TURKISH JOURNAL OF SCIENCE
VOLUME 6, ISSUE 3, 127–133
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

The Solution of Linear Volterra Integral Equation of the First Kind with ZZ-Transform

Erhan ÖZDEMİR[a], Ercan ÇELİK[b], Sıdıka Şule ŞENER[c]

[a]*Ataturk University, Faculty of Science, Department of Mathematics, Erzurum-TURKEY*
[b]*Ataturk University, Faculty of Science, Department of Mathematics, Erzurum-TURKEY*
[c]*Ataturk University, Faculty of Science, Department of Mathematics, Erzurum-TURKEY*

Abstract. In this paper, we apply ZZ-transform to solve linear Volterra integral equation of the first kind. The several examples solve by ZZ- Transform. This means that ZZ- transform is a powerful tool for solving linear Volterra integral equations of the first kind. The Convolution theorem for the ZZ- transform has been proved. ZZ- transform for the solution of linear Volterra integral equation of the first kind submitted in application section of this paper, some applications are given to demonstrate the effectiveness of proposed scheme.

1. Introduction

Integral transformations are encountered in many fields of engineering and science such as electrical networks, heat transfer, mixing problems, springs, signal processing, bending of beams, Newton's second law of motion, carbon dating problems, decay and exponential growth problems. In later times, many the scientist are related in solving the problems of engineering and science by introducing new integral transforms. The ZZ-Transform is integral transform. There are many integral transforms in the literature. Some of these transformations are Laplace transform, Fourier transform, Elzaki transform, Sumudu transform, Aboodh transform, Kamal transform [1, 9]. These transformations are used to solve for differential equations and integral equations. The ZZ-Transform was first presented by Zain UI Abadin Zafar in 2016 [10]. The linear Volterra integral equation of the first kind is given by $f(t) = \int\limits_0^x K(x,t)u(t)dt, u(x)$ is the unknown function and occurs only inside the integral sign. The function $f(x)$and the kernel $K(x,t)$ are real-valued functions [11]. The $ZZ-$transform of the function $f(t)$for $t \geq 0$is defined as;

$$Z(u,s) = Z\{f(t)\} = s\int_0^\infty f(ut)e^{-st}dt \tag{1}$$

or

$$Z(u,s) = Z\{f(t)\} = \frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt \tag{2}$$

Corresponding author: EÖ mail address: erhanozdmr1990@gmail.com ORCID:0000-0002-0415-7086, EÇ ORCID:0000-0002-1402-1457, SŞŞ ORCID:0000-0002-3807-7105

Received: 8 November 2021; Accepted: 29 November 2021; Published: 30 December 2021

Keywords. (Integral Equations, Linear Volterra Integral Equation of the First Kind, ZZ- Transform, Convolution Theorem, Inverse ZZ- Transform)

2010 *Mathematics Subject Classification*. 47N20, 44A10, 45N05

Cited this article as: Özdemir E, Çelik E, Şener SŞ. The Solution of Linear Volterra Integral Equation of the First Kind with ZZ-Transform, Turkish Journal of Science. 2021, 6(3), 127-133.

Where $Z(u,s)$ is transform operator. Assuming that the integral on the right side in (2) exists. The unique function $f(t)$ in (2) is called the inverse transform of $Z(u,s)$is indicated by

$$f(t) = Z^{-1}\{Z(u,s)\} \tag{3}$$

If $F(t)$ is piecewise continuous and of exponential order, the $ZZ-$transform of the function $F(t)$ for $t \geq 0$exist. These conditions are only sufficent conditions for the existence of $ZZ-$transform of the function $F(t)$.

1.1. Linearity Property of ZZ−Transform:

If $Z\{F(t)\} = A(u,s)$ and $Z\{G(t)\} = B(u,s)$then $Z\{aF(t) + bG(t)\} = aZ\{F(t)\} + bZ\{G(t)\} = aA(u,s) + bB(u,s)$,where $a, b$are arbitrary constants.

1.2. ZZ− Transform of Some Elementary Functions:

Table 1: ZZ− Transform of Some Elementary Functions:

No	$f(t)$	$Zf(t)$
1	1	1
2	t	$\frac{u}{s}$
3	e^{at}	$\frac{u}{s-au}$
4	$\sin at$	$\frac{aus}{s^2+a^2u^2}$
5	$\cos at$	$\frac{s^2}{s^2+a^2u^2}$
6	t^n	$n!\frac{u^n}{s^n}$
7	$e^{at}\sin bt$	$\frac{b\frac{s}{u}}{\left(\frac{s}{u}-a\right)^2+b^2}$
8	$e^{at}\cos bt$	$\frac{\frac{s^2}{u^2}-\frac{as}{u}}{\left(\frac{s}{u}-a\right)^2+b^2}$
9	$t\cos at$	$\frac{\frac{s}{u}\left(\frac{s^2}{u^2}-a^2\right)}{\left(\frac{s^2}{u^2}+a^2\right)^2}$
10	$t\sin at$	$\frac{2a\frac{s^2}{u^2}}{\left(\frac{s^2}{u^2}+a^2\right)^2}$

1.3. Existence of ZZ−Transform

Theorem 1.1. *If $f(t)$ is piecewise continuous in interval $0 \leq t \leq K$and of exponential order γ for $t > K$, then its ZZ− transform $Z(u,s)$ exists for all $s > \gamma$,$u > \gamma$.*

Proof. We have for every positive number K,

$$\frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt = \frac{s}{u}\int_0^K f(t)e^{-\frac{s}{u}t}dt + \frac{s}{u}\int_K^\infty f(t)e^{-\frac{s}{u}t}dt$$

Since $f(t)$ is piecewise continuous in every finite interval $0 \leq t \leq K$, the first integral on the right side exists. Also the second integral on the right side exists. So $f(t)$ is of exponential order γ for $t > K$. To see this we have only to observe that in such case:

$$\left[\frac{s}{u}\int_{K}^{\infty} f(t)e^{-\frac{s}{u}t}dt\right] \leq \frac{s}{u}\left[f(t)e^{-\frac{s}{u}t}\right]dt$$

$$\leq \frac{s}{u}\int_{0}^{\infty} e^{-\frac{s}{u}t}(f(t))dt \leq \frac{s}{u}\int_{0}^{\infty} e^{-\frac{s}{u}t}Me^{\gamma t}dt$$

$$\leq \frac{sM}{u}\int_{0}^{\infty} e^{-\frac{s}{u}t}e^{\gamma t}dt \leq \frac{sM}{u}\int_{0}^{\infty} e^{-(\frac{s}{u}-\gamma)t}dt$$

$$= \frac{sM}{u}\frac{e^{-(\frac{s}{u}-\gamma)t}}{\left(-\frac{s}{u}-\gamma\right)}\Bigg|_{0}^{\infty} = \frac{sM}{s-\gamma u}$$

□

1.4. Convolution of two Functions:

Convolution of $F(t)$ and $G(t)$ functions is defined by

$$F(t)\otimes G(t) = F\otimes G = \int_{0}^{t} F(x)G(t-x)dx = \int_{0}^{t} F(t-x)G(x)dx.$$

$$\frac{u}{s}\frac{s^2}{s^2+u^2}Z\left\{x(t)\right\} = 2\frac{s^2}{u^2}\frac{u^4}{\left(s^2+u^2\right)^2}$$

1.5. Convolution Theorem for ZZ−Transforms:

Theorem 1.2. *If* $Z\{F(t)\} = A(u,s)$ *and* $Z\{G(t)\} = B(u,s)$ *then*

$$Z(f\otimes g) = \frac{u}{s}Z(f)Z(g)$$

$$Z\left\{F(t)\otimes G(t)\right\} = \frac{u}{s}Z\left\{F(t)\right\}Z\left\{G(t)\right\} = \frac{u}{s}A(u,s)B(u,s)$$

Proof.

$$Z(f)Z(g) = \frac{s}{u}\int_{0}^{\infty} f(\tau)e^{-\frac{s}{u}\tau}d\tau\frac{s}{u}\int_{0}^{\infty} g(\vartheta)e^{-\frac{s}{u}\vartheta}d\vartheta$$

$$Z(f)Z(g) = \frac{s^2}{u^2}\int_{0}^{\infty} f(\tau)e^{-\frac{s}{u}\tau}d\tau\int_{0}^{\infty} g(\vartheta)e^{-\frac{s}{u}\vartheta}d\vartheta \tag{4}$$

$t = \vartheta + \tau$ and $\vartheta = t - \tau$

$$Z(g) = \int_{\tau}^{\infty} g(t-\tau)e^{-\frac{s}{u}(t-\tau)}dt$$

$$= \int_{0}^{\infty} g(t-\tau)e^{-\frac{s}{u}t}e^{\frac{s}{u}\tau}dt$$

$$= \int_{\tau}^{\infty} g(t-\tau)e^{-\frac{s}{u}t}e^{\frac{s}{u}\tau}dt$$

$$= e^{\frac{s}{u}\tau}\int_{\tau}^{\infty} g(t-\tau)e^{-\frac{s}{u}t}dt$$

Thus

$$\begin{aligned}
&= \frac{s^2}{u^2}\int_{0}^{\infty} f(\tau)e^{-\frac{s}{u}\tau}d\tau e^{\frac{s}{u}\tau}\int_{\tau}^{\infty} e^{-\frac{s}{u}t}g(t-\tau)dt \\
&= \frac{s^2}{u^2}\int_{0}^{\infty} f(\tau)\int_{\tau}^{\infty} e^{-\frac{s}{u}t}g(t-\tau)dtd\tau \\
&= \frac{s^2}{u^2}\int_{0}^{\infty} e^{-\frac{s}{u}t}\int_{0}^{t} f(t)g(t-\tau)d\tau dt \\
&= \frac{s^2}{u^2}\int_{0}^{\infty} e^{-\frac{s}{u}t}(f\otimes g)(t)dt \\
&= \frac{s}{u}Z(f\otimes g) \\
Z(f\otimes g) &= \frac{u}{s}Z(f)Z(g)
\end{aligned}$$

□

1.6. Inverse of ZZ−Transforms:

If $Z\{F(t)\} = Z\{u,s\}$then $F(t)$ is called the inverse $ZZ-$transform of $Z\{u,s\}$and it is defined as $F(t) = Z^{-1}\{Z(u.s)\}$, where Z^{-1} is the inverse $ZZ-$transform operator.

1.7. Applications:

In this chapter, some applications are given to show the effectiveness of $ZZ-$transform for solving of linear Volterra integral equation of the first kind.

Example 1.3. *Consider linear Volterra integral equation of the first kind:*

$$x = \int_{0}^{x} u(t)dt \tag{5}$$

Applying the ZZ− transform to both sides of (5), we have:

$$Z\{x\} = Z\left\{\int_{0}^{x} u(t)dt\right\} \tag{6}$$

Using convolution theorem of ZZ−transform on (6), we have:

$$Z\{x\} = \frac{u}{s}Z\{1\}Z\{u(x)\}$$

$$\frac{u}{s} = \frac{u}{s}.1.Z\{u(x)\}$$

$$Z\{u(x)\} = 1 \tag{7}$$

Operating inverse ZZ−transform on both sides of (7), we have:

$$Z^{-1}\{Z\{u(x)\}\} = Z^{-1}(1)$$

$$u(x) = 1.$$

This is the exact solution of equation (5).

Example 1.4. *Consider linear Volterra integral equation of the first kind:*

$$x^2 = \frac{1}{2}\int_0^x (x-t)u(t)dt \tag{8}$$

Applying the ZZ−transform to both sides of (8), we have:

$$Z\left\{x^2\right\} = Z\left\{\frac{1}{2}\int_0^x (x-t)u(t)dt\right\} \tag{9}$$

Using convolution theorem of ZZ−transform on (9), we have:

$$2\frac{u^2}{s^2} = \frac{1}{2}\frac{u}{s}Z\{x\}Z\{u(x)\}$$

$$2\frac{u^2}{s^2} = \frac{1}{2}\frac{u}{s}\frac{u}{s}Z\{u(x)\}$$

$$Z\{u(x)\} = 4 \tag{10}$$

operating inverse ZZ−transform on both sides of (10), we have:

$$Z^{-1}\{Z\{u(x)\}\} = Z^{-1}\{4\}$$

$$u(x) = 4$$

This is the exact solution of equation (8).

Example 1.5. *Consider linear Volterra integral equation of the first kind:*

$$y(t) = t^2 + \int_0^t y(u)\sin(t-u)du \tag{11}$$

$$Z\{y(t)\} = Z\left\{t^2 + \int_0^t y(u)\sin(t-u)du\right\}$$

From the linearity property of the inverse ZZ−transform

$$Z\{y(t)\} = Z\left\{t^2\right\} + Z\left\{\int_0^t y(u)\sin(t-u)du\right\} \tag{12}$$

Using convolution theorem of transform on (12), we have:

$$Z\{y(t)\} = Z\left\{t^2\right\} + \frac{u}{s} Z\{y(t)\} Z\{\sin t\}$$

$$Z\{y(t)\} = 2\frac{u^2}{s^2} + \frac{u}{s} Z\{y(t)\} \frac{us}{s^2+u^2}$$

$$Z\{y(t)\} - Z\{y(t)\}\left(\frac{su^2}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\}\left(1 - \frac{su^2}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\}\left(\frac{s^3}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\}\left(\frac{s^2}{s^2+u^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\} = 2\frac{u^2}{s^2}.\frac{s^2+u^2}{s^2}.$$

$$Z\{y(t)\} = 2\frac{u^2}{s^2} + 2\frac{u^4}{s^4} \tag{13}$$

operating inverse ZZ−transform on both sides of (13), we have:

$$Z^{-1}\{Z\{y(t)\}\} = Z^{-1}\left\{2\frac{u^2}{s^2} + 2\frac{u^4}{s^4}\right\}$$

From the linearity property of the inverse ZZ−transform

$$y(t) = Z^{-1}\left\{2\frac{u^2}{s^2}\right\} + Z^{-1}\left\{2\frac{u^4}{s^4}\right\}$$

$$y(t) = t^2 + \frac{t^4}{12}$$

This is the exact solution of equation (11).

Example 1.6. *Consider linear Volterra integral equation of the first kind:*

$$\int_0^t \cos(t-s)x(s)ds = t\sin t \tag{14}$$

Applying the ZZ−transform to both sides of (14), we have:

$$Z\left\{\int_0^t \cos(t-s)x(s)ds\right\} = Z\{t\sin t\} \tag{15}$$

Using convolution theorem of ZZ−transform on (15), we have:

$$\frac{u}{s} Z\{\cos t\} Z\{x(t)\} = \frac{2\frac{s^2}{u^2}}{\left(\frac{s^2}{u^2}+1\right)^2}$$

$$\frac{u}{s}\frac{s^2}{s^2+u^2}Z\{x(t)\} = 2\frac{s^2}{u^2}\frac{u^4}{(s^2+u^2)^2}$$

$$Z\{x(t)\} = 2\frac{us}{s^2+u^2} \tag{16}$$

operating inverse ZZ−transform on both sides of (16), we have:

$$Z^{-1}\{Z\{x(t)\}\} = Z^{-1}\left\{2\frac{us}{s^2+u^2}\right\}$$

$$x(t) = \sin t$$

This is the exact solution of equation (14).

2. Conclusion

In this study, we have discussed the $ZZ-$transform for the solution of linear volterra integral equation of the first kind. The given examples show that the exact solution have been obtained spending a very little time and using very less computational work.

References

[1] Aggarwal S., Gupta AR., Singh DP., et al. Application of Laplace transform for solving population growth and decay problems. International Journal of Latest Technology in Engineering, Management and Applied Science, 7(9): 141-145, 2018.

[2] Aggarwal S., Singh A., Kumar A., et al. Application of Laplace transform for solving improper integrals whose integrand consisting error function. Journal of Advanced Research in Applied Mathematics and Statistics, 4(2): 1-7, 2019.

[3] Aboodh KS., Farah RA., Almardy IA et al. Solving delay differential equations by Aboodh transformation method. International Journal of Applied Mathematics and Statistical Sciences, 7(2): 55-64, 2018.

[4] Eltayeb H., and Kilicman A., A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, Volume 4, no.22,1089-1098, 2010.

[5] Tarig M. Elzaki, The New Integral Transform "Elzaki Transform" Global Journal of Pure and Applied Mathematics, ISSN 0973-1768, Number 1, pp. 57-64, 2011.

[6]] Aggarwal S, Chauhan R, and Sharma N., A new application of Kamal transform for solving linear Volterra integral equations. International Journal of Latest Technology in Engineering, Management and Applied Science , 7(4): 138-140, 2018.

[7] Mohand M. Abdelrahim M., and Tarig M. E., Elzaki Transform And Power Series Expansion On A Bulge Heaviside Step Function, Global Journal of Pure and Applied Mathematics. ISSN 0973-1768 Volume 11, Number 3, pp. 1179-1184, 2015.

[8] Tarig M. E., and Salih M. E., On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554 Volume 6, Number 1, pp. 13-18, 2011.

[9] Najafi, R., Kucuk, G. D., Çelik, E. Modified Iteration Method for Solving Fractional Gas Dynamics Equation, Mathematical Methods in the Applied Sciences, 40(4), 939-946, 2017.

[10] Zain UI Abadin Zafar, ZZ Transform Method, International Journal of Advanced Engineering and Global Technology, 4(1), 1605-1611, 2016.

[11] Sudhanshu A., and Nidhi S., Laplace Transform for the Solution of First Kind Linear Volterra Integral Equation, Journal of Advanced Research in Applied Mathematics and Statistics, Volume 4, Issue 3/4, Pg. No. 16-23, 2019.