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Continuous Dependence on Data for a Solution of determination of an unknown source of Heat Conduction of Poly(methyl methacrylate) (PMMA)

İrem Bağlan[a], Timur Canel[b]

[a]*Department of Mathematics, Kocaeli University, Kocaeli-Turkey*
[b]*Department of Physics, Kocaeli University, Kocaeli-TURKEY*

Abstract. In this paper,we consider a coefficient problem of an inverse problem of a quasilinear parabolic equation with periodic boundary and integral over determination conditions. It showed the stability of the solution by iteration method and examined numerical solution.

1. Introduction

The inverse problem of determining unknown coefficient in a quasi-linear parabolic equation has generated an increasing amount of interest from engineers and scientist during the last few decades.Inverse Problem is a research area dealing with inversion of models or data. An inverse problem is a mathematical framework that is used to obtain information about a physical object or system from observed measurements. It is called an inverse problem because it starts with the results and then calculates the causes. This is the inverse of a direct problem, which starts with the causes and then calculates the results. Thus, inverse problems are some of the most important and well-studied mathematical problems in science and mathematics because they provide us about parameters that we cannot directly observe[1–3]. There are many different applications including medical imaging, geophysics, computer vision, astronomy, nondestructive testing, and many others. Nevertheless the inverse coefficient problems with periodic boundary and integral over determination conditions are not investigated by many researchers because of the difficulties of these conditions [1–3, 5–8]. This kind of conditions arise from many important applications in heat transfer, life sciences, etc. The inverse problem of unknown coefficients in a quasi-linear parabolic equations with periodic boundary conditions was studied by Kanca and Baglan [9, 10]. Over the last years, considerable efforts have been put into develop either approximate analytical solution and numerical solution to non-local boundary value problems [3]. Cannon implemented implicit finite difference scheme to obtain numerical solution of the one dimensional non-local boundary value problems [1]. Liu studied non-local boundary value problems and concluded that the presence of integral terms in boundary conditions can greatly complicate the application of standard numerical techniques such as finite difference schemes and finite element techniques [4]. Several researchers have discussed numerical solutions for non-local boundary value problems in one dimension.The one-dimensional case of this problem has been the guiding force behind considerable research in numerical methods such as finite difference method and finite element method.

Corresponding author: İB mail address: isakinc@kocaeli.edu.tr ORCID:https://orcid.org/0000-0002-2877-9791, TC ORCID: https://orcid.org/0000-0002-4282-1806

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Explicit and implicit finite difference schemes were used by many researchers to obtain numerical solutions of one-dimensional non-local boundary value problem. Finite difference method to a class of parabolic inverse problems is investigated. This method is very effective for solving various kinds of partial differential equations.

Consider the equation

$$u_t = u_{xx} + l(t) f(x,t,u),\ (x,t) \in D, \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x),\ \ x \in [0,\pi], \tag{2}$$

the periodic boundary condition

$$u(0,t) = u(\pi,t),\ u_x(0,t) = u_x(\pi,t),\ 0 \le t \le T, \tag{3}$$

and the over determination data

$$g(t) = u(\pi,t), 0 \le t \le T, \tag{4}$$

for a quasilinear parabolic equation with the nonlinear source term $f = f(x,t,u)$.

Here $D := \{0 < x < \pi,\ 0 < t < T\}$.The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,\pi]$ and $\bar{D} \times (-\infty,\infty)$, respectively.

The problem of finding the pair $\{l(t), u(x,t)\}$ in (1)-(4) will be called an inverse problem.

Definition 1.1. *The pair $\{l(t), u(x,t)\}$ from the class $C[0,T] \times (C^{2,1}(D) \cap C^{1,0}(\overline{D}))$ for which conditions (1)-(4) are satisfied is called the classical solution of the inverse problem (1)-(4).*

The paper organized as follows:

In Section 2, the existence and uniqueness of the solution of the inverse problem (1)-(4) is proved by using the Fourier method and iteration method. In Section 3, the continuous dependence upon the data of the inverse problem is shown. In Section 4, the numerical procedure for the solution of the inverse problem is given.

2. Existence and Uniqueness of the Solution of the Inverse Problem

The main result on the existence and the uniqueness of the solution of the inverse problem (1)-(4) is presented as follows:

We have the following assumptions on the data of the problem (1)-(4).

(A1) $g(t) \in C^1[0,T], l(t) \in C[0,T]$.

(A2) $\varphi(x) \in C^3[0,\pi]$, $\varphi(0) = \varphi(\pi)$, $\varphi'(0) = \varphi'(\pi)$, $\varphi''(0) = \varphi''(\pi)$,

(A3) Let the function $f(x,t,u)$ is continuous with respect to all arguments in $\bar{D} \times (-\infty,\infty)$ and satisfies the following condition

(1)

$$\left| \frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n} \right| \le b(t,x)\,|u - \tilde{u}|\ \ ,n = 0,1,2,$$

where $b(x,t) \in L_2(D),\ b(x,t) \ge 0$,

(2) $f(x,t,u) \in C^3[0,\pi]$, $t \in [0,T]$,

(3) $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=\pi}$, $f_x(0,t,u)|_{x=0} = f_x(\pi,t,u)|_{x=\pi}$, $f_{xx}(0,t,u)|_{x=0} = f_{xx}(\pi,t,u)|_{x=\pi}$,

(4) $\int_0^\pi f(x,t,u)\,dx \ne 0,\ \ \forall t \in [0,T]$.

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3) for arbitrary $l(t) \in C[0,T]$:

$$u(x,t)=\frac{u_0(t)}{2}+\sum_{k=1}^{\infty}\left[u_{ck}(t)\cos 2kx+u_{sk}(t)\sin 2kx\right],$$

$$\begin{aligned}
u_0(t) &= \varphi_0+\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)f\left(\xi,\tau,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right]\right)d\xi d\tau,\\
u_{ck}(t) &= \varphi_{ck}\, e^{-(2k)^2t}+\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)f\left(\xi,\tau,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right]\right)\cos 2k\xi\; e^{-(2k)^2(t-\tau)}d\xi d\tau,\\
u_{sk}(t) &= \varphi_{sk}\, e^{-(2k)^2t}+\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)f\left(\xi,\tau,\frac{u_0(\tau)}{2}+\sum_{k=1}^{\infty}\left[u_{ck}(\tau)\cos 2k\xi+u_{sk}(\tau)\sin 2k\xi\right]\right)\sin 2k\xi\; e^{-(2k)^2(t-\tau)}d\xi d\tau.
\end{aligned}$$

$$\begin{aligned}
u(x,t) &= \varphi_0+\int_0^t l(\tau)f_0(\tau,u)\,d\tau\\
&+\sum_{k=1}^{\infty}\cos 2kx\left[\varphi_{ck}\, e^{-(2k)^2t}+\int_0^t l(\tau)f_{ck}(\tau,u)\, e^{-(2k)^2(t-\tau)}d\tau\right]\\
&+\sum_{k=1}^{\infty}\sin 2kx\left[\varphi_{sk}\, e^{-(2k)^2t}+\int_0^t l(\tau)f_{sk}(\tau,u)\, e^{-(2k)^2(t-\tau)}d\tau\right],
\end{aligned}\tag{5}$$

where $\varphi_0=\frac{2}{\pi}\int_0^\pi\varphi(x)dx,\varphi_{ck}=\frac{2}{\pi}\int_0^\pi\varphi(x)\cos 2kxdx,\varphi_{sk}=\frac{2}{\pi}\int_0^\pi\varphi(x)\sin 2kxdx.$
Under the condition (A1)-(A3), differentiating (4), we obtain

$$u_t(\pi,t)=g'(t),0\le t\le T.\tag{6}$$

(5) and (6) yield

$$l(t)=\frac{g'(t)+\sum\limits_{k=1}^{\infty}(4k^2)\left(\varphi_{ck}e^{-(2k)^2t}+\int_0^t l(\tau)f_{ck}(\tau,u)\, e^{-(2k)^2(t-\tau)}d\tau\right)}{f_0(t)+\sum\limits_{k=1}^{\infty}f_{ck}(t)}.\tag{7}$$

Definition 2.1. *Denote the set* $\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),k=1,...,n\}$, *of continuous on* $[0,T]$ *functions satisfying the condition* $\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)<\infty$, *by* $\mathbf{B}_1$. *Let* $\|u(t)\|=\max\limits_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{ck}(t)|+\max\limits_{0\le t\le T}|u_{sk}(t)|\right)$, *be the norm in* $\mathbf{B}_1$.
It can be shown that $\mathbf{B}_1$ *are the Banach spaces.*

3. Continuous Dependence of (l,u) upon the data

Theorem 3.1. *Under assumption (A1)-(A3) the solution* (l,u) *of the problem (1)-(4) depends continuously upon the data* φ,g.

Proof. Let $\Phi = \{\varphi,\ g,\ f\}$ and $\overline{\Phi} = \{\overline{\varphi},\ \overline{g},\ f\}$ be two sets of the data, which satisfy the assumptions $(A_1) - (A_3)$. Suppose that there exist positive constants M_i, $i = 1,2$ such that

$$\|g\|_{C^1[0,T]} \le M_1, \|\overline{g}\|_{C^1[0,T]} \le M_1, \|\varphi\|_{C^3[0,\pi]} \le M_2, \|\overline{\varphi}\|_{C^3[0,\pi]} \le M_2.$$

Let us denote $\|\Phi\| = (\|g\|_{C^1[0,T]} + \|\varphi\|_{C^3[0,\pi]} + \|f\|_{C^{3,0}(\overline{D})})$. Let (l,u) and $\left(\overline{l},\overline{u}\right)$ be solutions of inverse problems (1)-(4) corresponding to the data $\Phi = \{\varphi,\ g,\ f\}$ and $\overline{\Phi} = \{\overline{\varphi},\ \overline{g},\ f\}$ respectively. According to (5)

$$\begin{aligned} u-\overline{u} &= \frac{(\varphi_0-\overline{\varphi_0})}{2} + \sum_{k=1}^{\infty}\cos 2k\xi\ (\varphi_{ck}-\overline{\varphi_{ck}})\,e^{-(2k)^2t} + \sum_{k=1}^{\infty}\sin 2k\xi\ (\varphi_{sk}-\overline{\varphi_{sk}})\,e^{-(2k)^2t} \\ &+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right]d\xi d\tau\right) \\ &+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi \left(l(\tau)-\overline{l}(\tau)\right) f(\xi,\tau,\overline{u}(\xi,\tau))d\xi d\tau\right) \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi \left(l(\tau)-\overline{l}(\tau)\right)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi l(\tau)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^t\int_0^\pi \left(l(\tau)-\overline{l}(\tau)\right)\left[f(\xi,\tau,u(\xi,\tau)) - f(\xi,\tau,\overline{u}(\xi,\tau))\right]\cos 2k\xi\ e^{-(2k)^2(t-\tau)}d\xi d\tau. \end{aligned}$$

By using same estimations, we obtain:

$$\begin{aligned} |u-\overline{u}| &\le M_3\left\|\Phi-\overline{\Phi}\right\| \\ &+M_4\left(\int_0^t\int_0^\pi l^2(\tau)b^2(\xi,\tau)\left|u(\tau)-\overline{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} \end{aligned} \tag{8}$$

$$\begin{aligned} |a-\overline{a}| &\le M_5\left\|\Phi-\overline{\Phi}\right\| \\ &+M_6\left|r(t)\right|\left|u(t)-\overline{u(t)}\right|, \end{aligned}$$

applying Gronwall's inequality to (8), we obtain:

$$\begin{aligned} |u-\overline{u}|^2 &\le 2M_3^2\left\|\Phi-\overline{\Phi}\right\|^2 \\ &\times\exp 2M_4^2\left(\int_0^t\int_0^\pi l^2(\tau)b^2(\xi,\tau)d\xi d\tau\right). \end{aligned}$$

For $\Phi \to \overline{\Phi}$ then $u \to \overline{u}$. Hence $l \to \overline{l}$. □

4. Numerical Procedure for the nonlinear problem (1)-(4)

We construct an iteration algorithm for the linearization of the problem (1)-(4):

$$\frac{\partial u^{(n)}}{\partial t} = \frac{\partial^2 u^{(n)}}{\partial x^2} + l(t) f(x,t,u^{(n-1)}), \quad (x,t) \in D \tag{9}$$
$$u^{(n)}(0,t) = u^{(n)}(\pi,t), \quad t \in [0,T] \tag{10}$$
$$u_x^{(n)}(0,t) = u_x^{(n)}(\pi,t) = 0, \; t \in [0,T] \tag{11}$$
$$u^{(n)}(x,0) = \varphi(x), \quad x \in [0,\pi]. \tag{12}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $f(x,t,u^{(n-1)}) = \widetilde{f}(x,t)$. Then the problem (9)-(12) can be written as a linear problem:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + l(t)\widetilde{f}(x,t) \quad (x,t) \in D \tag{13}$$
$$v(0,t) = v(\pi,t), \quad t \in [0,T] \tag{14}$$
$$v_x(0,t) = v_x(\pi,t), \quad t \in [0,T] \tag{15}$$
$$v(x,0) = \varphi(x), \quad x \in [0,\pi]. \tag{16}$$

We use the method of the linearization then we use the finite difference method to solve (13)-(16).

We subdivide the intervals $[0,\pi]$ and $[0,T]$ into subintervals N_x and N_t of equal lengths $h = \frac{\pi}{N_x}$ and $\tau = \frac{T}{N_t}$, respectively. We choose the implicit scheme which is absolutely stable and has a second-order accuracy in h and a first-order accuracy in τ. The implicit scheme for (13)-(16) is as follows:

$$\frac{1}{\tau}\left(v_i^{j+1} - v_i^j\right) = \frac{1}{2h^2}\left(v_{i-1}^{j+1} - 2v_i^{j+1} + v_{i+1}^{j+1}\right) + \frac{1}{2h^2}\left(v_{i-1}^{j} - 2v_i^{j} + v_{i+1}^{j}\right) + l^j \widetilde{f}_i^j, \tag{17}$$

$$v_i^0 = \phi_i, \tag{18}$$

$$v_0^j = v_{N_x+1}^j, \tag{19}$$

$$\frac{v_1^j + v_{N_x}^j}{2} = v_{N_x+1}^j, \tag{20}$$

where $1 \le i \le N_x$ and $0 \le j \le N_t$ are the indices for the spatial and time steps respectively, $v_i^j = v(x_i,t_j)$, $\phi_i = \varphi(x_i)$, $\widetilde{f}_i^j = \widetilde{f}(x_i,t_j)$, $x_i = ih$, $t_j = j\tau$. At the level $t = 0$, adjustment should be made according to the initial condition and the compatibility requirements.

Now, let us construct the predicting-correcting mechanism. First, integrating the equation (1) with respect to x from 0 to 1 and using (3) and (4), we obtain

$$l(t) = \frac{g'(t) - v_{xx}(\pi,t)}{\widetilde{f}(x,t)}. \tag{21}$$

The finite difference approximation of (21) is

$$l^j = \frac{-\left(\left(g^{j+1} - g^j\right)/\tau\right) + \frac{1}{2h^2}\left(v_{N_x-1}^{j+1} - 2v_{N_x}^{j+1} + v_{N_x+1}^{j+1}\right) + \frac{1}{2h^2}\left(v_{N_x-1}^{j} - 2v_{N_x}^{j} + v_{N_x+1}^{j}\right)}{(\widetilde{fi})^j}.$$

and the values of ϕ_i provide us to start our computation. We denote the values of l^j, v_i^j at the s-th iteration step

.and the values of ϕ_i provide us to start our computation. We denote the values of l^j, v_i^j at the s-th iteration step $l^{j(s)}$,

$v_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $l^{j+1(0)} = l^j$, $v_i^{j+1(0)} = v_i^j$, $j = 0,1,2,....N_t$, $i = 1,2,...,N_x$. At each $(s+1)$-th iteration step we first determine $l^{j+1(s+1)}$ from the formula

$$l^{j+1(s+1)} = \frac{-\left(\left(g^{j+2} - g^{j+1}\right)/\tau\right) + \frac{1}{2h^2}\left(v_{N_x-1}^{j+1(s)} - 2v_{N_x}^{j+1(s)} + v_{N_x+1}^{j+1(s)}\right) + \frac{1}{2h^2}\left(v_{N_x-1}^{j(s)} - 2v_{N_x}^{j(s)} + v_{N_x+1}^{j(s)}\right)}{(\widetilde{f_i})^{j+1}}.$$

Then from (17)-(20) we obtain

$$\frac{1}{\tau}\left(v_i^{j+1(s+1)} - v_i^{j+1(s)}\right) = \frac{1}{h^2}\left(v_{i-1}^{j+1(s+1)} - 2v_i^{j+1(s+1)} + v_{i+1}^{j+1(s+1)}\right) + l^{j+1(s+1)}\widetilde{f}_i^{j+1}, \tag{22}$$

$$v_0^{j(s)} = v_{N_x+1}^{j(s)}, \tag{23}$$

$$\frac{v_1^{j(s)} + v_{N_x}^{j(s)}}{2} = v_{N_x+1}^{j(s)}. \tag{24}$$

The system of equations (22)-(24) can be solved by the Gauss elimination method and $v_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $l^{j+1(s+1)}$, $v_i^{j+1(s+1)}(i = 1,2,...,N_x)$ as l^{j+1}, $v_i^{j+1}(i = 1,2,...,N_x)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

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