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The Hadamard-type Padovan-p Sequences

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Abstract. In this paper, we define the Hadamard-type Padovan-p sequence by using the Hadamard-type product of characteristic polynomials of the Padovan sequence and the Padovan-p sequence. Also, we derive the generating matrices for these sequences. Then using the roots of characteristic polynomial of the Hadamard-type Padovan-p sequence, we produce the Binet formula for the Hadamard-type Padovan-p numbers. Also, we give the permanental, determinantal, combinatorial, exponential representations and the sums of the Hadamard-type Padovan-p numbers.

1. Introduction

It is well-known that Padovan sequence is defined by the following equation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$, where $P(0) = P(1) = P(2) = 1$.

Deveci and Karaduman defined [8] the Padovan p-numbers as shown:

$$Pap(n+p+2) = Pap(n+p) + Pap(n)$$

for any given $p\,(p = 2, 3, 4, \ldots)$ and $n \geq 1$ with initial conditions $Pap(1) = Pap(2) = \cdots = Pap(p) = 0$, $Pap(p+1) = 1$ and $Pap(p+2) = 0$.

It is clear that the characteristic polynomials of Padovan sequence and the Padovan-p sequence are $P(x) = x^3 - x - 1$ and $P_p(x) = x^{p+2} - x^p - 1$, respectively.

Akuzum and Deveci [1] defined the Hadamard-type product of polynomials f and g as follows:

$$f(x) * g(x) = \sum_{i=0}^{\infty} (a_i * b_i)\, x^i, \text{ where } a_i * b_i = \begin{cases} a_i b_i & \text{if} \quad a_i b_i \neq 0 \\ a_i + b_i & \text{if} \quad a_i b_i = 0 \end{cases},$$

such that $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$.

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

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where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [13], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

$$A = \left[a_{i,j}\right]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument, he obtained that

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Recently, many authors studied number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant [2, 5–12, 14–20]. In [1], Akuzum and Deveci defined the Hadamard-type product of two polynomials and they obtained the Hadamard-type k-step Fibonacci sequence by the aid of this the Hadamard-type product. Then they studied properties of this sequence in detail. In this paper, we define the Hadamard-type Padovan-p sequence by using the definition of Hadamard-type product in [1]. Also, we produce the generating matrix of this sequence. Then we give relationships between the Hadamard-type Padovan-p numbers and the permanents and the determinants of certain matrices which are produced by using the generating matrix of the Hadamard-type Padovan-p sequence. Also, we obtain the combinatorial representations, the generating function, the exponential representation and the sums of the Hadamard-type Padovan-p numbers.

2. The Hadamard-type Padovan-p Sequences

We define a new sequence which is defined by using Hadamard-type product of characteristic polynomials of Padovan sequence and the Padovan-p sequence and is called the Hadamard-type Padovan-p sequence. This sequence is defined by integer constants $P_0^h = P_1^h = \cdots = P_p^h = 0$ and $P_{p+1}^h = 1$ and the recurrence relation

$$P_{n+p+2}^h = P_{n+p}^h - P_{n+3}^h + P_{n+1}^h - P_n^h \tag{1}$$

for the integers $n \geq 0$ and $p \geq 4$.

By relation (1), we can write the following companion matrix:

$$M_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+2)\times(p+2)}.$$

The matrix M_p is said to be a Hadamard-type Padovan-p matrix.

It can be readily established by an inductive argument that

$$\left(M_p\right)^n=\begin{bmatrix} P^h_{n+p+1} & P^h_{n+p+2} & & P^h_{n+p-1}-P^h_{n+p-2} & P^h_{n+p}-P^h_{n+p-1} & -P^h_{n+p} \\ P^h_{n+p} & P^h_{n+p+1} & & P^h_{n+p-2}-P^h_{n+p-3} & P^h_{n+p-1}-P^h_{n+p-2} & -P^h_{n+p-1} \\ P^h_{n+p-1} & P^h_{n+p} & & P^h_{n+p-3}-P^h_{n+p-4} & P^h_{n+p-2}-P^h_{n+p-3} & -P^h_{n+p-2} \\ \vdots & \vdots & M^*_p & \vdots & \vdots & \vdots \\ P^h_{n+1} & P^h_{n+2} & & P^h_{n-1}-P^h_{n-2} & P^h_{n}-P^h_{n-1} & -P^h_{n} \\ P^h_{n} & P^h_{n+1} & & P^h_{n-2}-P^h_{n-3} & P^h_{n-1}-P^h_{n-2} & -P^h_{n-1} \end{bmatrix} \tag{2}$$

where M^*_p is a $(p-3)\times(p-3)$ matrix as follows:

$$\begin{bmatrix} P^h_{n+p+3}-P^h_{n+p+1} & P^h_{n+p+4}-P^h_{n+p+2} & \cdots & P^h_{n+2p-1}-P^h_{n+2p-3} \\ P^h_{n+p+2}-P^h_{n+p} & P^h_{n+p+3}-P^h_{n+p+1} & \cdots & P^h_{n+2p-2}-P^h_{n+2p-4} \\ P^h_{n+p+1}-P^h_{n+p-1} & P^h_{n+p+2}-P^h_{n+p} & \cdots & P^h_{n+2p-3}-P^h_{n+2p-5} \\ \vdots & \vdots & & \vdots \\ P^h_{n+3}-P^h_{n+1} & P^h_{n+4}-P^h_{n+2} & \cdots & P^h_{n+p-1}-P^h_{n+p-3} \\ P^h_{n+2}-P^h_{n} & P^h_{n+3}-P^h_{n+1} & \cdots & P^h_{n+p-2}-P^h_{n+p-4} \end{bmatrix}$$

for $n\geq 3$. Also, It is easy to see that $\det M_p=(-1)^p$.

Now we concentrate on finding a Binet formula for the Hadamard-type Padovan-p numbers.

Lemma 2.1. *The characteristic equation of the Hadamard-type Padovan-p sequence* $x^{p+2}-x^p+x^3-x+1=0$ *does not have multiple roots.*

Proof. Let $f(x)=x^{p+2}-x^p+x^3-x+1$*. It is clear that* $f(0)\neq 0$ *and* $f(1)\neq 0$ *for all* $p\geq 4$*. Let* λ *be a multiple root of* $f(x)$*, then* $\lambda\notin\{0,1\}$*. If it is possible that* λ *is a multiple root of* $f(x)$ *then it follows that* $f(\lambda)=0$ *and* $f'(\lambda)=0$*. Now, we consider* $f(\lambda)=\lambda^{p+2}-\lambda^p+\lambda^3-\lambda+1$*. So, we obtain*

$$\lambda^p=\frac{-\lambda^3+\lambda-1}{\lambda^2-1}. \tag{3}$$

Moreover, we may write $f'(\lambda)=(p+2)\lambda^{p+1}-p\lambda^{p-1}+3\lambda^2-1$ *and hence we get*

$$\lambda^p=\frac{-3\lambda^3+\lambda}{(p+2)\lambda^2-p}. \tag{4}$$

From (3) and (4), the following equation can be obtained:

$$p=1+\frac{3\lambda^2-1}{-\lambda^5+2\lambda^3-\lambda^2-\lambda+1}.$$

Using appropriate softwares such as Mathematica Wolfram 10.0 [21], we obtain that there is no solution for $p\geq 4$*. Since all p's are integers with* $p\geq 4$*, it is a contradiction. So, the equation* $f(x)=0$ *does not have multiple roots.* □

If $x_1, x_2, \ldots, x_{p+2}$ are roots of the equation $x^{p+2}-x^p+x^3-x+1$, then by Lemma 2.1, it is known that x_1, $x_2, \ldots, x_{p+2}$ are distinct. Define the $(p+2)\times(p+2)$ Vandermonde matrix V^{p+2} as shown:

$$V^{p+2}=\begin{bmatrix} (x_1)^{p+1} & (x_2)^{p+1} & \cdots & \left(x_{p+2}\right)^{p+1} \\ (x_1)^{p} & (x_2)^{p} & \cdots & \left(x_{p+2}\right)^{p} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & & x_{p+2} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Assume that

$$W^{p+2}(i,j)=\begin{bmatrix} x_1^{n+p+2-i} \\ x_2^{n+p+2-i} \\ \vdots \\ x_{p+2}^{n+p+2-i} \end{bmatrix}$$

and $V^{p+2}(i,j)$ is a $(p+2)\times(p+2)$ matrix obtained from V^{p+2} by replacing the jth column of V^{p+2} by $W^{p+2}(i,j)$.

Theorem 2.2. *Let $(M_P)^n=\left[m_{i,j}^{p,n}\right]$, then*

$$m_{i,j}^{p,n}=\frac{\det V^{p+2}(i,j)}{\det V^{p+2}},$$

for $n\geq 3$ and $p\geq 4$.

Proof. Since the eigenvalues of the matrix M_P, $x_1, x_2, \ldots, x_{p+2}$ are distinct, the matrix M_P is diagonalizable. Let $D^{p+2}=\left(x_1,x_2,\ldots,x_{p+2}\right)$, then we easily see that $M_PV^{p+2}=V^{p+2}D^{p+2}$. Since V^{p+2} is invertible, we can write $\left(V^{p+2}\right)^{-1}M_PV^k=D^{p+2}$. Then, the matrix M_P is similar to D^{p+2} and so $(M_P)^n\,V^{p+2}=V^{p+2}\left(D^{p+2}\right)^n$. Hence we have the following linear system of equations:

$$\begin{cases} m_{i,1}^{p,n}x_1^{p+1}+m_{i,2}^{p,n}x_1^{p}+\cdots+m_{i,p+2}^{p,n}=x_1^{n+p+2-i} \\ m_{i,1}^{p,n}x_2^{p+1}+m_{i,2}^{p,n}x_2^{p}+\cdots+m_{i,p+2}^{p,n}=x_2^{n+p+2-i} \\ \vdots \\ m_{i,1}^{p,n}x_{p+2}^{p+1}+m_{i,2}^{p,n}x_{p+2}^{p}+\cdots+m_{i,p+2}^{p,n}=x_{p+2}^{n+p+2-i} \end{cases}$$

Therefore, for each $i,j=1,2,\ldots,k$, we obtain

$$m_{i,j}^{p,n}=\frac{\det V^{p+2}(i,j)}{\det V^{p+2}}.$$

□

From this result we immediately deduce:

Corollary 2.3. *Let P_n^h be the nth the Hadamard-type Padovan-p number, then*

$$P_n^h=\frac{\det V^{p+2}(p+2,1)}{\det V^{p+2}}=-\frac{\det V^{p+2}(p+1,p+2)}{\det V^{p+2}}$$

for $n\geq 3$ and $p\geq 4$.

Now we concentrate on finding the permanental representations of the Hadamard-type Padovan-p numbers.

Definition 2.4. *A $u\times v$ real matrix $M=\left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1,x_2,\ldots,x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k}\neq 0, m_{j,k}\neq 0$ and $i\neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j+m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [3], Brualdi and Gibson obtained that $per(M)=per(N)$ if M is a real matrix of order $\alpha>1$ and N is a contraction of M.

Let $\alpha \geq p+2$ be a integer and let $A^{p,\alpha} = \left[a_{i,j}^{p,\alpha}\right]$ be the $\alpha \times \alpha$ super-diagonal matrix, defined by

$$a_{i,j}^{p,\alpha} = \begin{cases} 1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+1 \text{ for } 1 \leq r \leq \alpha - 1, \\ i = r \text{ and } j = r-1 \text{ for } 2 \leq r \leq \alpha \\ \text{and} \\ i = r \text{ and } j = r+p \text{ for } 1 \leq r \leq \alpha - p, \end{array} \\ -1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+p-2 \text{ for } 1 \leq r \leq \alpha - p + 2 \\ \text{and} \\ i = r \text{ and } j = r+p+1 \text{ for } 1 \leq r \leq \alpha - p - 1, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following Theorem.

Theorem 2.5. *For $\alpha \geq p+2$ and $p \geq 4$,*

$$perA^{p,\alpha} = P^h_{\alpha+p+1}.$$

Proof. The assertion may be proved by induction on α. Let the equation be hold for $\alpha \geq p+2$, then we show that the equation holds for $\alpha+1$. If we expand the $perA^{p,\alpha}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$perA^{p,\alpha+1} = perA^{p,\alpha-1} - perA^{p,\alpha-p+2} + perA^{p,\alpha-p} - perA^{p,\alpha-p-1}.$$

Since $perA^{p,\alpha-1} = P^h_{\alpha+p}$, $perA^{p,\alpha-p+2} = P^h_{\alpha+3}$, $perA^{p,\alpha-p} = P^h_{\alpha+1}$ and $perA^{p,\alpha-p-1} = P^h_{\alpha}$, it is easy to see that $perA^{p,\alpha+1} = P^h_{\alpha+p+2}$. Thus, the proof is complete. □

Let $\alpha \geq p+2$ and let $B^{p,\alpha} = \left[b_{i,j}^{p,\alpha}\right]$ be the $\alpha \times \alpha$ matrix, defined by

$$b_{i,j}^{p,\alpha} = \begin{cases} 1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+1 \text{ for } 1 \leq r \leq \alpha - p - 1, \\ i = r \text{ and } j = r-1 \text{ for } 2 \leq r \leq \alpha \\ \text{and} \\ i = r \text{ and } j = r+p \text{ for } 1 \leq r \leq \alpha - p - 1, \end{array} \\ -1 & \begin{array}{c} \text{if } i = r \text{ and } j = r+p-2 \text{ for } 1 \leq r \leq \alpha - p - 1 \\ \text{and} \\ i = r \text{ and } j = r+p+1 \text{ for } 1 \leq r \leq \alpha - p - 1, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Now we define the $\alpha \times \alpha$ matrix $C^{p,\alpha} = \left[c_{i,j}^{p,\alpha}\right]$ as follows:

$$C^{p,\alpha} = \begin{bmatrix} 1 & \cdots & \overset{(\alpha-p-2)\text{th}}{\overset{\downarrow}{1}} & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & B^{p,\alpha-1} & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}.$$

Then we can give the following Theorem by using the permanental representations.

Theorem 2.6. (i). *For $\alpha \geq p+2$,*

$$perB^{p,\alpha} = -P^h_{\alpha-1}.$$

(ii) *. For $\alpha > p+2$,*

$$perC^{p,\alpha} = -\sum_{i=0}^{\alpha-2} P^h_i.$$

Proof. (i) .Let the equation be hold for $\alpha \geq p+2$, then we show equation hold for $\alpha+1$. If we expand the $perB^{p,\alpha}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$\begin{aligned} perB^{p,\alpha+1} &= perB^{p,\alpha-1} - perB^{p,\alpha-p+2} + perB^{p,\alpha-p} - perB^{p,\alpha-p-1} \\ &= -P^h_{\alpha-2} + P^h_{\alpha-p+1} - P^h_{\alpha-p-1} + P^h_{\alpha-p-2}. \end{aligned}$$

So, we have the conclusion.

(ii) . If we expand the $perC^{p,\alpha}$ with respect to the first row, we write

$$perC^{p,\alpha} = perC^{p,\alpha-1} + perB^{p,\alpha-1}.$$

From Theorem 2.5 and Theorem 2.6. (i) and induction on α, the proof follows directly. □

Let the notation $M \circ K$ denotes the Hadamard product of M and K. A matrix M is called convertible if there is an $u \times u$ (1, -1)-matrix K such that per $M = \det(M \circ K)$.

Let G be the $\alpha \times \alpha$ matrix, defined by

$$G = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}$$

for $\alpha > p+2$.

Corollary 2.7. *For $\alpha > p+2$ and $p \geq 4$*

$$\det(A^{p,\alpha} \circ G) = P^h_{\alpha+p+1},$$

$$\det(B^{p,\alpha} \circ G) = -P^h_{\alpha-1}$$

and

$$\det(C^{p,\alpha} \circ G) = -\sum_{i=0}^{\alpha-2} P^h_i.$$

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.8. *(Chen and Louck [4]).The (i, j) entry $k^{(u)}_{i,j}(k_1, k_2, \ldots, k_v)$ in the matrix $K^u(k_1, k_2, \ldots, k_v)$ is given by the following formula:*

$$k^{(u)}_{i,j}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \tag{5}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = u - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (5) are defined to be 1 if $u = i - j$.

Then we have the following Corollary for the Hadamard-type Padovan-p numbers.

Corollary 2.9. *For $p \geq 4$, let P_n^h be the nth the Hadamard-type Padovan-p number. Then*
i.

$$P_n^h = \sum_{(t_1,t_2\ldots,t_{p+2})} \binom{t_1+\cdots+t_{p+2}}{t_1,\ldots,t_{p+2}} (-1)^{t_{p-1}+t_{p+2}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+2)\,t_{p+2} = n - p - 1$.
ii.

$$P_n^h = -\sum_{(t_1,t_2\ldots,t_k)} \frac{t_{p+2}}{t_1+t_2+\cdots+t_{p+2}} \times \binom{t_1+\cdots+t_{p+2}}{t_1,\ldots,t_{p+2}} (-1)^{t_{p-1}+t_{p+2}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+2)\,t_{p+2} = n + 1$.

Proof. In Theorem 2.8, If we take $i = p+2$ and $j = 1$, for case *i.* and $i = p+1$, $j = p+2$, for case *ii.*, then the proof is immediately seen from $\left(M_p\right)^n$. □

The generating function of the Hadamard-type Padovan-p sequence is given by:

$$f_p(x) = \frac{x^{p+1}}{1 - x^2 + x^{p-1} - x^{p+1} + x^{p+2}}.$$

It can be readily established that the Hadamard-type Padovan-p sequences have the following exponential representation.

Theorem 2.10. *The Hadamard-type Padovan-p numbers have the following exponential representation:*

$$f_p(x) = x^{p+1} \exp\left(\sum_{i=1}^{\infty} \frac{\left(x^2\right)^i}{i} \left(1 - x^{p-3} + x^{p-1} - x^p\right)^i \right)$$

where $p \geq 4$.

Proof. It is clear that

$$\ln \frac{f_p(x)}{x^{p+1}} = -\ln\left(1 - x^2 + x^{p-1} - x^{p+1} + x^{p+2}\right)$$

and

$$\begin{aligned} -\ln\left(1 - x^2 + x^{p-1} - x^{p+1} + x^{p+2}\right) &= -[-x^2\left(1 - x^{p-3} + x^{p-1} - x^p\right) - \\ &\quad \frac{1}{2}x^4\left(1 - x^{p-3} + x^{p-1} - x^p\right)^2 - \cdots - \\ &\quad \frac{1}{n}x^{2n}\left(1 - x^{p-3} + x^{p-1} - x^p\right)^n - \cdots]. \end{aligned}$$

A simple calculation shows that

$$\ln \frac{f_p(x)}{x^{p+1}} = \sum_{i=1}^{\infty} \frac{\left(x^2\right)^i}{i} \left(1 - x^{p-3} + x^{p-1} - x^p\right)^i.$$

Thus the conclusion is obtained. □

Now we consider the sums of the Hadamard-type Padovan-p numbers.
Let

$$T_n = \sum_{i=0}^{n} P_n^h$$

for $n \geq 3$ and $p \geq 4$, and let Q_p be the $(p+3) \times (p+3)$ matrix, such that

$$Q_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & M_p & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then it can be shown by induction that

$$\left(Q_p\right)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ T_{n+p} & & & \\ T_{n+p-1} & & \left(M_p\right)^n & \\ \vdots & & & \\ T_{n-1} & & & \end{bmatrix}.$$

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