TURKISH JOURNAL OF SCIENCE
VOLUME 9, ISSUE 1, 6–18
ISSN: 2587–0971

https://www.tjoscience.com

Rough Ideal Convergence in 2-Normed Spaces

Mukaddes ARSLAN[a], Erdinç DÜNDAR[b]

[a]*Ministry of National Education, İbn-i Sina Vocational and Technical Anatolian High School, İhsaniye, Afyonkarahisar, Turkey.*
[b]*Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey.*

Abstract. In this study, using the concepts of $\mathcal{I}$-convergence and rough convergence, we introduced the notion of rough $\mathcal{I}$-convergence and giving example investigated the relation between $\mathcal{I}$-convergence and rough $\mathcal{I}$-convergence in 2-normed space. Also, we defined the set of rough $\mathcal{I}$-limit points of a sequence in 2-normed space and obtained two rough $\mathcal{I}$-convergence criteria associated with this set in 2-normed space. Then, we proved that this set is closed and convex in 2-normed space. Also, we examined the relations between the set of $\mathcal{I}$-cluster points and the set of rough $\mathcal{I}$-limit points of a sequence in 2-normed space.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [15] and Schoenberg [36]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [28] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$.

The concept of 2-normed spaces was initially introduced by Gähler [16, 17] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [21] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Gürdal and Açık [23] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [34] studied statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Arslan and Dündar [2, 3] investigated the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [9, 22, 30, 35, 37–39]).

The idea of rough convergence was first introduced by Phu [31] in finite-dimensional normed spaces. In [31], he showed that the set $\mathrm{LIM}^r x$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathrm{LIM}^r x$ on the roughness degree r. In another paper [32] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous at every point $x \in X$ under the assumption $dim Y < \infty$ and $r > 0$ where X and Y are normed spaces. In [33], he extended the results given in [31] to infinite-dimensional normed spaces. Aytar [7] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence

Corresponding author: ED mail address: edundar@aku.edu.tr ORCID:0000-0002-0545-7486, MA ORCID:0000-0002-5798-670X
Received: 12 January 2024; Accepted: 27 February 2024; Published: 30 April 2024
Keywords. Rough convergence, Rough ideal convergence, 2-normed space
2010 *Mathematics Subject Classification*. 40A05, 40A35
Cited this article as: Arslan, M., & Dündar, E. (2024). Rough Ideal Convergence in 2-Normed Spaces. Turkish Journal of Science, 9(1), 6-18.

and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [8] studied that the r-limit set of the sequence is equal to the intersection of these sets and that r-core of the sequence is equal to the union of these sets. Recently, Dündar and Çakan [11–13] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notions of rough convergence, $\mathcal{I}_2$-convergence and the sets of rough limit points and rough $\mathcal{I}_2$-limit points of a double sequence. Arslan and Dündar [4, 5] introduced rough convergence and investigated some properties in 2-normed spaces. Also, Arslan and Dündar [6] investigated rough statistical convergence.

In this paper, using the concepts of $\mathcal{I}$-convergence and rough convergence, we introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence in 2-normed space and obtained two rough $\mathcal{I}$-convergence criteria associated with this set. Then, we proved that this set is closed and convex in 2-normed space. Also, we examined the relations between the set of $\mathcal{I}$-cluster points and the set of rough $\mathcal{I}$-limit points of a sequence in 2-normed space. We note that our results and proof techniques presented in this paper are analogues of those in Phu's [31] paper. Namely, the actual origin of most of these results and proof techniques is them papers. The following our theorems and results are the extension of theorems and results in [4, 5, 31].

Now, we recall the some fundamental definitions and notations about the our issue. (See [1–4, 6–8, 10, 14, 18–29, 31–34, 38–42]).

Throughout the paper, let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_n) \subset \mathbb{R}^n$.

The sequence $x = (x_n)$ is said to be r-convergent to L, denoted by $x_n \xrightarrow{r} L$ provided that $\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}: \ n \geq n_\varepsilon \Rightarrow \|x_n - L\| < r + \varepsilon$.

The set $\mathrm{LIM}^r x = \{L \in \mathbb{R}^n : x_n \xrightarrow{r} L\}$ is called the r-limit set of the sequence $x = (x_n)$. A sequence $x = (x_n)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_n)$. For $r = 0$, we get the ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}$, and let us denote the set $\{k \in K : k \leq n\}$ by K_n. Then the natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

where $|K_n|$ denotes the number of elements in K_n. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c := \mathbb{N} \setminus K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

A sequence $x = (x_n)$ is said to be r-statistically convergent to L, denoted by $x_n \xrightarrow{r-st} L$, provided that the set $\{n \in \mathbb{N} : \|x_n - L\| \geq r + \varepsilon\}$ has natural density zero for $\varepsilon > 0$; or equivalently, if the condition $st - \limsup \|x_n - L\| \leq r$ is satisfied. In addition, we can write $x_n \xrightarrow{r-st} L$ if and only if the inequality $\|x_n - L\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all n.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula $\|x, y\| = |x_1 y_2 - x_2 y_1|$; $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, y\| = 0$, for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

i) $\emptyset \in \mathcal{I}$, ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.
$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$, for each $x \in X$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:
i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 1.1. *[28] If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter on X, called the filter associated with $\mathcal{I}$.*

Example 1.2 ([28], Example 3.1.). *Denote by $\mathcal{I}_\delta$ the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$. Then $\mathcal{I}_\delta$ is non-trivial admissible ideal and $\mathcal{I}_\delta$-convergence coincides with the statistical convergence.*

Throughout the paper we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

A sequence $x = (x_i)$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}^n$, written as $\mathcal{I}\text{-}\lim x = L$, provided that $\{i \in \mathbb{N} : \|x_i - L\| \geq \varepsilon\} \in \mathcal{I}$, for every $\varepsilon > 0$. In this case, L is called the $\mathcal{I}$-limit of the sequence x.

$c \in \mathbb{R}^n$ is called a $\mathcal{I}$-cluster point of a sequence $x = (x_i)$ provided that $\{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \notin \mathcal{I}$, for every $\varepsilon > 0$. We denote the set of all $\mathcal{I}$-cluster points of the sequence x by $\mathcal{I}(\Gamma_x)$.

A sequence $x = (x_i)$ is said to be $\mathcal{I}$-bounded if there exists a positive real number M such that $\{i \in \mathbb{N} : \|x_i\| \geq M\} \in \mathcal{I}$.

For a sequence $x = (x_i)$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$\mathcal{I} - \limsup x = \begin{cases} \sup B_x & , \ if \ B_x \neq \emptyset \\ -\infty & , \ if \ B_x = \emptyset \end{cases}$$

and

$$\mathcal{I} - \liminf x = \begin{cases} \inf A_x & , \ if \ A_x \neq \emptyset \\ +\infty & , \ if \ A_x = \emptyset \end{cases},$$

where $A_x = \{a \in \mathbb{R} : \{i \in \mathbb{N} : x_i < a\} \notin \mathcal{I}\}$ and $B_x = \{b \in \mathbb{R} : \{i \in \mathbb{N} : x_i > b\} \notin \mathcal{I}\}$.

A sequence $x = (x_i)$ is said to be rough $\mathcal{I}$-convergent to x_*, denoted by $x_i \xrightarrow{r-\mathcal{I}} x_*$ provided that $\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$, for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I} - \limsup \|x_i - x_*\| \leq r \tag{1}$$

is satisfied. In addition, we can write $x_i \xrightarrow{r-\mathcal{I}} x_*$ iff the inequality $\|x_i - x_*\| < r + \varepsilon$, holds for every $\varepsilon > 0$ and almost all i.

A sequence (x_n) in $(X, \|.,.\|)$ is said to be rough convergent (r-convergent) to L, denoted by $x_n \xrightarrow{\|.,.\|}_r L$, if

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow \|x_n - L, z\| < r + \varepsilon \tag{2}$$

or equivalently, if for every $z \in X$

$$\limsup \|x_n - L, z\| \leq r. \tag{3}$$

If (2) holds L is an r-limit point of (x_n), which is usually no more unique (for $r > 0$). So, we have to consider the so-called r-limit set (or shortly r-limit) of (x_n) defined by

$$\mathrm{LIM}_2^r x := \{L \in X : x_n \xrightarrow{\|.,.\|}_r L\}. \tag{4}$$

The sequence (x_n) is said to be rough convergent if $\mathrm{LIM}_2^r x \neq \emptyset$. In this case, r is called a convergence degree of (x_n). For $r = 0$ we have the classical convergence in 2-normed space again. But our proper interest is case $r > 0$. There are several reasons for this interest. For instance, since an orginally convergent sequence (y_n) (with $y_n \to L$) in 2-normed space often cannot be determined (i.e., measured or calculated) exactly, one has to do with an approximated sequence (x_n) satisfying $\|x_n - y_n, z\| \leq r$, for all n and for every $z \in X$, where $r > 0$ is an upper bound of approximation error. Then, (x_n) is no more convergent in the classical sense, but for every $z \in X$, $\|x_n - L, z\| \leq \|x_n - y_n, z\| + \|y_n - L, z\| \leq r + \|y_n - L, z\|$ implies that is r-convergent in the sense of (2).

Example 1.3. *Let* $X = \mathbb{R}^2$. *The sequence* $x = (x_n) = ((-1)^n, 0)$ *is not convergent in* $(X, \|.,.\|)$ *but it is rough convergent for every* $z \in X$. *It is clear that* $\mathrm{LIM}_2^r x = \{y = (y_1, y_2) \in X : |y_1| \leq r - 1,\ |y_2| \leq r\}$. *In other words*

$$\mathrm{LIM}_2^r x = \begin{cases} \emptyset & , \ if \ r < 1 \\ \overline{B}_r((-1,0)) \cap \overline{B}_r((1,0)) & , \ if \ r \geq 1, \end{cases}$$

where $\overline{B}_r(L) := \{y \in X : \|y - L, z\| \leq r\}$.

A sequence $x = (x_n)$ in $(X, \|.,.\|)$ is said to be rough statistically convergent (r_2st-convergent) to L, denoted by $x_n \xrightarrow{\|.,.\|}_{r_2st} L$, provided that the set $\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}$ has natural density zero, for every $\varepsilon > 0$ and each nonzero $z \in X$; or equivalently, if the condition $st - \limsup \|x_n - L, z\| \leq r$ is satisfied. In addition, we can write $x_n \xrightarrow{\|.,.\|}_{r_2st} L$, if and only if, the inequality $\|x_n - L, z\| < r + \varepsilon$, holds for every $\varepsilon > 0$, each nonzero $z \in X$ and almost all n.

In this convergence, r is called the statistical convergence degree. For $r = 0$, rough statistically convergent coincide ordinary statistical convergence.

In general, the rough statistical limit of a sequence $x = (x_n)$ may not be unique for the roughness degree $r > 0$. So, we have to consider the so-called r-statistically limit set of the sequence x in X, which is defined by

$$st - \mathrm{LIM}_2^r x := \{L \in X : x_n \xrightarrow{\|.,.\|}_{r_2st} L\}. \tag{5}$$

The sequence x is said to be r-statistically convergent provided that $st - \mathrm{LIM}_2^r x \neq \emptyset$.

Lemma 1.4 ([4], Theorem 2.2). *Let* $(X, \|.,.\|)$ *be a 2-normed space and consider a sequence* $x = (x_n) \in X$. *The sequence* (x_n) *is bounded if and only if there exist an* $r \geq 0$ *such that* $\mathrm{LIM}_2^r x \neq \emptyset$. *For all* $r > 0$, *a bounded sequence* (x_n) *is always contains a subsequence* x_{n_k} *with* $\mathrm{LIM}_2^{(x_{n_k}),r} x_{n_k} \neq \emptyset$.

Lemma 1.5 ([4], Theorem 2.3). *Let* $(X, \|.,.\|)$ *be a 2-normed space and consider a sequence* $x = (x_n) \in X$. *For all* $r \geq 0$, *the r-limit set* $\mathrm{LIM}_2^r x$ *of an arbitrary sequence* (x_n) *is closed.*

Lemma 1.6 ([4], Theorem 2.4). *Let* $(X, \|.,.\|)$ *be a 2-normed space and consider a sequence* $x = (x_n) \in X$. *If* $y_0 \in \mathrm{LIM}_2^{r_0} x$ *and* $y_1 \in \mathrm{LIM}_2^{r_1} x$, *then* $y_\alpha := (1 - \alpha) y_0 + \alpha y_1 \in \mathrm{LIM}_2^{(1-\alpha)r_0 + \alpha r_1} x$, *for* $\alpha \in [0, 1]$.

2. Main Results

Definition 2.1. *A sequence* $x = (x_n)$ *said to be rough ideal convergence* ($r_2\mathcal{I}$*-convergent) to L in 2-normed space X, denoted by* $x_n \xrightarrow{\|.,.\|}_{r_2\mathcal{I}} L$, *if for every* $\varepsilon > 0$ *and each nonzero* $z \in X$

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \in \mathcal{I}$$

or equivalently, if the condition

$$\mathcal{I} - \limsup \|x_n - L, z\| \leq r \tag{6}$$

is satisfied. In addition, we can write $x_n \xrightarrow{\|.,.\|}_{r_2\mathcal{I}} L$, *if and only if, the inequality*

$$\|x_n - L, z\| < r + \varepsilon,$$

holds for every $\varepsilon > 0$, *each nonzero* $z \in X$ *and almost all* n.

Remark 2.2. *If* $\mathcal{I}$ *is an admissible ideal, then classical rough convergence implies rough* $\mathcal{I}$*-convergence in* 2*-normed space.*

In this convergence, r is called the roughness degree. For $r = 0$, rough ideal convergence coincide ordinary ideal convergence in 2-normed space.

In a similar fashion to the idea of classical rough convergence, the idea of rough ideal convergence of a sequence in 2-normed space can be interpreted as follows.

Suppose that a sequence $y = (y_n)$ in X is $\mathcal{I}$-convergent and cannot be measured or calculated exactly, one has to do with an approximated (or $\mathcal{I}$ approximated) sequence $x = (x_n)$ in X satisfying $\|x_n - y_n, z\| \leq r$, for all n and each nonzero $z \in X$, (or for almost all n, that is, $\{n \in \mathbb{N} : \|x_n - y_n, z\| \geq r\} \in \mathcal{I}$.) Then, the sequence $x = (x_n)$ is not $\mathcal{I}$-convergent in 2-normed space anymore, but since the inclusion

$$\{n \in \mathbb{N} : \|y_n - L', z\| \geq \varepsilon\} \supseteq \{n \in \mathbb{N} : \|x_n - L', z\| \geq r + \varepsilon\} \tag{7}$$

holds for each nonzero $z \in X$ and we have

$$\{n \in \mathbb{N} : \|y_n - L', z\| \geq r + \varepsilon\} \in \mathcal{I}$$

and so

$$\{n \in \mathbb{N} : \|x_n - L', z\| \geq r + \varepsilon\} \in \mathcal{I}$$

that is, the sequence x is rough $\mathcal{I}$-convergent in 2-normed space $(X, \|.,.\|)$ in the sense of Definition 2.1

In general, the rough-$\mathcal{I}$ limit of a sequence $x = (x_n)$ may not be unique for the roughness degree $r > 0$ in 2-normed space $(X, \|.,.\|)$. So, we have to consider the so-called rough-$\mathcal{I}$ limit set of the sequence x in X, which is defined by

$$\mathcal{I} - \mathrm{LIM}_2^r x := \{L \in X : x_n \xrightarrow{\|.,.\|}_{r_2\mathcal{I}} L\}. \tag{8}$$

The sequence x is said to be rough $\mathcal{I}$-convergent provided that $\mathcal{I} - \mathrm{LIM}_2^r x \neq \emptyset$.

We have that $\mathrm{LIM}_2^r x = \emptyset$ for an unbounded sequence $x = (x_n)$. But such a sequence might be rough $\mathcal{I}$-convergent. For instance, let $\mathcal{I}$ be the $\mathcal{I}_\delta$ of $\mathbb{N}$ and define

$$x_n := \begin{cases} ((-1)^n, 0) & , \ if \ \ n \neq k^2 \ \ (k \in \mathbb{N}) \\ (n, n) & , \ otherwise \end{cases} \tag{9}$$

in X. Because the set $\{1, 4, 9, 16, \dots\}$ belongs to $\mathcal{I}$, we have

$$\mathcal{I} - \mathrm{LIM}_2^r x := \begin{cases} \emptyset & , \ if \ \ r < 1, \\ \overline{B}_r((-1, 0)) \cap \overline{B}_r((1, 0)) & , \ if \ \ r \geq 1, \end{cases}$$

and $\mathrm{LIM}_2^r x = \emptyset$ for all $r \geq 0$.

From the example above, we have $\mathrm{LIM}_2^r x = \emptyset$ but $\mathcal{I} - \mathrm{LIM}_2^r x \neq \emptyset$. Because $\mathcal{I}$ is an admissible ideal, $\mathrm{LIM}_2^r x \neq \emptyset$ implies $\mathcal{I} - \mathrm{LIM}_2^r x \neq \emptyset$, that is, if $x = (x_n) \in \mathrm{LIM}_2^r x$, then, by Remark 2.2, $x = (x_n) \in \mathcal{I} - \mathrm{LIM}_2^r x$, for each sequence $x = (x_n)$. Also, if we define all the rough convergent sequences by $\mathrm{LIM}_2^r x$ and if we define all the rough $\mathcal{I}$-convergent sequences by $\mathcal{I} - \mathrm{LIM}_2^r x$, then we have

$$\mathrm{LIM}_2^r x \subseteq \mathcal{I} - \mathrm{LIM}_2^r x.$$

That is, we have the fact

$$\{r \geq 0 : \mathrm{LIM}_2^r x \neq \emptyset\} \subseteq \{r \geq 0 : \mathcal{I} - \mathrm{LIM}_2^r x \neq \emptyset\}$$

and so

$$\inf\{r \geq 0 : \mathrm{LIM}_2^r x \neq \emptyset\} \geq \inf\{r \geq 0 : \mathcal{I} - \mathrm{LIM}_2^r x \neq \emptyset\}.$$

It also directly yields

$$diam(\mathrm{LIM}_2^r x) \leq diam(\mathcal{I} - \mathrm{LIM}_2^r x).$$

As mentioned above, we cannot say that the rough $\mathcal{I}$-limit of a sequence is unique for the degree of roughness $r > 0$. The following conclusion related to this fact.

Theorem 2.3. *For a sequence $x = (x_n)$ in $(X, \|.,.\|)$, we have $diam(\mathcal{I} - \text{LIM}_2^r x) \leq 2r$. Also, generally, $diam(\mathcal{I} - \text{LIM}_2^r x)$ has no smaller bound.*

Proof. Suppose that $diam(\mathcal{I} - \text{LIM}_2^r x) > 2r$. Then, there exist $y, t \in \mathcal{I} - \text{LIM}_2^r x$ such that $\|y - t, z\| > 2r$, for each nonzero $z \in X$. Choose $\varepsilon \in \left(0, \frac{\|y-t,z\|}{2} - r\right)$. Since $y, t \in \mathcal{I} - \text{LIM}_2^r x$ we have

$$T_1 = T_1(\varepsilon) \in \mathcal{I} \text{ and } T_2 = T_2(\varepsilon) \in \mathcal{I},$$

where

$$T_1 = T_1(\varepsilon) = \{n \in \mathbb{N} : \|x_n - y, z\| \geq r + \varepsilon\}$$

and

$$T_2 = T_2(\varepsilon) = \{n \in \mathbb{N} : \|x_n - t, z\| \geq r + \varepsilon\}$$

for every $\varepsilon > 0$ and each nonzero $z \in X$. By the properties of $\mathcal{F}(\mathcal{I})$, we have $(T_1^c \cap T_2^c) \in \mathcal{F}(\mathcal{I})$ and so for all $n \in T_1^c \cap T_2^c$, and each nonzero $z \in X$, we can write

$$\begin{aligned} \|y - t, z\| &\leq \|x_n - y, z\| + \|x_n - t, z\| \\ &< 2(r + \varepsilon) \\ &< 2\left(r + \frac{\|y - t, z\|}{2} - r\right) \\ &= \|y - t, z\| \end{aligned}$$

which is a contradiction.

Now let's do the second part of the proof. Let a sequence $x = (x_n)$ in $(X, \|.,.\|)$ such that $\mathcal{I} - \lim x_n = L$. Then, for every $\varepsilon > 0$ and each nonzero $z \in X$, we can write

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\} \in \mathcal{I}.$$

So, for each nonzero $z \in X$, we have

$$\begin{aligned} \|x_n - y, z\| &\leq \|x_n - L, z\| + \|L - y, z\| \\ &\leq \|x_n - L, z\| + r, \end{aligned}$$

for each $y \in \overline{B}_r(L) := \{y \in X : \|y - L, z\| \leq r\}$. Then, for every $\varepsilon > 0$ and each nonzero $z \in X$ we get

$$\|x_n - y, z\| < r + \varepsilon,$$

for each $n \in \{n \in \mathbb{N} : \|x_n - L, z\| < \varepsilon\}$. Since the sequence x is $\mathcal{I}$-convergent to L, for each nonzero $z \in X$, we have

$$\{n \in \mathbb{N} : \|x_n - L, z\| < \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Hence, we have $y \in \mathcal{I} - \text{LIM}_2^r x$. As a result, we can write

$$\mathcal{I} - \text{LIM}_2^r x = \overline{B}_r(L).$$

Since $diam(\overline{B}_r(L)) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $\mathcal{I} - \text{LIM}_2^r x$ can no longer be reduced. □

By [[4], Theorem 2.2], there exists a nonnegative real number r such that $\text{LIM}_2^r x \neq \emptyset$ for a bounded sequence. Because the fact $\text{LIM}_2^r x \neq \emptyset$ implies $\mathcal{I} - \text{LIM}_2^r x \neq \emptyset$, we have the following result.

Result 2.4. *If a sequence $x = (x_n)$ is bounded, then there exists a nonnegative real number r such that $\mathcal{I} - \text{LIM}_2^r x \neq \emptyset$.*

The opposite implication of the above result is not valid. If we let the sequence to be $\mathcal{I}$-bounded in 2-normed space, then we have the converse of Result 2.4. Hence, we give the following theorem.

Theorem 2.5. *A sequence $x = (x_n)$ is $\mathcal{I}$-bounded if and only if there exists a nonnegative real number r such that $\mathcal{I} - \text{LIM}_2^r x \neq \emptyset$. Also, for all $r > 0$ and an $\mathcal{I}$-bounded sequence $x = (x_n)$ always contains a subsequence (x_{n_k}) with $\mathcal{I} - \text{LIM}_2^{(x_{n_k}),r} x_{n_k} \neq \emptyset$.*

Proof. Let $x = (x_n)$ be a $\mathcal{I}$-bounded sequence. Then, there exists a positive real number M such that for each nonzero $z \in X$,

$$\{n \in \mathbb{N} : \|x_n, z\| \geq M\} \in \mathcal{I}.$$

Now, we let $r_1 := \sup\{\|x_n, z\| : n \in T^c\}$, where $T := \{n \in \mathbb{N} : \|x_n, z\| \geq M\}$, for each nonzero $z \in X$. Then, the set $\mathcal{I} - \text{LIM}_2^{r_1} x$ contains the origin of X. Therefore, we have $\mathcal{I} - \text{LIM}_2^{r_1} x \neq \emptyset$.

If $\mathcal{I} - \text{LIM}_2^r x \neq \emptyset$ for some $r \geq 0$, then there exists an L such that $L \in \mathcal{I} - \text{LIM}_2^r x$, i.e.,

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \in \mathcal{I},$$

for each $\varepsilon > 0$ and each nonzero $z \in X$. Then, we say that almost all x_n's are contained in some ball with any radius grater than r. So the sequence x is $\mathcal{I}$-bounded. □

By [[4], Proposition 2.1], we know that if $x' = (x_{n_k})$ is a subsequence of $x = (x_n)$, then $\mathcal{I} - \text{LIM}_2^r x \subseteq \mathcal{I} - \text{LIM}_2^r x'$. But this fact does not hold in the theory of ideal convergence. For instance, let $\mathcal{I}$ be the $\mathcal{I}_\delta$ of $\mathbb{N}$ and define

$$x_n := \begin{cases} (n, n) & , \ if \ \ n = k^3, (k \in \mathbb{N}) \\ (0, (-1)^n) & , \ otherwise \end{cases}$$

of real numbers. Then, the sequence $x' := ((1, 1), (8, 8), (27, 27), \cdots)$ is a subsequence of x. We have $\mathcal{I} - \text{LIM}_2^r x = \overline{B}_r((0, -1)) \cap \overline{B}_r((0, 1))$ and $\mathcal{I} - \text{LIM}_2^r x' = \emptyset$, for $r \geq 1$.

So we can present the statistical analogue of Arslan and Dündar's result [[4], Proposition 2.1] in the following theorem without proof.

Theorem 2.6. *If $x' = (x_{n_k})$ is a nonthin subsequence of $x = (x_n)$, then*

$$\mathcal{I} - \text{LIM}_2^r x \subseteq \mathcal{I} - \text{LIM}_2^r x'.$$

Now, we give the topological and geometrical properties of the rough $\mathcal{I}$-limit set of a sequence in 2-normed space.

Theorem 2.7. *The rough $\mathcal{I}$-limit set of a sequence $x = (x_n)$ in 2-normed space is closed.*

Proof. If $\mathcal{I} - \text{LIM}_2^r x = \emptyset$, proof is clear. Let $\mathcal{I} - \text{LIM}_2^r x \neq \emptyset$. Then, we can choose a sequence

$$(y_n) \subseteq \mathcal{I} - \text{LIM}_2^r x$$

such that $y_n \to L$, for $n \to \infty$. For the proof we have to show that $L \in \mathcal{I} - \text{LIM}_2^r x$.

Since $y_n \to L$, for every $\varepsilon > 0$ there exists an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$\|y_n - L, z\| < \frac{\varepsilon}{2},$$

for all $n > n_{\frac{\varepsilon}{2}}$ and each nonzero $z \in X$. Now choose an $n_0 \in \mathbb{N}$ such that $n_0 > n_{\frac{\varepsilon}{2}}$. Then, we can write $\|y_{n_0} - L, z\| < \frac{\varepsilon}{2}$. On the other hand, since $(y_n) \subseteq \mathcal{I} - \text{LIM}_2^r x$, we have $y_{n_0} \in \mathcal{I} - \text{LIM}_2^r x$, that is,

$$\left\{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| \geq r + \frac{\varepsilon}{2}\right\} \in \mathcal{I}.$$

Now let us show that the inclusion

$$\left\{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| < r + \frac{\varepsilon}{2}\right\} \subseteq \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\} \tag{10}$$

holds for each nonzero $z \in X$. Let $k \in \{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| < r + \frac{\varepsilon}{2}\}$. Hence, for each nonzero $z \in X$ we have

$$\|x_k - y_{n_0}, z\| < r + \frac{\varepsilon}{2}$$

and so

$$\|x_k - L, z\| \leq \|x_k - y_{n_0}, z\| + \|y_{n_0} - L, z\| < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon,$$

that is,

$$k \in \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\},$$

which proves (10). So we have

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \subseteq \left\{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| \geq r + \frac{\varepsilon}{2}\right\},$$

for each nonzero $z \in X$. Since $\left\{n \in \mathbb{N} : \|x_n - y_{n_0}, z\| \geq r + \frac{\varepsilon}{2}\right\} \in \mathcal{I}$, for each nonzero $z \in X$ we have

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \in \mathcal{I},$$

(i.e. $L \in \mathcal{I} - \text{LIM}_2^r x$), which completes the proof. □

Theorem 2.8. *The rough $\mathcal{I}$-limit set of a sequence in 2-normed space is convex.*

Proof. Let $y_0, y_1 \in \mathcal{I} - \text{LIM}_2^r x$ for the sequence $x = (x_n)$. For every $\varepsilon > 0$ and each nonzero $z \in X$, we define

$$T_1(\varepsilon) := \{n \in \mathbb{N} : \|x_n - y_0, z\| \geq r + \varepsilon\} \text{ and } T_2(\varepsilon) := \{n \in \mathbb{N} : \|x_n - y_1, z\| \geq r + \varepsilon\}.$$

Since $y_0, y_1 \in \mathcal{I} - \text{LIM}_2^r x$, we have $T_1(\varepsilon) \in \mathcal{I}$ and $T_2(\varepsilon) \in \mathcal{I}$. Hence, for each $n \in T_1^c(\varepsilon) \cap T_2^c(\varepsilon)$ we have

$$\|x_n - [(1-\lambda)y_0 + \lambda y_1], z\| = \|(1-\lambda)(x_n - y_0) + \lambda(x_n - y_1), z\| < r + \varepsilon$$

for each $\lambda \in [0,1]$ and each nonzero $z \in X$. Since, $T_1^c(\varepsilon) \cap T_2^c(\varepsilon) \in \mathcal{F}(\mathcal{I})$ by definition $\mathcal{F}(\mathcal{I})$, we have

$$\{n \in \mathbb{N} : \|x_n - [(1-\lambda)(y_0) + \lambda y_1], z\| \geq r + \varepsilon\} \in \mathcal{I},$$

that is,

$$[(1-\lambda)(y_0) + \lambda y_1] \in \mathcal{I} - \text{LIM}_2^r x,$$

for each nonzero $z \in X$. This proves the convexity of the set $\mathcal{I} - \text{LIM}_2^r x$. □

Theorem 2.9. *A sequence $x = (x_n)$ is rough $\mathcal{I}$-convergent to L, if and only if there exists a sequence $y = (y_n)$ such that $\mathcal{I} - \lim y = L$ and $\|x_n - y_n, z\| \leq r$, for each $n \in \mathbb{N}$ and each nonzero $z \in X$.*

Proof. Let $x_n \xrightarrow{\|.,.\|}_{r_2\mathcal{I}} L$. Then, by definition for each nonzero $z \in X$ we have

$$\mathcal{I} - \limsup \|x_n - L, z\| \leq r. \tag{11}$$

Now, for each nonzero $z \in X$ we define

$$y_n := \begin{cases} L & , \ if \ \|x_n - L, z\| \leq r \\ x_n + r\frac{L - x_n}{\|x_n - L, z\|} & , \ otherwise. \end{cases} \tag{12}$$

Then, for each nonzero $z \in X$ we can write

$$\|y_n - L, z\| = \begin{cases} 0 & , \ if \ \|x_n - L, z\| \leq r \\ \|x_n - L, z\| - r & , \ otherwise \end{cases} \tag{13}$$

and by definition of y_n, we have

$$\|x_n - y_n, z\| \leq r, \text{ for all } n \in \mathbb{N}.$$

By (11) and the definition of y_n, for all $n \in \mathbb{N}$ we have $\mathcal{I} - \limsup \|y_n - L, z\| = 0$, which implies that $\mathcal{I} - \lim y_n = L$.

Conversely, since $\mathcal{I} - \lim y_n = L$, we have

$$\{n \in \mathbb{N} : \|y_n - L, z\| \geq \varepsilon\} \in \mathcal{I},$$

for each $\varepsilon > 0$ and each nonzero $z \in X$ and so, it is easy to see that the inclusion

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \subseteq \{n \in \mathbb{N} : \|y_n - L, z\| \geq \varepsilon\}$$

holds. Since

$$\{n \in \mathbb{N} : \|y_n - L, z\| \geq \varepsilon\} \in \mathcal{I},$$

for each nonzero $z \in X$, we have

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \in \mathcal{I},$$

which completes the proof. □

If we replace the condition

$$"\|x_n - y_n, z\| \leq r, \text{ for all } n \in \mathbb{N} \text{ and for each nonzero } z \in X,"$$

in the hypothesis of the above theorem with the condition

$$"\{n \in \mathbb{N} : \|x_n - y_n, z\| > r\} \in \mathcal{I}",$$

then the theorem will also be valid.

Definition 2.10. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ an admissible ideal. $c \in X$ is called a ideal cluster point of a sequence $x = (x_n)$ provided that the set*

$$\{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\} \notin \mathcal{I}$$

for every $\varepsilon > 0$ and each nonzero $z \in X$. We denote the set of all $\mathcal{I}$-cluster points of the sequence x by $\mathcal{I}(\Gamma_x^2)$.

Now, we give an important property of the set of rough $\mathcal{I}$-limit points of a sequence.

Lemma 2.11. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ an admissible ideal. For an arbitrary $c \in \mathcal{I}(\Gamma_x^2)$ of a sequence $x = (x_n)$, we have $\|L - c, z\| \leq r$, for all $L \in \mathcal{I} - \text{LIM}_2^r x$ and for each nonzero $z \in X$.*

Proof. Assume on the contrary that there exists a point $c \in \mathcal{I}(\Gamma_x^2)$ and $L \in \mathcal{I} - \text{LIM}_2^r x$ such that

$$\|L - c, z\| > r,$$

for each nonzero $z \in X$. Define $\varepsilon := \frac{\|L-c,z\|-r}{3}$. Then, for each nonzero $z \in X$ we can write

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \supseteq \{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\}. \tag{14}$$

Since $c \in \mathcal{I}(\Gamma_x^2)$, for each nonzero $z \in X$ we have

$$\{n \in \mathbb{N} : \|x_n - c, z\| < \varepsilon\} \notin \mathcal{I}.$$

But from the definition of $\mathcal{I}$-convergence, since

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\} \in \mathcal{I},$$

so by (14), for each nonzero $z \in X$ we have

$$\{n \in \mathbb{N} : \|x_n - c, z\| \geq \varepsilon\} \in \mathcal{I},$$

which contradicts the fact $c \in \mathcal{I}(\Gamma_x^2)$. On the other hand, if $c \in \mathcal{I}(\Gamma_x^2)$ then,

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}$$

must not belong to $\mathcal{I}$, which contradits the fact $L \in \mathcal{I} - \text{LIM}_2^r x$. This completes the proof. □

Now we give two $\mathcal{I}$-convergence criteria associated with the rough $\mathcal{I}$-limit set.

Theorem 2.12. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ an admissible ideal. A sequence $x = (x_n)$ is ideal convergent to L if and only if $\mathcal{I} - \mathrm{LIM}_2^r x = \overline{B}_r(L)$.*

Proof. Since $x = (x_n)$ is ideal convergent to L, by the proof of Theorem 2.3 we have

$$\mathcal{I} - \mathrm{LIM}_2^r x = \overline{B}_r(L).$$

Since $\mathcal{I} - \mathrm{LIM}_2^r x = \overline{B}_r(L) \neq \emptyset$, then by Theorem 2.5 we can say that the sequence x is $\mathcal{I}$-bounded. Assume on the contrary that the sequence x has another $\mathcal{I}$-cluster point L' different from L. Then, the point

$$\overline{L} := L + \frac{r}{\|L - L', z\|}(L - L')$$

satisfies

$$\|\overline{L} - L', z\| = \left(\frac{r}{\|L - L', z\|} + 1\right)\|L - L', z\| = r + \|L - L', z\| > r.$$

Since L' is a $\mathcal{I}$-cluster point of the sequence x, by Lemma 2.11 this inequality implies that $\overline{L} \notin \mathcal{I} - \mathrm{LIM}_2^r x$. This contradicts the fact

$$\|\overline{L} - L, z\| = r \text{ and } \mathcal{I} - \mathrm{LIM}_2^r x = \overline{B}_r(L).$$

Therefore, L is the unique $\mathcal{I}$-cluster point of the sequence x and so, we can say that the sequence x is $\mathcal{I}$-convergent to L. Hence L is the unique $\mathcal{I}$-cluster point of the sequence x as a bounded sequence (by Theorem 2.5) in some finite-dimensional normed space. Consequently, we can say that

$$x_n \xrightarrow{\|.,.\|}_{\mathcal{I}} L.$$

This completes the proof. □

Theorem 2.13. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ an admissible ideal, $(X, \|.,.\|)$ be a strictly convex space and $x = (x_n)$ be a sequence in this space. If there exist $t_1, t_2 \in \mathcal{I} - \mathrm{LIM}_2^r x$ such that $\|t_1 - t_2, z\| = 2r$ for each nonzero $z \in X$, then this sequence is $\mathcal{I}$-convergent to $\frac{1}{2}(t_1 + t_2)$.*

Proof. Assume that $t \in \mathcal{I}(\Gamma_x^2)$. Then, $t_1, t_2 \in \mathcal{I} - \mathrm{LIM}_2^r x$ implies that

$$\|t_1 - t, z\| \leq r \text{ and } \|t_2 - t, z\| \leq r \tag{15}$$

for each nonzero $z \in X$, by Lemma 2.11. On the other hand, for each nonzero $z \in X$, we have

$$2r = \|t_1 - t_2, z\| \leq \|t_1 - t, z\| + \|t_2 - t, z\|, \tag{16}$$

and so

$$\|t_1 - t, z\| = \|t_2 - t, z\| = r,$$

combining the inequalities (15) and (16). Since for each nonzero $z \in X$,

$$\frac{1}{2}(t_2 - t_1) = \frac{1}{2}[(t - t_1) + (t_2 - t)] \tag{17}$$

and $\|t_1 - t_2, z\| = 2r$, we have

$$\|\frac{1}{2}(t_2 - t_1), z\| = r.$$

By the strict convexity of the space and from the equality (17), we get

$$\frac{1}{2}(t_2 - t_1) = t - t_1 = t_2 - t,$$

for each nonzero $z \in X$, which implies that

$$t = \frac{1}{2}(t_1 + t_2).$$

Hence, t is the unique $\mathcal{I}$-cluster point of the sequence $x = (x_n)$. On the other hand, the assumption $t_1, t_2 \in \mathcal{I} - \text{LIM}_2^r x$ implies that

$$\mathcal{I} - \text{LIM}_2^r x \neq \emptyset.$$

By Theorem 2.5, the sequence x is $\mathcal{I}$-bounded. Consequently, the sequence x is $\mathcal{I}$-convergent, that is,

$$\mathcal{I} - \lim x = \frac{1}{2}(t_1 + t_2).$$

□

The following Theorem is the ideal extension of [[5], Theorem 2.5].

Theorem 2.14. ***(i)*** *If* $c \in \mathcal{I}(\Gamma_x^2)$ *then,*

$$\mathcal{I} - \text{LIM}_2^r x \subseteq \overline{B}_r(c). \tag{18}$$

(ii)

$$\mathcal{I} - \text{LIM}_2^r x = \bigcap_{c \in \mathcal{I}(\Gamma_x^2)} \overline{B}_r(c) = \{L \in X : \mathcal{I}(\Gamma_x^2) \subseteq \overline{B}_r(L)\}. \tag{19}$$

Proof. (i) Let $c \in \mathcal{I}(\Gamma_x^2)$. Then, by Lemma 2.11, for each nonzero $z \in X$ we have

$$\|L - c, z\| \leq r, \text{ for all } L \in \mathcal{I} - \text{LIM}_2^r x,$$

otherwise we get

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}) \notin \mathcal{I},$$

for $\varepsilon := \frac{\|L-c,z\|-r}{3}$. Since c is an $\mathcal{I}$-cluster point of (x_n), this contradicts the fact $L \in \mathcal{I} - \text{LIM}_2^r x$.

(ii) From the inclusion (18), we get

$$\mathcal{I} - \text{LIM}_2^r x \subseteq \bigcap_{c \in \mathcal{I}(\Gamma_x^2)} \overline{B}_r(c). \tag{20}$$

Now, let $y \in \bigcap_{c \in \mathcal{I}(\Gamma_x^2)} \overline{B}_r(c)$. Then, for each nonzero $z \in X$, we have

$$\|y - c, z\| \leq r,$$

for all $c \in \mathcal{I}(\Gamma_x^2)$, which is equivalent to

$$\mathcal{I}(\Gamma_x^2) \subseteq \overline{B}_r(y),$$

that is,

$$\bigcap_{c \in \mathcal{I}(\Gamma_x^2)} \overline{B}_r(c) \subseteq \{L \in X : \mathcal{I}(\Gamma_x^2) \subseteq \overline{B}_r(L)\}. \tag{21}$$

Now, let $y \notin \mathcal{I} - \text{LIM}_2^r x$. Then, there exists an $\varepsilon > 0$ such that for each nonzero $z \in X$,

$$\{n \in \mathbb{N} : \|x_n - y, z\| \geq r + \varepsilon\} \notin \mathcal{I},$$

which implies the existence of a $\mathcal{I}$-cluster point c of the sequence x with

$$\|y-c,z\| \geq r+\varepsilon,$$

that is,

$$\mathcal{I}(\Gamma_x^2) \nsubseteq \overline{B}_r(y) \text{ and } y \notin \{L \in X : \mathcal{I}(\Gamma_x^2) \subseteq \overline{B}_r(L)\}.$$

Hence,

$$y \in \mathcal{I}-\mathrm{LIM}_2^r x$$

follows from

$$y \in \{L \in X : \mathcal{I}(\Gamma_x^2) \subseteq \overline{B}_r(L)\},$$

that is,

$$\{L \in X : \mathcal{I}(\Gamma_x^2) \subseteq \overline{B}_r(L)\} \subseteq \mathcal{I}-\mathrm{LIM}_2^r x. \tag{22}$$

Therefore, the inclusions (20)-(22) ensure that (19) holds, that is,

$$\mathcal{I}-\mathrm{LIM}_2^r x = \bigcap_{c\in\mathcal{I}(\Gamma_x^2)} \overline{B}_r(c) = \{L \in X : \mathcal{I}(\Gamma_x^2) \subseteq \overline{B}_r(L)\}.$$

□

We end this work by giving the relation between the set of $\mathcal{I}$-cluster points and the set of rough $\mathcal{I}$-limit points of a sequence.

Example 2.15. *Consider the sequence $x=(x_n)$ defined in (9) and let $\mathcal{I}$ be the $\mathcal{I}_\delta$ of $\mathbb{N}$. Then, we have*

$$\mathcal{I}(\Gamma_x^2) = \{(-1,0),(1,0)\}.$$

It follows from (19) that

$$\mathcal{I}-\mathrm{LIM}^r x = \overline{B}_r((-1,0)) \cap \overline{B}_r((1,0)).$$

In this last part of the study, we give the relation between the set of $\mathcal{I}$-cluster points and the set of rough $\mathcal{I}$-limit points of a sequence in 2-normed space.

Theorem 2.16. *Let $x=(x_n)$ be a $\mathcal{I}$-bounded sequence in X. If*

$$r = diam(\mathcal{I}(\Gamma_x^2)),$$

then we have

$$\mathcal{I}(\Gamma_x^2) \subseteq \mathcal{I}-\mathrm{LIM}_2^r x.$$

Proof. Let $c_1 \notin \mathcal{I}-\mathrm{LIM}_2^r x$. Then, there exists an $\varepsilon_1>0$ such that, for each nonzero $z\in X$

$$\{n\in\mathbb{N} : \|x_n-c_1,z\| \geq r+\varepsilon_1\} \notin \mathcal{I}. \tag{23}$$

Since the sequence is $\mathcal{I}$-bounded and from the inequality (23), there exists another $\mathcal{I}$-cluster point c_2 such that, for each nonzero $z\in X$,

$$\|c_1-c_2,z\| > r+\varepsilon_2,$$

where $\varepsilon_2 := \frac{\varepsilon_1}{2}$. Hence, we get

$$diam(\mathcal{I}(\Gamma_x^2)) > r+\varepsilon_2,$$

which proves the theorem. □

References

[1] R. Antal, M. Chawla, V. Kumar, *Rough statistical convergence in probabilistic normed spaces*, Thai J. Math. **20**(4) (2023), 1707–1719.
[2] M. Arslan, E. Dündar, $\mathcal{I}$-Convergence and $\mathcal{I}$-Cauchy Sequence of Functions In 2-Normed Spaces, Konuralp J. Math. **6**(1) (2018), 57–62.
[3] M. Arslan, E. Dündar, On $\mathcal{I}$-Convergence of sequences of functions in 2-normed spaces, Southeast Asian Bull. Math. **42** (2018) 491–502.
[4] M. Arslan, E. Dündar, Rough convergence in 2-normed spaces, Bull. Math. Anal. Appl. **10**(3) (2018) 1–9.
[5] M. Arslan, E. Dündar, On rough convergence in 2-normed spaces and some properties, Filomat **33**(16) (2019), 5077–5086.
[6] M. Arslan, E. Dündar, Rough statistical convergence in 2-normed spaces, Honam Mathematical J. **43**(3) (2021), 417–431.
[7] S. Aytar, Rough statistical convergence, Numer. Funct. Anal. Optim. **29**(3-4) (2008) 291–303.
[8] S. Aytar, The rough limit set and the core of a real requence, Numer. Funct. Anal. Optim. **29**(3-4) (2008) 283–290.
[9] H. Çakallı and S. Ersan, New types of continuity in 2-normed spaces, Filomat **30**(3) (2016) 525–532.
[10] K. Demirci, *$\mathcal{I}$-limit superior and limit inferior*, Math. Commun. **6** (2001), 165–172.
[11] E. Dündar, C. Çakan, Rough $\mathcal{I}$-convergence, Gulf J. Math. **2**(1) (2014) 45–51.
[12] E. Dündar, C. Çakan, Rough convergence of double sequences, Demonstr. Math. **47**(3) (2014) 638–651.
[13] E. Dündar, On Rough $\mathcal{I}_2$-convergence, Numer. Funct. Anal. Optim. **37**(4) (2016) 480–491.
[14] E. Dündar, M. Arslan, S. Yegül, On $\mathcal{I}$-Uniform Convergence Of Sequences Of Functions in 2-Normed Spaces, Rocky Mountain J. Math. **50**(5) (2020), 1637–1646
[15] H. Fast, Sur la convergence statistique, Colloq. Math. **2** (1951), 241–244.
[16] S. Gähler, 2-metrische Räume und ihre topologische struktur, Math. Nachr. **26** (1963), 115–148.
[17] S. Gähler, 2-normed spaces, Math. Nachr. **28** (1964), 1–43.
[18] H. Gunawan, M. Mashadi, On *n*-normed spaces, Int. J. Math. Math. Sci. **27**(10) (2001) 631–639.
[19] H. Gunawan, M. Mashadi, On finite dimensional 2-normed spaces, Soochow J. Math. **27**(3) (2001) 321–329.
[20] M. Gürdal, S. Pehlivan, The statistical convergence in 2-Banach spaces, Thai J. Math. **2**(1) (2004) 107–113.
[21] M. Gürdal, S. Pehlivan, Statistical convergence in 2-normed spaces, Southeast Asian Bull. Math. **33** (2009) 257–264.
[22] M. Gürdal, On ideal convergent sequences in 2-normed spaces, Thai J. Math. **4**(1) (2006) 85–91.
[23] M. Gürdal, I. Açık, On $\mathcal{I}$-Cauchy sequences in 2-normed spaces, Math. Inequal. Appl. **11**(2) (2008) 349–354.
[24] M. Gürdal, E. Kaya, E. Savaş, *Lacunary statistical convergence of rough triple sequence via ideals*. Asian-European J. Math. **16**(07) (2023), https://doi.org/10.1142/S1793557123501322
[25] Ö. Kişi, E. Dündar, Rough $\mathcal{I}_2$-lacunary statistical convergence of double sequences, J. Inequal. Appl. **2018:230** (2018) 16 pages, https://doi.org/10.1186/s13660-018-1831-7
[26] Ö. Kişi, E. Dündar, *Rough $\Delta\mathcal{I}$-Statistical Convergence*, J. Appl. Math. & Informatics, **40** (2022), 619–632.
[27] Ö. Kişi, C. Choudhury, *Some results on rough ideal convergence of triple sequences in gradual normed linear spaces*, Adv. Math. Sci. Appl. **32**(1) (2023), 179–201.
[28] P. Kostyrko, T. Salat and W. Wilczyński, *I-convergence*, Real Anal. Exchange, **26(2)** (2000), 669–686.
[29] P. Kostyrko, M. Macaj, T. Salat and M. Sleziak,*I-convergence and extremal I-limit points*, Math. Slovaca, **55**(2005), 443–464.
[30] M. Mursaleen, A. Alotaibi, On $\mathcal{I}$-convergence in random 2-normed spaces, Math. Slovaca **61**(6) (2011) 933–940.
[31] H. X. Phu, Rough convergence in normed linear spaces, Numer. Funct. Anal. Optim. **22** (2001) 199–222.
[32] H. X. Phu, Rough continuity of linear operators, Numer. Funct. Anal. Optim. **23** (2002) 139–146.
[33] H. X. Phu, Rough convergence in infinite dimensional normed spaces, Numer. Funct. Anal. Optim. **24** (2003) 285–301.
[34] S. Sarabadan, S. Talebi, Statistical convergence and ideal convergence of sequences of functions in 2-normed spaces, Int. J. Math. Math. Sci. **2011** (2011) 10 pages, doi:10.1155/2011/517841.
[35] E. Savaş, M. Gürdal, Ideal Convergent Function Sequences in Random 2-Normed Spaces, Filomat **30**(3) (2016) 557–567.
[36] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly **66** (1959) 361–375.
[37] A. Sharma, K. Kumar, Statistical convergence in probabilistic 2-normed spaces, Math. Sci. (Springer) **2**(4) (2008) 373–390.
[38] A. Şahiner, M. Gürdal, S. Saltan, H. Gunawan, Ideal convergence in 2-normed spaces, Taiwanese J. Math. **11**(5) (2007) 1477–1484.
[39] S. Yegül, E. Dündar, On Statistical Convergence of Sequences of Functions In 2-Normed Spaces, J. Class. Anal. **10**(1) (2017) 49–57.
[40] S. Yegül, E. Dündar, Statistical Convergence of Double Sequences of Functions and Some Properties In 2-Normed Spaces, Facta Univ. Ser. Math. Inform. **33**(5) (2018) 705–719.
[41] S. Yegül, E. Dündar, $\mathcal{I}_2$-Convergence of Double Sequences of Functions In 2-Normed Spaces, Univ. J.Math. Appl. **2**(3) (2019) 130–137.
[42] S. Yegül, E. Dündar, On $\mathcal{I}_2$-Convergence and $\mathcal{I}_2$-Cauchy Double Sequences Of Functions in 2-Normed Spaces, Facta Univ. Ser. Math. Inform. **35**(3) (2020) 801–814.