana, T. (2009). On the twisted Dirac operators. *Bull. Math. Soc. Sci. Math. Roumanie Tome*, 52(3), 383–386.

[19] Xiao, Z. (1998). Lower bounds for eigenvalues of hypersurface Dirac operators. *Mathematical Research Letters*, 5(2), 199–210.

[20] Witten, E. (1981). A new proof of the positive energy theorem. *Communications in Mathematical Physics*, 80(3), 381–402.