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Structure of nearly $\alpha-$cosymplectic manifolds

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Abstract. The main purpose of this paper is to study the structure of nearly $\alpha-$ cosymplectic manifolds and some basic curvature relations of this manifolds satisfying some conditions where α is real defined.

1. Introduction

In more recent times,the geometry of cosymplectic manifolds has an increasing interest. The topology of cosymplectic manifolds and curvature properties of almost cosymplectic manifolds have been examined by Blair and Goldberg[1], Yano[14], Olszak[6], Kirichenko[17], Endo[9] and others. The category of almost cosymplectic manifolds is much wider than other structures. Many other authors also have applications to characterize and analize the properties of almost cosymplectic manifolds (see [18, 22, 25]).

In addition to geometric studies of cosymplectic manifolds, recent interest in the subject of the geometry of nearly contact structures has become favorite. Many mathematicians have began to examine nearly structures on various manifolds by examining new curvature properties. Some of these are nearly Kaehler, nearly Sasakian, nearly Kenmotsu and nearly cosymplectic manifolds etc. Now we will try to give some of these works in a chronological order.

Nearly Kaehler manifolds are presented by Gray in [4, 5]. Blair et al. has introduced nearly Sasakian manifolds [2] and also Olszak has improved this kind of manifolds [7]. In another study of Olzsak, the properties of five dimensional nearly Sasakian and non-Sasakian manifolds have been given [8]. Parallel to Olszak's works, Endo has analyzed and has studied the geometry and curvature properties of nearly cosymplectic manifolds [10]. In addition to these important works, nearly cosymplectic manifolds and some curvature conditions on nearly cosymplectic structures have been studied by many authors and they have also introduced some of the remarkable properties of nearly cosymplectic structures [16, 20, 23, 24].

Starting from the previous studies, in this study we define nearly $\alpha-$ cosymplectic manifolds and obtain some basic curvature properties of nearly $\alpha-$cosymplectic manifolds. By means of this paper, we will elaborate on the subject using the notations and terminology of nearly $\alpha-$cosymplectic manifolds.

2. Preliminaries

Throughout this study, M is considered as C^∞ class manifolds and we accept $X, Y, U', V', U, V \in \chi(M)$ as vector fields unless otherwise stated.

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Let (M, ϕ, ξ, η, g) be $(2n+1)-$ dimensional differentiable almost contact metric manifold with $(1,1)-$tensor field ϕ, a characteristic vector field ξ, $1-$form η and the Riemannian metric g. M, with this structure (ϕ, ξ, η, g) is called an almost contact metric structure. By the way, an almost contac metric structure satisfies the following conditions here with [1];

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

$$\phi^2 X = -X + \eta(X)\xi,$$
$$\eta(\xi) = 1,$$
$$rank\phi = 2n.$$

where $X, Y \in \chi(M)$. Also an almost contact metric structure (ϕ, ξ, η, g) satisfies;

$$\eta(X) = g(X, \xi),$$
$$\phi(\xi) = 0,$$
$$\eta(\phi X) = 0,$$

$$g(X, \phi Y) + g(Y, \phi X) = 0. \tag{2}$$

In the above equations, ϕ is skew-symmetric operator with respect to g and Φ is the bilininear fundamental $2-$form such that $\Phi(X, Y) = g(X, \phi Y)$ on M [15]. An almost contact metric manifold with $d\eta = 2\Phi$ is called a contact metric manifold on M. Moreover, almost contact metric manifolds in which both Φ and η are closed are called almost cosymplectic manifolds with $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. Finally, a normal almost cosymplectic manifold is called a cosymplectic manifold (see [1–3] for further details).

By the way, Kenmotsu manifolds, as it is named, were defined and studied by Katsueı Kenmotsu in 1972 [13]. Later, nearly Kenmotsu manifolds were studied by Shukla [11]. Shukla, A. defined an almost contact manifold (M, ϕ, ξ, η, g) as a nearly Kenmotsu manifold with the following relation;

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y \tag{3}$$

where ∇ is Levi-Civita connection of g.

Recently, many other authors [12, 19, 21] have studied the geometric properties of nearly Kenmotsu manifolds. If we mentioning about nearly Kenmotsu manifolds briefly, we can describe the skew-symmetric $(1,1)-$tensor field H, with $d\eta(X, Y) = g(HX, Y)$. When $H = 0$, M is said to be a nearly Kenmotsu manifold. Now, it is easy to see that every Kenmotsu manifold is nearly Kenmotsu manifold but converse is not true.

On the other hand, a nearly cosymplectic manifold is an almost contact metric manifold (M, ϕ, ξ, η, g) such that

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0, \quad X, Y \in \chi(M),$$

where ∇ denote the Levi-Civita connection with respect to the Riemannian metric g on M [10]. For a nearly cosymplectic manifold, the vector field ξ is Killing and satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$ conditions.

As we know that with the normality condition $([\phi, \phi] + 2\eta d \otimes \xi) = 0)$, a nearly cosymplectic structure is a cosymplectic structure [10].

Beside, for an $\alpha-$cosymplectic manifold the following condition holds [22];

$$(\nabla_X \phi)Y = \alpha \left[g(\phi X, Y)\zeta - \eta(Y)\phi X \right] \tag{4}$$

for any vector field X and Y on M.

Now from the equation above, by the sum of $(\nabla_X \phi)Y$ and $(\nabla_Y \phi)X$, we define a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g) , with the following definition;

Definition 2.1. *Let* (M, ϕ, ξ, η, g) *be* $(2n+1)-$ *dimensional differentiable almost contact metric manifold with* $(1,1)-$*tensor field* ϕ*, a characteristic vector field* ξ*,* $1-$*form* η *and the Riemannian metric* g*.Then if* M *satisfies the following relation ;*

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = \alpha[-\eta(Y)\phi X - \eta(X)\phi Y] \tag{5}$$

then, M is a said to be a nearly $\alpha-$cosymplectic manifold where ∇ is Levi-Civita connection of g and $\alpha \in \mathbb{R}$.

3. Nearly $\alpha-$cosymplectic manifolds

In this section, for a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), some basic structures are given.

Proposition 3.1. *For a nearly* $\alpha-$*cosymplectic manifold* (M, ϕ, ξ, η, g) *we have;*

$$\begin{aligned} g(\nabla_{U'}\xi, V') + g(U', \nabla_{V'}\xi) &= 2\alpha g(\phi U', \phi V'), \\ \nabla_{U'}\xi &= -\alpha\phi^2 U' + HU', \end{aligned} \tag{6}$$

$$\begin{aligned} \phi H + H\phi &= 0, \\ \nabla_\xi \phi &= \phi H, \\ H\xi &= 0, \\ \nabla_\xi \xi &= 0, \end{aligned} \tag{7}$$

where H is the skew-symmetric $(1,1)$-tensor field.

Proof. By (5), $(\nabla_\xi \phi)\xi = -\phi(\nabla_\xi \xi) = 0$, hence $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$. Now by making use of equation (1) we have

$$\begin{aligned} 0 &= g((\nabla_\xi \phi)U', \phi V') + g((\nabla_\xi \phi)V', \phi U') \\ &= -g((\nabla_{U'}\phi)\xi, \phi V') - g((\nabla_{V'}\phi)\xi, \phi U') - 2g(\phi U', \phi V') \\ &= g(\nabla_{U'}\xi, V') + g(\nabla_{V'}\xi, U') - 2\alpha g(\phi U', \phi V'). \end{aligned}$$

With help of definition of H, we get $\nabla_{U'}\xi = -\alpha\phi^2 U' + HU'$.
By $\phi\xi = 0$ and $\eta(\phi U') = 0$,we have

$$0 = (\nabla_{U'}\phi)\xi + \phi\nabla_{U'}\xi = -(\nabla_\xi \phi)U' + \phi HU', \tag{8}$$

$$\begin{aligned} 0 &= \eta((\nabla_{U'}\phi)V') + \eta((\nabla_{V'}\phi)U') \\ &= -g(U', (\nabla_{V'}\phi)\xi) - g(V', (\nabla_{U'}\phi)\xi) \\ &= g((\nabla_\xi \phi)U', V') + g((\nabla_\xi \phi)V', U') \\ &= g(U', \phi HY) + g(V', \phi HU') \\ &= g((\phi HU' + H\phi U'), V'). \end{aligned}$$

□

4. Curvature properties of nearly $\alpha-$cosymplectic manifolds

In this section, for a nearly $\alpha-$cosymplectic manifold (M,ϕ,ξ,η,g), some curvature relations are given. R is the Riemannian curvature tensor and it is defined by

$$R(U',V')U = (\nabla^2_{U',V'}U) - (\nabla^2_{V',U'}U) = [\nabla_{U'},\nabla_{V'}]U - \nabla_{[U',V']}U.$$

At the same time, the $(0,4)-$type tensor field is defined as

$$R(U',V',U,V) = g(R(U',V')U,V).$$

Theorem 4.1. *For a nearly $\alpha-$cosymplectic manifold (M,ϕ,ξ,η,g), following curvature relations are hold;*

$$R(U',V',\phi U,V) + R(U',V',U,\phi V) + R(U',\phi V',U,V) + R(\phi U',V',U,V) = 0, \tag{9}$$

$$\begin{aligned} R(\xi,U',V',U) &= \alpha\left[-2\eta(U')g(U,HY) + \eta(V')g(HU,U') - \eta(U)g(HY,U')\right] \\ &\quad + \alpha^2\left[\eta(V')g(U',U) - \eta(U)g(V',U')\right] - g((\nabla_{U'}H)V',U), \end{aligned} \tag{10}$$

$$R(\phi U',\phi V',U,V) = R(U',V',\phi U,\phi V), \tag{11}$$

$$R(\phi U',\phi V',\phi U,\phi V) = R(U',V',U,V) - \eta(U')R(\xi,V',U,V) + \eta(V')R(\xi,U',U,V). \tag{12}$$

Proof. Let define a $(1,3)-$type tensor field T_s as follow

$$(\nabla^2_{U',V'}\phi)U - (\nabla^2_{U',U}\phi)V' = T_s(U',V',U), \tag{13}$$

which satisfies $T_s(U',V',U) = T_s(U',U,V')$. To put it simples, we can write the $(0,4)-$type tensor field T_s, with respect to g, as follows;

$$T_s(U',V',U,V) = g(T_s(U',V',U),V).$$

If we use the Ricci identity, then we obtain

$$0 = R(U',V',U,\phi V) - R(U',V',V,\phi U) - g((\nabla^2_{U',V'}\phi)U,V) + g(((\nabla^2_{V',U'}\phi)U,V).$$

Also by the first Bianchi identity and (13), we get

$$\begin{aligned} R(U',V',U,\phi V) &= R(U',V',V,\phi U) + g((\nabla^2_{U',V'}\phi)U,V) - g(((\nabla^2_{V',U'}\phi)U,V) \\ &= R(U',V',V,\phi U) - g((\nabla^2_{U',U}\phi)V',V) + g((\nabla^2_{V',U}\phi)U',V) \\ &\quad + T_s(U',U,V',V) - T_s(V',U,U',V), \end{aligned}$$

and thus, we have

$$\begin{aligned} R(U',V',U,\phi V) &= R(U',U,V',\phi V) - R(V',U,U',\phi V) \\ &= R(U',U,V',\phi V) - R(V',U,V,\phi U') \\ &\quad + g((\nabla^2_{U,V'}\phi)U',V) - g((\nabla^2_{V',U}\phi)U',V), \end{aligned}$$

If we equalize the right sides of equations above, we get

$$\begin{aligned} &R(U',V',U,\phi V) - R(U',V',V,\phi U) - R(V',U,V,\phi U') + g((\nabla^2_{U',U}\phi)V',V) \\ &+ g((\nabla^2_{U,V'}\phi)U',V) + T_s(V',U,U',V) - T_s(U',U,V',V) = 2g((\nabla^2_{V',U}\phi)U',V). \end{aligned} \tag{14}$$

and we note that

$$g((\nabla^2_{U',U}\phi)V',V)+g((\nabla^2_{U,V'}\phi)U',V)=R(U',U,V',\phi V)-R(U',U,V,\phi V')+T_s(U,U',V',V),$$

$$g((\nabla^2_{V',U}\phi)U',V)=g((\nabla^2_{V',V}\phi)U,U')-T_s(V',V,U,U').$$

By considering this in (14), we have

$$\begin{aligned}&2R(U',U,V',\phi V)-R(U',V',V,\phi U)-R(V',U,V,\phi U')-R(U',U,V,\phi V')\\&-T_s(U',V',U,V)+T_s(V',U,U',V)+T_s(U,U',V',V)+2T_s(V',V,U,U')\\&=2g((\nabla^2_{V',V}\phi)U,U').\end{aligned}\tag{15}$$

If we apply (5) to (15), we obtain

$$\begin{aligned}T_s(U',V',U,V)=&\ \alpha(-g(V',U'+HU')+\eta(U')\eta(V'))g(\phi U,V)\\&+\alpha^2(-g(U,U'+HU')+\eta(U')\eta(U))g(\phi V',V)\\&-\eta(V')g((\nabla_{U'}\phi)U,V)-\eta(U)g((\nabla_{U'}\phi)V',V),\end{aligned}$$

and after a straight forward computation, we get

$$\begin{aligned}&T_s(V',U,U',V)+T_s(U,U',V',V)-T_s(U',V',U,V)+2T_s(V',V,U,U')=\\&\alpha\left[G(U',V',U,V)+2g(\phi V',V)g(HU',U)+2g(\phi U,V)g(HU',V')\ +2g(\phi U',V)g(HY,U)+2g(\phi U',U)g(HY,V)\right]\\&+\alpha^2\left[2g(\phi U,U')g(V',\phi^2V)+\eta(U')\eta(V')g(\phi U,V)-\eta(U)\eta(V')g(\phi U',V)\right],\end{aligned}$$

where

$$G(U',V',U,V)=\alpha\left[-\eta(V')g((\nabla_U\phi)U',V)\ +\eta(V')g((\nabla_{U'}\phi)U,V)-2\eta(V)g((\nabla_{V'}\phi)U,U')\right].$$

The anti-symmetrization of (15) in V' and V and also using the first Bianchi identity, we have

$$\begin{aligned}&3R(\phi U',U,V',V)+3R(U',\phi U,V',V)+3R(U',U,\phi V',V)+3R(U',U,V',\phi V)\\&+\alpha\left[4g(\phi V',V)g(HU',U)+2g(\phi U,V)g(HU',V')-2g(\phi U,V')g(HU',V)\right.\\&\left.+4g(\phi U',U)g(HY,V)+2g(\phi U',V)g(HY,U)-2g(\phi U',V')g(HV,U)\right]=0,\end{aligned}$$

which implies equation (9) if one assumes $H=0$. Now we will show that $H=0$. For $U'=\xi$, $(H\xi=\phi\xi=0)$, we get

$$R(\xi,\phi U,V',V)+R(\xi,U,\phi V',V)+R(\xi,U,V',\phi V)=0,\tag{16}$$

and

$$-R(\xi,U,\phi V',V)-R(\xi,\phi U,V',V)+R(\xi,\phi U,\phi V',\phi V)+\eta(V')R(\xi,\phi U,\xi,V)=0.\tag{17}$$

Hence we have

$$R(\xi,U,V',\phi V)+R(\xi,\phi U,\phi V',\phi V)+\eta(V')R(\xi,\phi U,\xi,V)=0,\tag{18}$$

and

$$\begin{aligned}&-R(\xi,\phi U,V',V)+R(\xi,U,\phi V',V)+\eta(V)R(\xi,U,\xi,\phi V')\\&-\eta(V)R(\xi,\phi U,\xi,V')+\eta(V')R(\xi,\phi U,\xi,V)=0.\end{aligned}\tag{19}$$

From (16) and (19), we have

$$2R(\xi,\phi U,V',V)+R(\xi,U,V',\phi V)=\eta(V')R(\xi,\phi U,\xi,V)+\eta(V)\left[R(\xi,U,\xi,\phi V')-R(\xi,\phi U,\xi,V')\right].\tag{20}$$

and changing U by ϕU and V by ϕV in (20), we get

$$-2R(\xi, U, V', \phi V) - R(\xi, \phi U, V', V) = -\eta(V')R(\xi, U, \xi, \phi V) + \eta(V)R(\xi, \phi U, \xi, V'). \tag{21}$$

Taking the sum of the last two equations above, we obtain

$$R(\xi, \phi U, V', V) - R(\xi, U, V', \phi V) = \eta(V')R(\xi, \phi U, \xi, V) + \eta(V)R(\xi, U, \xi, \phi V') - R(\xi, U, \xi, \phi V). \tag{22}$$

From the equations (19) and (22), we get (16) as

$$3R(\xi, \phi U, V', V) = \eta(V')\left[2R(\xi, \phi U, \xi, V) - R(\xi, U, \xi, \phi V)\right] + \eta(V)\left[2R(\xi, U, \xi, \phi V') - R(\xi, \phi U, \xi, V')\right],$$

$$R(\xi, U, \phi V', \phi V) = 0.$$

Applying $\nabla\xi = -\alpha\phi^2 + H$, we have

$$R(V', U, \xi, U') = -g((\nabla_{V'}H)U', U) + g((\nabla_U H)U', V') - g((\nabla^2_{V'}\phi)U', U) + g((\nabla^2_U\phi)U', V').$$

Using the first Bianchi identity by applying the cyclic sum on U', V', U, we obtain

$$g((\nabla_{U'}H)V', U) - g((\nabla_{V'}H)U', U) + g((\nabla_U H)U', V') = 0,$$

in this way we have

$$\begin{aligned} R(V', U, \xi, U') = &- g((\nabla_{V'}\phi^2)U', U) + g((\nabla_U\phi^2)U', V') - g((\nabla_{U'}H)V', U) \\ &+ \alpha\left[-2\eta(U')g(U, HY) + \eta(V')g(U', HU) - \eta(U)g(U', HY)\right] \\ &+ \alpha^2\left[\eta(V')g(U', U) - \eta(U)g(U', V')\right] - g((\nabla_{U'}H)V', U), \end{aligned} \tag{23}$$

$$\begin{aligned} 0 = R(\xi, U', \phi V', \phi U) &= -2\alpha\eta(U')g(H\phi V', \phi U) - g((\nabla_{U'}H)\phi V', \phi U) \\ &= \alpha\left[-2\eta(U')g(HY, U)\right] - g((\nabla_{U'}H)\phi V', \phi U). \end{aligned} \tag{24}$$

If we take V' , a unit eigenvector field on M such that $\eta(V') = 0$ and $H^2V' = \lambda V'$; in this way, note that $H^2\phi V' = \lambda\phi V'$, as $\phi H = -H\phi$. Then

$$\begin{aligned} 0 = R(\xi, U', \phi V', \phi HY) &= -2\alpha\lambda\eta(U') - g((\nabla_{U'}H)\phi V', \phi HY) - \frac{1}{2}g((\nabla_{U'}H^2)\phi V', \phi V') \\ &= 2\alpha\lambda\eta(U') - \frac{1}{2}d\lambda(U') = 0, \end{aligned} \tag{25}$$

so that $d\lambda = -4\alpha\lambda\eta$, where U' is arbitrary vector field on M.

As a result, $\lambda = 0$ or $d\eta = 0$, means $H = 0$. Then from (23), we obtain (10).

Stating the left hand side of (9) by R_*, we will prove (11).

Then, if we applying this regulation in (10), we have

$$\begin{aligned} 0 &= R_*(U', \phi V', U, V) - R_*(U', V', \phi U, V) - R_*(U', V', U, \phi V) + R_*(\phi U', V', U, V) \\ &= -2R(U', V', \phi U, \phi V) + 2R(\phi U', \phi V', U, V). \end{aligned}$$

Now, it is immediate to see (12). □

Proposition 4.2. *For a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), following relation holds;*

$$-2\alpha g(\phi U', V')\xi + (\nabla_{U'}\phi)V' + \alpha\eta(V')\phi U' + (\nabla_{\phi U'}\phi)\phi V' = 0.$$

Proof. By $\phi^2 = -Id + \eta \otimes \xi$, we have

$$g((\nabla_{U'}\phi)\phi V', U) = \alpha\left[\eta(V')g(U', U) + \eta(U)g(U', V') - 2\eta(U')\eta(V')\eta(U)\right] + \eta(V')g(HU', U) + \eta(U)g(HU', V') + g((\nabla_{U'}\phi)V', \phi U),$$

taking into account (5), we get

$$g((\nabla_{\phi U'}\phi)V', U) = \alpha\left[2\eta(V')g(U', U) - \eta(U)g(U', V') - \eta(U')\eta(V')\eta(U)\right] + \eta(U')g(HU, V') + \eta(U)g(HU', V') + g((\nabla_{U'}\phi)V', \phi U). \tag{26}$$

From the equations above, the expression we are trying to show is obtained. □

Proposition 4.3. *For a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), following curvature relations are hold;*

$$Ric(U', \xi) = \alpha^2\left[-2n\eta(U')\right], \tag{27}$$

$$Ric(\phi V', \phi U) = \alpha^2\left[2n\eta(V')\eta(U)\right] + Ric(V', U), \tag{28}$$

$$Ric(U, \phi V') + Ric(\phi U, V') = 0, \tag{29}$$

where Ric is the Ricci tensor of M.

Proof. In for dimension M is $2n + 1$ and $(E_0 = \xi, E_1, ..., E_n, E_{n+1}, ..., E_{2n})$, orthonormal ϕ-frame satisfies $\phi E_i = E_{i+n}$, $\phi E_{i+n} = -E_i$, $i = 1, ..., n$. If we evaluate the ϕ-basis with (10), we can give the *Ricci tensor* $Ric(U', \xi)$ by (27).

Then from the equation (12) we get;

$$\begin{aligned} Ric(U', V') &= \sum_{i=1}^{n}(R(E_i, U', V', E_i) + R(E_{i+n}, U', V', E_{i+n})) + R(\xi, U', V', \xi) \\ &= Ric(\phi U', \phi V') + \eta(U')Ric(\xi, V') - R(\xi, \phi U', \phi V', \xi) + R(\xi, U', V', \xi) \\ &= Ric(\phi U', \phi V') + \eta(U')Ric(\xi, V') = Ric(\phi U', \phi V') - 2\alpha^2 n\eta(U')\eta(V'), \end{aligned} \tag{30}$$

in which we applied (27). From the direct consequence of (28), we obtain (29). □

Proposition 4.4. *The fundamental form of a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g) satisfies;*

$$3d\Phi(U', V', U) = \alpha\Big[-2\eta(U')g(\phi V', U) - \eta(V')g(\phi U', U) + \eta(U)g(\phi U', V')\Big] - 3g((\nabla_{U'}\phi)V', U). \tag{31}$$

$$d\Phi(U', V', U) = 2\alpha(\eta \wedge \Phi)(U', V', U) + \frac{1}{4}g([\phi, \phi](U', V'), \phi U). \tag{32}$$

Proof. From the well known following identities

$$3d\Phi(U', V', U) = (\nabla_{U'}\Phi)(V', U) + (\nabla_{V'}\Phi)(U, U') + (\nabla_U\Phi)(U', V')$$

and

$$[\phi, \phi](U', V') = -\phi(\nabla_{U'}\phi)V' + \phi(\nabla_{V'}\phi)U' + (\nabla_{\phi U'}\phi)V' - (\nabla_{\phi V'}\phi)U',$$

we have

$$\begin{aligned} 3d\Phi(U', V', U) &= -g((\nabla_{U'}\phi)V', U) + g((\nabla_{V'}\phi)U', U) - g((\nabla_U\phi)U', V') \\ &= \alpha\Big[-2\eta(U')g(\phi V', U) + \eta(V')g(\phi U, U') - \eta(U)g(\phi V', U')\Big] - 3g((\nabla_{U'}\phi)V', U), \end{aligned} \tag{33}$$

$$\frac{1}{2}[\phi,\phi](U',V') = \alpha\left[-\eta(U')V' + \eta(V')U'\right] - \phi(\nabla_{U'}\phi)V' + \phi(\nabla_{V'}\phi)U'. \tag{34}$$

Hence

$$\begin{aligned} 6d\Phi(U',V',U) &= \alpha\left[-\eta(U')g(\phi V',U) + \eta(V')g(\phi U',U) + 2\eta(U)g(\phi U',V')\right] - 3g((\nabla_{U'}\phi)V' - (\nabla_{V'}\phi)U',U) \\ &= 4\alpha\left[\eta(U')g(V',\phi U) + \eta(V')g(U,\phi U') + \eta(U)g(U',\phi V')\right] + \frac{3}{2}g([\phi,\phi](U',V'),\phi U) \\ &= 12\alpha(\eta\wedge\Phi)(U',V',U) + \frac{3}{2}g([\phi,\phi](U',V'),\phi U). \end{aligned}$$

□

Theorem 4.5. *Every normal nearly $\alpha-$cosymplectic manifold (M,ϕ,ξ,η,g) is cosymplectic manifold.*

Proof. We know that $d\eta = 0$ and if and only if $N = 0$, the structure is normal. According to Proposition 4.4, in the case of $N = 0$ we have

$$3d\Phi(U',V',Z) = 2\alpha(\eta\wedge\Phi)(U',V',Z)$$

and

$$d\Phi = 2\alpha\eta\wedge\Phi.$$

That is to say, M is almost $\alpha-$cosymplectic. Namely, we can see that a normal almost $\alpha-$cosymplectic manifold is $\alpha-$cosymplectic. □

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